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Exponentiated log-logistic distribution: A Bayesian approach

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Abstract

In this paper, the Markov chain Monte Carlo (MCMC) method is used to estimate the parameters of exponentiated log-logistic distribution based on a complete sample. The procedures are developed to perform full Bayesian analysis of the exponentiated log-logistic distribution using MCMC simulation method in OpenBUGS, established software. We have obtained the Bayes estimates of the parameters and their probability intervals are presented. We have also discussed the estimation of reliability function. A real data set is considered for illustration under independent gamma priors.

Keywords - Exponentiated log-logistic distribution, maximum likelihood estimation, bayesian estimation, markov chain Monte Carlo, Model validation, OpenBUGS.

1. Introduction

The log-logistic distribution is very useful in survival analysis since it has a nonmonotonic hazard function, (Bennett, 1983) and (Tadikamalla and Johnson, 1982). The shape of this distribution is very similar to that of the log-normal, but has a more tractable form than that of the log-normal which makes it more convenient than the log-normal distribution when dealing with censored data. (Srivastava and Shukla, 2008) studied the log-logistic distribution as step-stress model. (Balakrishnan and Malik, 1987) gave the moments of order statistics from the truncated log-logistic distribution. This distribution has been also studied by (Howlader and Weiss, 1992). (Lawless, 2003), (Lee and Wang, 2003) and (Murthy *et al.,* 2004) provide an excellent review for the log-logistic distribution.In recent years, new classes of models have been proposed based on modifications such as adding parameters to the existing models. Adding one or more parameters to a distribution makes it richer and more flexible for modeling data. There are different ways for adding parameter(s) to a distribution. (Marshall and Olkin, 1997) added one positive parameter to a given (general) survival function. As described by (Marshall and Olkin, 2007) and (Klugman *et al.,* 2012), an exponentiated distribution can be easily constructed. It is based on the observation that by raising any baseline cumulative distribution function (cdf) $F_{baseline}(x)$ to an arbitrary power $\alpha > 0$, a new cdf $F(x) = \left(F_{baseline}(x)\right)^{\alpha}$; $\alpha > 0$ (1.1) is obtained with the additional parameter α. Following this idea, several authors have considered extensions from usual survival

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distributions. For instance, (Mudholkar and Srivastava, 1993) considered the exponentiated Weibull distribution as a generalization of the Weibull distribution. (Gupta and Kundu, 1999) introduced the exponentiated exponential distribution as a generalization of the usual exponential distribution and (Nadarajah and Kotz, 2006) proposed exponentiated type distributions extending the Frchet, gamma, Gumbel and Weibull distributions. (Rosaiah, *et al*. (2006, 2007)) studied the reliability test plan for exponentiated log-logistic distribution. (Santana *et al*., 2012) introduced the Kumaraswamy-log-logistic distribution, which includes exponentiated log-logistic distribution.

The cdf of the log-logistic distribution is given by

$$F_{LL}(x;\ \beta,\lambda)=\frac{(x/\lambda)^{\beta}}{1+(x/\lambda)^{\beta}}\ ;(\beta,\lambda)>0,\quad x>0 \tag{1.2}$$

where $\beta>0$ is the shape and $\lambda>0$ is the scale parameter.

The cdf of the exponentiated log-logistic(ELL) distribution is defined by raising $F_{LL}(x)$ to the power of α, namely $F(x)=(F_{LL}(x))^{\alpha}$. The distribution function of ELL distribution with three parameters is given by

$$F(x;\alpha,\beta,\lambda)=\left\{\frac{(x/\lambda)^{\beta}}{1+(x/\lambda)^{\beta}}\right\}^{\alpha}\ ;(\alpha,\beta,\lambda)>0,\quad x>0 \tag{1.3}$$

where $\alpha>0$ and $\beta>0$ are shape parameters and $\lambda>0$ is a scale parameter. For $\alpha=1$, the model reduces to the log-logistic distribution. Since λ is the scale parameter, we can assume $\lambda=1$ without any loss of generality. For $\lambda=1$, we have two-parameter ELL distribution and we shall denote it as $ELL(\alpha,\beta)$.

The rest of the article is organized as follows. The model and its features are introduced in Section 2. In Section 3, we have discussed the Bayesian model formulation including the priors, posterior, Gibbs sampler and its implementation in OpenBUGS. The real data set and its exploratory data analysis, maximum likelihood estimation (MLE) and model validation are described in Section 4. The full Bayesian analysis under independent gamma of priors for the data set using Markov chain Monte Carlo (MCMC) simulation method in OpenBUGS, an established software, is presented in Section 5. The Bayes estimates of the parameters and their probability intervals based on posterior samples are presented. The posterior analysis is performed and we have also estimated the reliability function. Conclusions are given in Section 6.

2. The exponentiated log-logistic model

The cumulative distribution function of exponentiated log-logistic (ELL) distribution with two parameters is given by

$$F(x;\alpha,\beta)=\left\{\frac{x^{\beta}}{1+x^{\beta}}\right\}^{\alpha}\ ;(\alpha,\beta)>0,\quad x>0 \tag{2.1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape parameters. The corresponding probability density function is given by

$$f(x;\alpha,\beta) = \frac{\alpha\ \beta\ x^{\alpha\beta}}{x\left[1 + x^{\beta}\right]^{\alpha+1}} \quad ;(\alpha,\beta) > 0, x > 0 \cdot \tag{2.2}$$

The reliability/survival function is

$$R(x;\alpha,\ \beta) = 1 - \left\{\frac{x^{\beta}}{1 + x^{\beta}}\right\}^{\alpha} \quad ;(\alpha,\beta) > 0, \quad x > 0 \cdot \tag{2.3}$$

The hazard rate function is

$$h(x;\alpha,\beta) = \frac{\alpha\ \beta\ x^{\alpha\beta}}{x\left(1 + x^{\beta}\right)^{\alpha+1}}\left(1 - \left\{\frac{x^{\beta}}{1 + x^{\beta}}\right\}^{\alpha}\right)^{-1} ;\ (\alpha,\beta) > 0,\ x > 0. \tag{2.4}$$

The quantile function is given by

$$x_p = \left(p^{-1/\alpha} - 1\right)^{-1/\beta} ; \quad 0 < p < 1 \cdot \tag{2.5}$$

The random deviate can be generated from $ELL(\alpha,\beta)$ by

$$x = \left(u^{-1/\alpha} - 1\right)^{-1/\beta} ; 0 < u < 1 \tag{2.6}$$

where u has the $U(0,1)$ distribution.

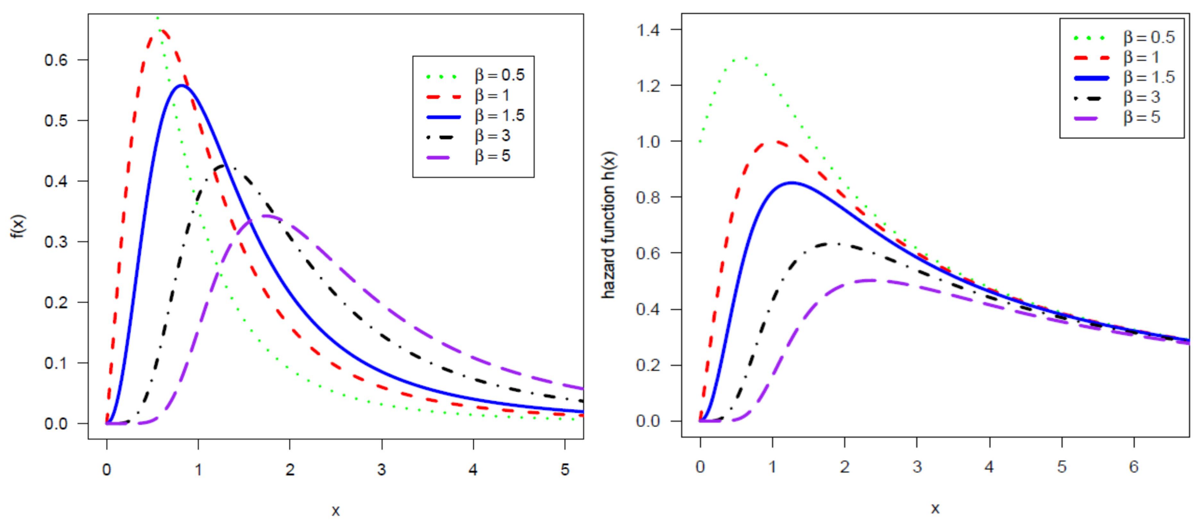


Figure 1 The probability density function (Left panel); The hazard function (Right panel) of $ELL(\alpha,\beta)$ distribution for $\alpha = 1$ and different values of β.

The r^{th} moment and mode are given by

$$\mu_r' = \frac{\alpha}{\Gamma(\alpha+1)}\,\Gamma\left(\alpha+\frac{r}{\beta}\right)\,\Gamma\left(1-\frac{r}{\beta}\right) \quad ; -\beta < r < \beta \tag{2.7}$$

and $$\text{Mode} = \left(\frac{\alpha\beta-1}{\beta+1}\right)^{1/\beta} \quad ; \alpha\beta \geq 1 \cdot$$

Some of the typical $ELL(\alpha,\beta)$ density functions for different values of β and for $\alpha = 1$ are depicted in Figure 1 (left panel). It is evident from the Figure 1 that the density function of the ELL distribution can take different shapes. Figure 1(right panel) exhibits the different hazard rate functions of $ELL(\alpha,\beta)$ distribution.

3. Bayesian model formulation

In this section, we provide the Bayes estimates of the shape parameters assuming independent gamma priors for both the parameters α and β. For the $ELL(\alpha,\beta)$, the Bayesian model is constructed by specifying a prior distribution for α and β, and then multiplying with the likelihood function to obtain the posterior distribution function. Given a set of data $\underline{x} = (x_1,\ldots,x_n)$, the likelihood function is

$$L(\alpha,\beta \mid \underline{x}) = \alpha^n \beta^n \prod_{i=1}^{n} \frac{x_i^{\alpha\beta-1}}{\left(1+x_i^{\beta}\right)^{\alpha-1}} \; . \tag{3.1}$$

Prior distributions

Denote the prior distribution of α and β as $p(\alpha,\beta)$. The joint posterior is

$$p(\alpha,\beta \mid \underline{x}) \propto L(\alpha,\beta \mid \underline{x})\; p(\alpha,\beta) \; . \tag{3.2}$$

We assume the independent gamma priors for $\alpha \sim G(a,\, b)$ and $\beta \sim G(c,\, d)$ as

$$p(\alpha) = \frac{b^a}{\Gamma(a)}\,\alpha^{a-1}\,e^{-b\alpha} \quad ; \alpha > 0, (a,b) > 0 \tag{3.3}$$

and

$$p(\beta) = \frac{d^c}{\Gamma(c)}\,\beta^{c-1}\,e^{-d\beta} \quad ; \beta > 0, (c,d) > 0. \tag{3.4}$$

Posterior distribution

Combining the likelihood function with the prior via Bayes' theorem yields the posterior up to proportionality as

$$p(\alpha,\beta \mid \underline{x}) \propto \alpha^{a+n-1}\,\beta^{c+n-1}\,exp\left\{-\left(b\alpha+d\beta\right)\right\} T_1 \tag{3.5}$$

where $$T_1 = \prod_{i=1}^{n} \frac{x_i^{\alpha\beta-1}}{\left(1 + x_i^{\beta}\right)^{\alpha-1}} . \tag{3.6}$$

The posterior is obviously complicated and no close form inferences appear possible. We, therefore, propose to consider MCMC methods to simulate samples from the posterior so that sample-based inferences can be easily drawn. The Gibbs sampler is as an important Markov Chain Monte Carlo technique, which provides a way for extracting samples from the posteriors.. This sampling scheme was first introduced by (Geman and Geman, 1984), but the applicability to statistical modelling for Bayesian computation was demonstrated by (Gelfand and Smith, 1990). It generates a sample from an arbitrarily complex multidimensional distribution by sampling from each of the univariate full conditional distributions in turn. That is, every time a variate value is generated from a full conditional, it is influenced by the most recent values of all other conditioning variables and, after each cycle of iteration, it is updated by sampling a new value from its full conditional. The entire generating scheme is repeated unless the generating chain achieves a systematic pattern of convergence. It can be shown that after a large number of iterations the generated variates can be regarded as the random samples from the corresponding posteriors. (Gelman *et al*., 2004), (Albert, 2009), (Hamada *et al*., 2008), (Ntzoufras, 2009) and (Hoff, 2009) provide the details of the procedure and the related convergence diagnostic issues. Therefore to obtain the full conditional distribution of α (or β), we need only choose the terms in the posterior, which involve parameter α (or β). The full posterior conditional distributions for α and β, are

$$p\left(\alpha \mid \beta, \underline{x}\right) \propto \alpha^{a+n-1}\, exp\left(-b\,\alpha\right) T_1 \tag{3.7}$$

and

$$p\left(\beta \mid \alpha, \underline{x}\right) \propto \beta^{c+n-1}\, exp\left(-d\,\beta\right) T_1 \ . \tag{3.8}$$

As the exponentiated log-logistic(ELL) distribution is not available in OpenBUGS., it requires incorporation of a module in *ReliaBUGS,* (Kumar et al., 2010) and (Lunn, 2010), subsystem of *OpenBUGS* for ELL.

A module *dexpo.loglogistic_T(alpha, beta)* is written in Component Pascal for ELL, the corresponding computer program can be obtained from authors, to perform full Bayesian analysis in OpenBUGS using the method described in (Thomas *et al.,* 2006), (Thomas, 2010), (Kumar *et al.,* 2010) and (Lunn *et al.,* 2013). It is important to note that this module can be used for any set of suitable priors of the model parameters. Almost all aspects of the model in Bayesian framework can be studied using the developed module *dexpo.loglogistic_T (alpha, beta),* (Kumar, 2010).

Gibbs Sampler : Implementation

1. Select an initial value $\underline{\theta}^{(0)} = \left(\alpha^{(0)}, \beta^{(0)}\right)$ to start the chain.
2. Suppose at the *ith*-step, $\underline{\theta} = \left(\alpha, \beta\right)$ takes the value $\underline{\theta}^{(i)} = \left(\alpha^{(i)}, \beta^{(i)}\right)$ then from full conditionals, we generate

   $\alpha^{(i+1)}$ from $p\left(\alpha \mid \beta^{(i)}, \underline{x}\right)$ and

$\beta^{(i+1)}$ from $p\left(\beta \mid \alpha^{(i+1)}, \underline{x}\right)$.

3. This completes a transition from $\underline{\theta}^{(i)}$ to $\underline{\theta}^{(i+1)}$.
4. Repeat Step 2, N times.

MCMC output: Posterior sample

Monitor the convergence using convergence diagnostics (trace and ergodic mean plots). Suppose that convergence have been reached after 'B' iterations (the burn-in period). Discard the observations $\left(\underline{\theta}^{(1)}, \underline{\theta}^{(2)}, \ldots, \underline{\theta}^{(B)}\right)$ and retain the observations

$$\left(\underline{\theta}^{(B+1+(j-1)L)};\ B+1+(M-1)L \le N;\ j = 1, 2, \ldots, M;\ L \ge 1\right)$$

which are viewed as being an independent sample from the stationary distribution of the Markov chain that is typically the posterior distribution, where 'L' is the lag (or thin interval).

Consider $\left(\underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}, \ldots, \underline{\theta}^{(M)}\right)$ as the MCMC output (posterior sample) for the posterior analysis

$$\underline{\theta}^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}\right);\ j = 1, 2, \ldots, M \cdot$$

Thus MCMC output is referred as the sample after removing the initial iterations (produced during the burn-in period) and considering the appropriate lag. The Bayes estimates of $\underline{\theta} = \left(\alpha, \beta\right)$, under squared error loss function, using the ergodic theorem are given by

$$\hat{\alpha} = \frac{1}{M}\sum_{j=1}^{M} \alpha^{(j)} \quad \text{and} \quad \hat{\beta} = \frac{1}{M}\sum_{j=1}^{M} \beta^{(j)} \ . \tag{3.9}$$

An important advantage of sample-based approaches includes the routine developments for any linear and/or non-linear functions of the original parameters. It is to be noted that once the samples from the posterior is obtained, samples from the posterior of any linear and/or non-linear functions can be easily created merely by substitution. Some of such functions where reliability practitioners are often interested include reliability, hazard rate, mean time to failure, percentiles, etc.

4. Data, maximum likelihood estimation and model validation

The following real data set is considered for illustration of the proposed methodology. The data given below represent active repair times (in hours) for 46 repair times of an airborne communication transceiver. (Chhikara and Folks, 1977) fitted a two-parameter inverse Gaussian distribution. The data are presented below:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5

4.1Exploratory data analysis (EDA)

EDA is an approach to statistical analysis, heavily graphical in nature that attempts to maximize insight into data, (Tukey, 1977). It allows data to speak for themselves, without making assumptions and conducting formal analyses. The descriptive statistical methods quantitatively describe the main features of data.

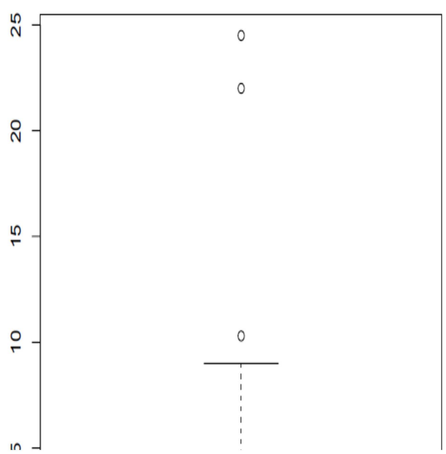


Table -1: Summary statistics

Minimum	0.200
First Quartile (Q_1)	0.800
Median	1.750
Mean	3.607
Third Quartile (Q_3)	4.375
Maximum	24.500
Kurtosis	8.295
Skewness	2.795

The main data features are (i) measures of central tendency(e.g. mean and median); (ii) measures of variability (e.g., standard deviation) and (iii) measures of relative standing (e.g., quantiles). The descriptive statistics for the above data set are presented in Table 1. We have plotted the boxplot in Figure 2, which shows that data set contains three "outliers". The estimation of the parameter of the proposed model is obtained by the method of maximum likelihood (ML) estimation.

4.2 Maximum likelihood estimation (MLE) and asymptotic confidence intervals

In this section, we briefly discuss the maximum likelihood estimators (MLE's) of the two-parameter ELL distribution and discuss their asymptotic properties to obtain approximate confidence intervals based on MLE's.

Let $\underline{x} = (x_1, \ldots, x_n)$ be a random sample of size n from $ELL(\alpha, \beta)$, then the log-likelihood function $\ell(\alpha, \beta)$ can be written as;

$$\ell(\alpha, \beta) = n \log \alpha + n \log \beta + \alpha \sum_{i=1}^{n} \log\left(x_i^{\beta}\right) - \sum_{i=1}^{n} \log x_i - (\alpha + 1) \sum_{i=1}^{n} \log\left(1 + x_i^{\beta}\right). \quad (4.1)$$

Therefore, to obtain the MLE's of α and β, we can maximize (9) directly with respect to α and β or we can solve the following two non-linear equations using iterative method e.g. Newton-Raphson method

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(x_i^{\beta}\right) - \sum_{i=1}^{n} log\left(1 + x_i^{\beta}\right) = 0$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \alpha \sum_{i=1}^{n} log\left(x_i\right) - \left(\alpha + 1\right) \sum_{i=1}^{n} \left(\frac{x_i^{\beta}}{1 + x_i^{\beta}}\right) log\left(x_i\right) = 0 \cdot$$

Let us denote the parameter vector by $\underline{\theta} = \left(\alpha, \beta\right)$ and the corresponding MLE of $\underline{\theta}$ as $\underline{\hat{\theta}} = \left(\hat{\alpha}, \hat{\beta}\right)$ then the asymptotic normality results in

$$\left(\underline{\hat{\theta}} - \underline{\theta}\right) \rightarrow N_2\left(0, \left(I\left(\underline{\theta}\right)\right)^{-1}\right) \tag{4.2}$$

where $I\left(\underline{\theta}\right)$ is the Fisher's information matrix given by

$$I\left(\underline{\theta}\right) = -\begin{bmatrix} E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \beta \, \partial \alpha}\right) & E\left(\frac{\partial^2 \ell}{\partial \beta^2}\right) \end{bmatrix}. \tag{4.3}$$

In practice, it is useless that the MLE has asymptotic variance $\left(I\left(\underline{\theta}\right)\right)^{-1}$ because we do not know $\underline{\theta}$. Hence, we approximate the asymptotic variance by "plugging in" the estimated value of the parameters. The common procedure is to use observed Fisher information matrix $O\left(\underline{\hat{\theta}}\right)$ (as an estimate of the information matrix $I\left(\underline{\theta}\right)$) given by

$$O\left(\underline{\hat{\theta}}\right) = -\left.\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \, \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}\right|_{\left(\hat{\alpha}, \hat{\beta}\right)} = -H\left(\theta\right)\Big|_{\underline{\theta} = \underline{\hat{\theta}}} \tag{4.4}$$

where H is the Hessian matrix, $\underline{\theta} = \left(\alpha, \beta\right)$ and $\underline{\hat{\theta}} = \left(\hat{\alpha}, \hat{\beta}\right)$. The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by

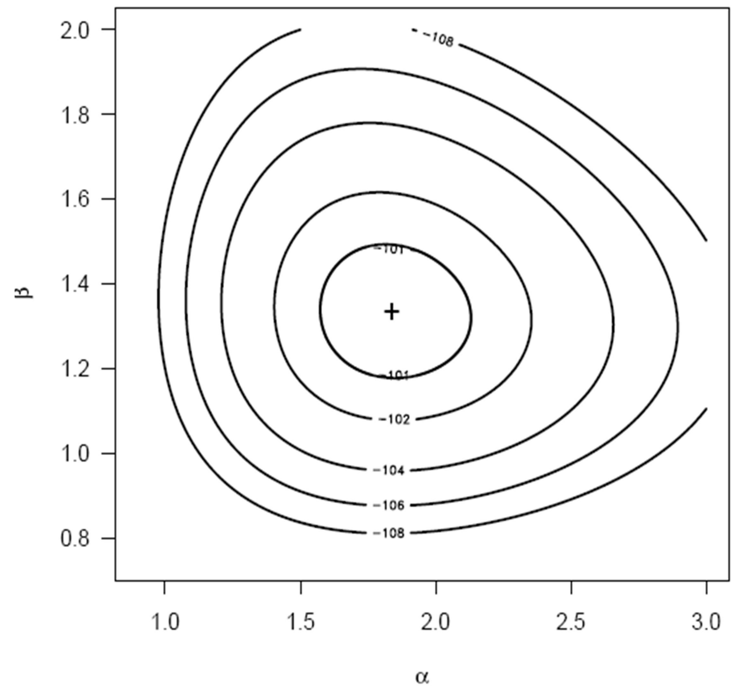
$$\left(-H\left(\underline{\theta}\right)\Big|_{\underline{\theta} = \underline{\hat{\theta}}}\right)^{-1} = \begin{pmatrix} var\left(\hat{\alpha}\right) & cov\left(\hat{\alpha}, \hat{\beta}\right) \\ cov\left(\hat{\beta}, \hat{\alpha}\right) & var\left(\hat{\beta}\right) \end{pmatrix}. \tag{4.5}$$

Hence, from the asymptotic normality of MLEs, approximate $100(1-\gamma)\%$ confidence intervals for α and β can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha})} \quad \text{and} \quad \hat{\beta} \pm z_{\gamma/2}\sqrt{var(\hat{\beta})} \tag{4.6}$$

where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

4.3 Computation of MLE



We have started the iterative procedure by maximizing the log-likelihood function given in equation.(9) directly with an initial guess for $\alpha = 0.5$ and $\beta = 0.5$, far away from the solution. We have used *optim*( ) function in R, (R Development Core Team, 2013) and (Rizzo, 2008), with option Newton-Raphson method. The iterative process stopped only after 26 iterations. We obtain $\hat{\alpha} = 1.838$ and $\hat{\beta} = 1.3297$ and the corresponding log-likelihood value is $\ell(\hat{\alpha}, \hat{\beta}) = -100.474$. We have plotted the contour plot of $\ell(\alpha, \beta)$ in Figure 3, the (+) indicates the MLE.

The 95% confidence interval is computed using (4.5) and (4.6). The Table 2 shows the ML estimates, standard error (SE) and 95 % Confidence Intervals for the parameters alpha and beta.
The Akaike information criterion (AIC) and Bayesian information criterion (BIC) are defined as

$$\text{AIC} = -2\,\ell(\underline{\hat{\theta}}) + 2\,p \quad \text{and} \quad \text{BIC} = -2\ell(\underline{\hat{\theta}}) + p\,log\left(n\right)$$

where $\underline{\hat{\theta}} = (\hat{\alpha}, \hat{\beta})$ is the ML estimate of $\underline{\theta} = (\alpha, \beta)$ and p is the number of parameters estimated in the model. The smaller the value of AIC and BIC, the better the model. The values of the information measures are AIC= 204.9 and BIC = 208.6, respectively.

Table 2 MLE, standard error and 95% confidence interval

Parameter	MLE	Std. Error	95% Confidence Interval
alpha	1.8381	0.27185	(1.3053, 2.3709)
beta	1.3297	0.15382	(1.0282, 1.6312)

4.4 Model validation

To check the validity of the model, we compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by method of maximum likelihood. The graphical methods Quantile-Quantile (QQ) and Probability–Probability (PP) plots are used for suitability of the model under consideration.

The value of K-S test statistic is 0.0899 and the corresponding p-value is given by 0.8514. The high p-value clearly indicates that ELL distribution can be used to analyze the given data set, and we have also plotted the empirical distribution function and the fitted distribution function in Figure 4. It is clear that the estimated ELL distribution provides reasonable fit to the given data, (Kumar and Ligges, 2011).

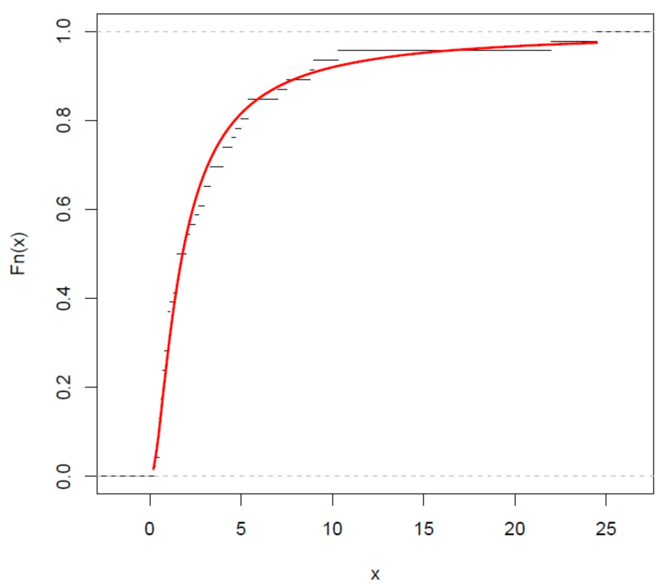


Figure 4 The empirical and fitted distribution function.

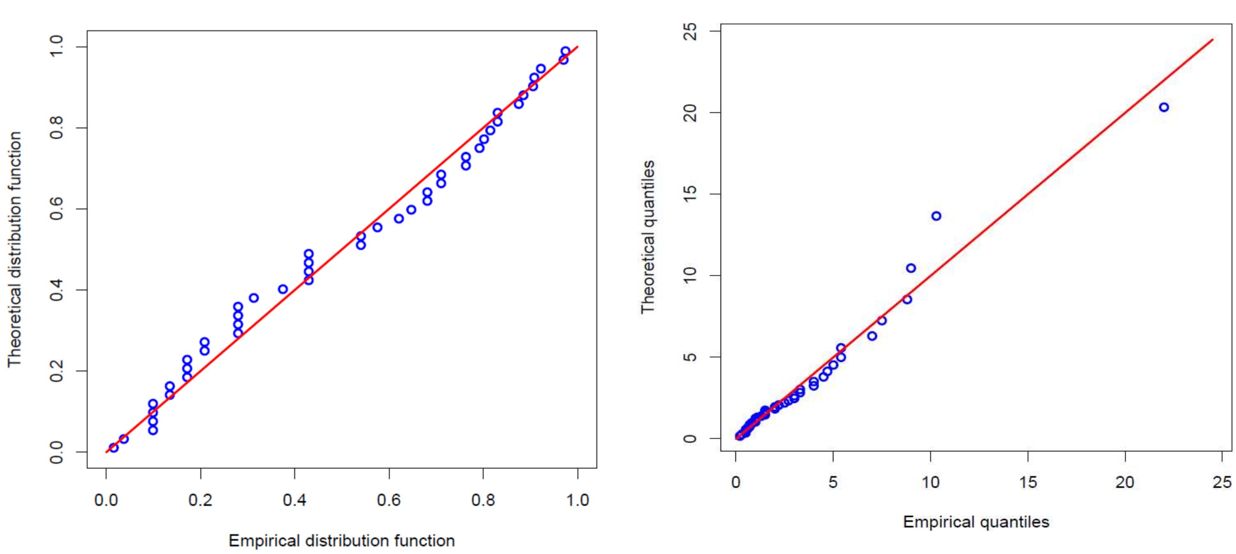


Figure 5 Probability-Probability (PP) plot (left panel); Quantile-Quantile (QQ) plot (right panel)

A further support for this finding can be obtained by inspecting the probability-probability (P–P) and quantile–quantile (Q–Q) plots. The P-P plot shows the empirical and theoretical distribution functions. The Q-Q plot shows the estimated versus the observed quantiles. As can be seen from the straight line pattern in Figure 5 the ELL fits the data well.

5. Bayesian analysis

The developed module *dexpo.loglogistic_T(alpha, beta)* in OpenBUGS is implemented for the full Bayesian analysis of the exponentiated log-logistic distribution using MCMC method. The following script can be used to get samples from the posterior arising from the model.

OpenBUGS script for the Bayesian analysis of ELL distribution

```
model

{for( i in 1 : N ) {x[i] ~ dexpo.loglogistic_T(alpha, beta)  # ELL distribution reliability[i] <-
R(x[i], x[i])  # to estimate reliability f[i] <- density(x[i], x[i])  # to estimate density}

# Prior distributions of the model parameters

        alpha ~ dgamma(0.001, 0.001)

        beta ~ dgamma(0.001, 0.001) }
```

Data

```
list(N=46, x = c(0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1,
1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0,
7.5, 8.8, 9.0, 10.3, 22.0, 24.5))
```

Initial values

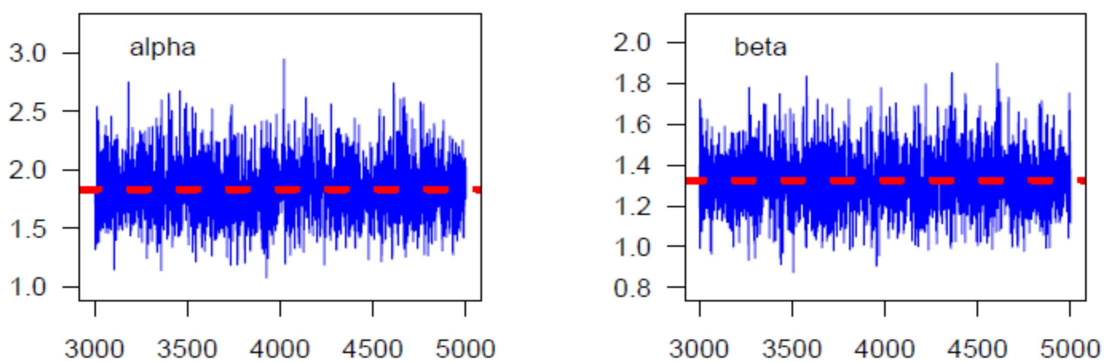
```
list(alpha=0.5, beta=0.5)      # Chain 1

list(alpha=3.0, beta=2.5)      # Chain 2
```

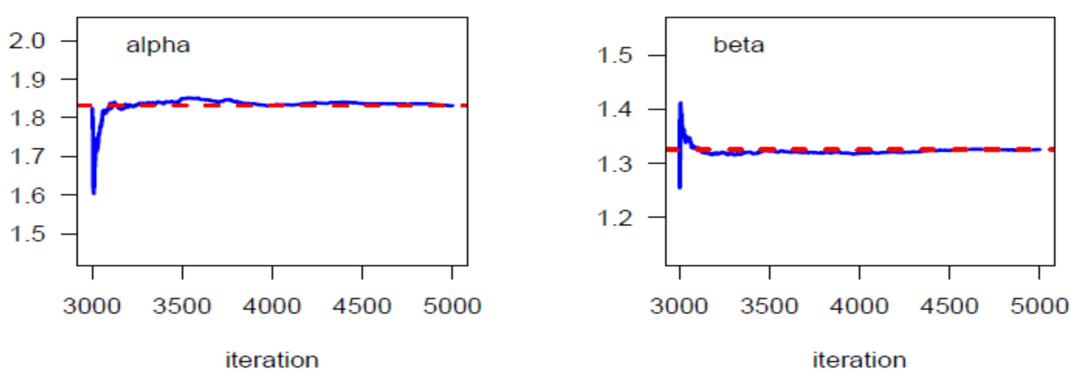
We run the model to generate two Markov Chains at the length of 40,000 with different starting points of the parameters. The convergence is monitored using trace and ergodic mean plots, we find that the Markov Chain converge together after approximately 2000 observations. Therefore, burn-in of 5000 samples is more than enough to erase the effect of starting point (initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome, i.e. thin=5, starting from 5001.This are done to minimize the auto correlation among the generated deviates.

Therefore, we have the posterior sample $\left(\alpha_1^{(j)}, \beta_1^{(j)}\right)$; $j = 1,\ldots,7000$ from chain 1 and $\left(\alpha_2^{(j)}, \beta_2^{(j)}\right)$; $j = 1,\ldots,7000$ from chain 2.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on posterior sample obtained from chain 1 whereas the numerical summary is presented for both the chains. The convergence is monitored by history/ trace and ergodic means plots. The sequential plot of parameters is the plot that most often exhibits difficulties in the Markov chain. Figure 6 shows the sequential realizations of the parameters of the model. In this case Markov chain seems to be mixing well enough and is likely to be sampling from the stationary distribution. The plot looks like a horizontal band, with no long upward or downward trends, we have evidence that the chain has converged.



The running mean (ergodic mean) plot is a time series(iteration number) plot of the running mean for each parameter in the chain. The running mean is computed as the mean of all sampled values up to and including that at a given iteration. The convergence pattern based on ergodic averages is shown in Figure 7 indicating the convergence of the chain.



We may consider an independent sample from the target distribution i.e. posterior. Thus, we can obtain the posterior summary statistics.

5.1 Posterior analysis

(a) Numerical Summary

The numerical summary is presented for $\left(\alpha_1^{(j)}, \beta_1^{(j)}\right)$; $j = 1, \ldots, 7000$ from chain 1 and $\left(\alpha_2^{(j)}, \beta_2^{(j)}\right)$; $j = 1, \ldots, 7000$ from chain 2.

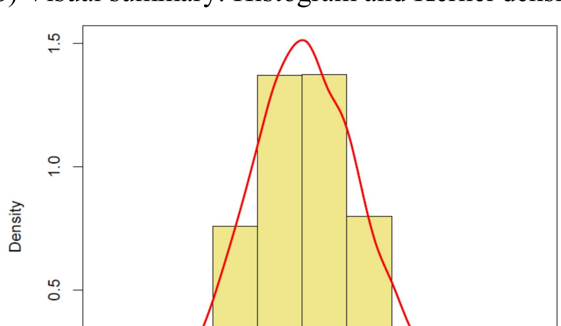
We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for ELL distribution. The MCMC results of the posterior mean, mode, standard deviation(SD), five point summary statistics (minimum, first quartile, median, third quartile and maximum), 2.5^{th} percentile, 97.5^{th} percentile, 95% symmetric and HPD credible intervals of the parameters α and β are displayed in Table 3. In fact, we have summarized the uncertainty about the parameters.

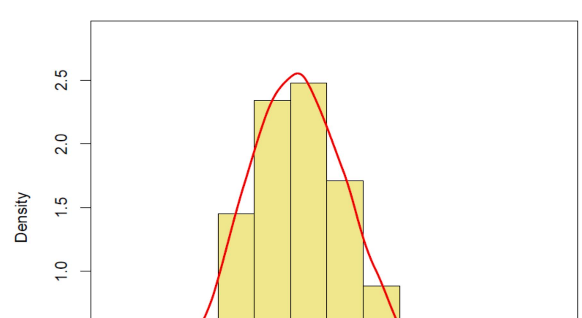
Table 3. Numerical summaries based on MCMC sample of posterior characteristics for ELL distribution under gamma priors

Characteristics	Chain 1		Chain 2	
	alpha	beta	alpha	beta
Mean	1.830	1.327	1.836	1.328
Standard Deviation(S.D.)	0.271	0.157	0.275	0.153
Monte Carlo(MC) error	0.003	0.002	0.003	0.002
Minimum	0.982	0.824	1.030	0.791
First Quartile (Q_1)	1.645	1.218	1.640	1.222
Median	1.817	1.322	1.826	1.320
Third Quartile (Q_3)	2.001	1.430	2.012	1.427
Maximum	3.179	2.062	2.904	2.200
Mode	1.801	1.323	1.857	1.303
2.5th Percentile($P_{2.5}$)	1.336	1.040	1.344	1.048
97.5th Percentile($P_{97.5}$)	2.400	1.654	2.412	1.632
95% Credible Interval	(1.336, 2.400)	(1.040, 1.654)	(1.344, 2.412)	(1.048, 1.642)
95% HPD Credible Interval	(1.318, 2.374)	(1.018, 1.627)	(1.329, 2.393)	(1.046, 1.640)

The posterior characteristics presented in Table 3 are very closed for chain 1 and chain 2. Therefore, one can use either of the chains for posterior analysis. The Highest probability density (HPD) intervals are computed the algorithm described by (Chen and Shao, 1999) under the assumption of unimodal marginal posterior distribution.

(b) Visual summary: Histogram and Kernel density estimates





The Figure 8 and Figure 9 represent the histogram and marginal posterior density for α (left panel) and for β (right panel). We have also plotted the actual realizations of parameter values along x-axis, which is known as "rug" plot. Histograms can provide insights on skewness, behaviour in the tails, presence of multi-modal behaviour, and data outliers; histograms can be compared to the fundamental shapes associated with standard analytic distributions. The kernel density estimates have been drawn using R software with the assumption of Gaussian kernel and properly chosen values of the bandwidths. It can be seen that α and β both are slightly positive skewed.

5.2 Comparison with MLE

We have used graphical method for the comparison of Bayes estimates with ML estimates. In Figure 10, the density functions $f(x;\hat{\alpha},\hat{\beta})$ using MLEs and Bayesian estimates, computed via MCMC samples under gamma priors, are plotted. It is clear from the Figure 10 that the MLEs and the Bayes estimates with respect to the gamma priors are quite close and fit the data very well.

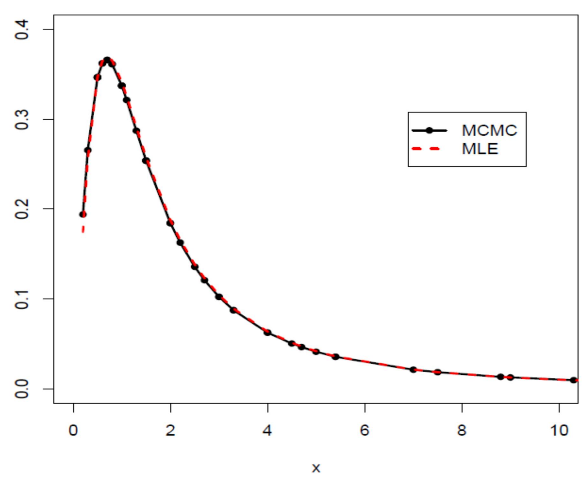


Figure 10 The density functions using ML and Bayesian estimates

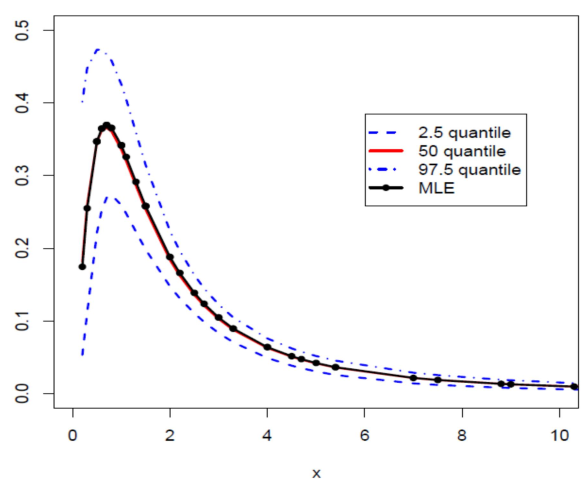


Figure 11 Density estimates

A further support for this finding can be obtained by inspecting the Figure 11. In Figure 15 we have plotted $2.5^{\text{th}}, 50^{th}$ and 97.5^{th} quantiles of the estimated density based on MCMC sample $\left(\alpha_1^{(j)}, \beta_1^{(j)}\right);\ j = 1, \ldots, 7000$. Here the density is computed at each data point for 7000 posterior samples. The density corresponding to MLE has been plotted using the "plug-in" estimates of the parameters. It shows that we have a fairly good model for the given data set.

5.3 Estimation of reliability function

In this section our main aim is to demonstrate the effectiveness of proposed methodology. For this we have estimated the reliability function using MCMC samples under gamma priors. Since we have an effective MCMC technique, we can estimate any function of the parameters. We have used the empirical reliability function to make the comparison more meaningful. The Figure 12, exhibits the estimated reliability function (dashed line: 2.5^{th} and 97.5^{th} quantiles; solid line: 50^{th} quantile) using Bayes estimate based on MCMC output under independent gamma priors for both the parameters and the empirical reliability function (solid line). The Figure 12 shows that reliability estimate based on MCMC is very closed to the empirical reliability estimates.

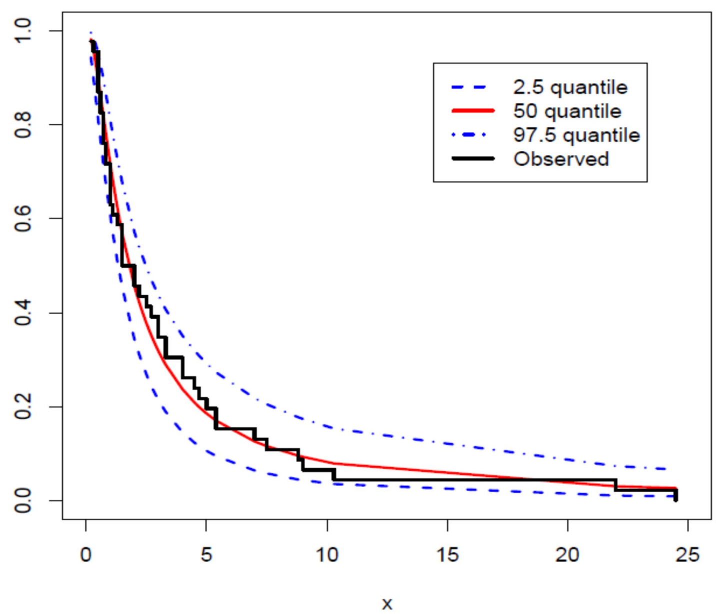


Figure 12 The empirical and estimated reliability function using MCMC

Conclusion

We have discussed the Markov chain Monte Carlo (MCMC) method to compute the Bayesian estimates the parameters and reliability functions of exponentiated log-logistic distribution based on a complete sample. We have obtained the probability intervals for parameters. The MCMC method provides an alternative method for parameter estimation of the exponentiated log-logistic distribution. It is more flexible when compared with the traditional methods such as MLE method. Moreover, 'exact' probability intervals are available rather than relying on estimates of the asymptotic variances. Indeed, the MCMC sample may be used to completely summarize posterior distribution about the parameters, through kernel estimation. This is also true for any function of the parameters such as reliability and hazard functions. We have applied the developed techniques on a real data set. The paper successfully describes the scope of Markov chain Monte Carlo (MCMC) technique in the exponentiated log-logistic distribution. Thus, the tools developed can be applied for full Bayesian analysis of exponentiated log-logistic distribution.

Reference

1. Albert, J. (2009). *Bayesian Computation with R*. 2nd edition. Springer, New York.
2. Balakrishnan, N. and Malik, H.J. (1987). Moments of order statistics from truncated log-logistic distribution. *Journal of Statistical Planning and Inference***.** 17: 251-267.
3. Bennette, S. (1983). Log-logistic regression models for survival data. Applied Statistics.32:165 – 171.
4. Chen, M. H. and Shao, Q. M. (1999). Monte Carlo estimation of Bayesian credible intervals and HPD intervals. *Journal of Computational and Graphical Statistics*. 8(1): 69-92.
5. Chhikara, R.S. and Folks, J.L. (1977). The inverse Gaussian distribution as a lifetime model. *Technometrics*. 19: 461-468.
6. Gelfand, A.E. and Smith, A.F.M. (1990). Sampling based approach to calculating marginal densities. *Journal of the American Statistical Association*. 85: 398-409.
7. Gelman, A.; Carlin, J.; Stern, H., and Rubin, D. (2004). *Bayesian Data Analysis*. Second Edition, Chapman & Hall, London.
8. Geman, S. and Geman, D. (1984). Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images. *IEEE Transactions of Pattern Analysis and Machine Intelligence*. 6: 721-741.
9. Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*. 41(2): 173 - 188.
10. Hamada, M.S.; Wilson, A.G.; Reese, C.S. and Martz, H.F.(2008). *Bayesian Reliability*. Springer, New York.
11. Hoff, P.D. (2009). *A First Course in Bayesian Statistical Methods*. Springer, New York.
12. Howlader, H.A. and Weiss, G.(1992). Log-logistic survival estimation based on failure-censored data. *Journal of Applied Statistics*. 19, No. 2 : 231-240.
13. Klugman, S.; Panjer, H. and Willmot, G. (2012). *Loss Models: From Data to Decisions.* (4th edition). John Wiley & Sons, New York.
14. Kumar, V. (2010). Bayesian analysis of exponential extension model. *J. Nat. Acad. Math*. 24 :109-128.
15. Kumar, V. and Ligges, U. (2011). *reliaR: A package for some probability distributions*. http://cran.r-project.org/web/packages/reliaR/index.html.
16. Kumar, V.; Ligges, U. and Thomas, A. (2010). *ReliaBUGS User Manual: A subsystem in OpenBUGS for some statistical models*. version 1.0, OpenBUGS 3.2.1, http://openbugs.info/w/Downloads/
17. Lawless, J. F., (2003). *Statistical Models and Methods for Lifetime Data*. 2nd ed., John Wiley and Sons, New York.
18. Lee, E.T. and Wang, T.W.(2003). Statistical Methods for Survival Data Analysis. 3rd Ed., John Wiley & Sons.
19. Lunn, D. (2010). Recent Developments in the BUGS software. *ISBA Bulletin*. 17(3):16-17.
20. Lunn, D.J., Andrew, A., Best, N. and Spiegelhalter, D. (2000). WinBUGS– A Bayesian modeling framework: Concepts, structure, and extensibility. *Statistics and Computing*. 10**:** 325–337

21. Lunn, D.J.; Jackson, C.; Best, N.; Andrew, A., and Spiegelhalter, D. (2013). *The BUGS Book : A Practical Introduction to Bayesian Analysis*. Chapman & Hall/CRC, London.
22. Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika.* 84(3):641–652.
23. Marshall, A. W. and Olkin, I. (2007). *Life Distributions: Structure of Nonparametric, Semiparametric and Parametric Families*. Springer, New York.
24. Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*. 42(2): 299–302.
25. Murthy, D.N.P., Xie, M. and Jiang, R. (2004). *Weibull Models*. Wiley, New York.
26. Nadarajah, S. and Kotz, S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae*. 92: 97-111.
27. Ntzoufras, I. (2009). *Bayesian Modeling using WinBUGS*. John Wiley & Sons, New York
28. R Development Core Team (2013). R: A language and environment for statistical computing. R Foundation for Statistical Computing. Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org.
29. Rizzo, M. L.(2008). *Statistical computing with R*. Chapman & Hall/CRC.
30. Rosaiah, K.; Kantam, R.R.L. and Kumar, Ch. S. (2006). Reliability test plans for exponentiated log- logistic distribution. *Economic Quality Control*. 21(2) : 165-175.
31. Rosaiah, K.; Kantam, R.R.L. and Santosh Kumar, Ch. (2007). Exponentiated log-logistic distribution - an economic reliability test plan. *Pakistan Journal of Statistics*. 23 (2) : 147 -156.
32. Santana, T.V.F.; Ortega, E.M.M.; Cordeiro, G.M. and Silva, G.O. (2012). The Kumaraswamy-Log-Logistic Distribution. *Journal of Statistical Theory and Applications*. 11(3) : 265-291.
33. Srivastava, P.W. and Shukla, R. (2008). A Log-Logistic Step-Stress Model. *IEEE Transactions on Reliability*. 57(3) : 431-434.
34. Tadikamalla, P. R. and Johnson, N.L. (1982). Systems of frequency curves generated by the transformation of logistic variables. *Biometrika*. 69: 461-465.
35. Thomas, A.; O'Hara, B., Ligges, U. and Sturtz, S. (2006). Making BUGS Open. *R News*. 6 : 12–17.
36. Thomas,A.(2010).*OpenBUGS Developer Manual*. ver.3.1.2, http://www.openbugs.info/
37. Tukey, J.W. (1977). *Exploratory Data Analysis*. Reading, Mess. Addison-Wesley.

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