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# On the special concircular h-curvature collineation in a bi-recurrent Finsler space

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## Abstract

*A con-circular transformation in Riemannian spaces was studied in a series of papers by Yano (1940, 42) and Okumura (1962), has developed a similar transformation in non-Riemannian symmetric space. The concept of special con-circular projective curvature collineation in a Recurrent Finsler space was developed by Singh (2001). He discussed some basic properties of afore-said transformation in Recurrent Finsler space. The object of present paper is to discuss the special con-circular H-curvature collineation in Bi-recurrent Finsler space. The necessary condition for existence of such transformation in Bi-recurrent Finsler space has been established. We have also discussed some fundamental properties of such transformation in afore-said space.*

## 1. Preliminaries

Consider an n-dimensional Finsler space $F_n$ (Rund, 1959) with Berwald's connection parameter $G^i_{jk}(x,\dot{x})$. The curvature tensor $H^i_{jkh}$ arising from connection $G^i_{jk}$, is homogeneous function of degree zero in $\dot{x}$ and hence, we have :

$$H^i_{jkh}\dot{x}^h = H^i_{jk} \tag{1.1}$$

$$\dot{\partial}_l H^i_{jkh}\dot{x}^l = 0 \tag{1.2}$$

The Berwald's covariant derivative of a tensor field $T^i_j(x,\dot{x})$ with respect to $x^k$, is given by Rund, 1959.

$$T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_k + T^s_j G^i_{sk} - T^i_s G^s_{jk}. \tag{1.3}$$

The commutation formula arising due to above covariant derivative is given by

$$2T^i_{j(h)(k)} = -\dot{\partial}_r T^i_j H^r_{hk} - T^i_s H^s_{jhk} + T^s_j H^i_{shk} \tag{1.4}$$

$$H^i_{hjk} = 2\{\partial_{[k} G^i_{j]h} - G^i_{rh[j} G^r_{k]} + G^r_{h[j} G^i_{k]r}\}, \tag{1.5}$$

where $H^i_{hjk}$ is Berwald curvature tensor field satisfy the following identities :

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0 \tag{1.6}$$

$$H^i_{hjk} = -H^i_{khj} \tag{1.7}$$

$$H^i_{hji} = -H_{hj} \tag{1.8}$$

$$H^i_{ikh} = 2H_{[hk]} . \tag{1.9}$$

**Definition (1.1).** A non flat Finsler space $F_n$, in which there exists a non-zero vector field $\beta_m$ which is positively homogeneous function of degree zero in $\dot{x}^i$ such that the curvature tensor field satisfy the relation

$$H^i_{jkh(m)} = \beta_m H^i_{jkh} , \tag{1.10}$$

is called a recurrent Finsler space (Moor,1963 ; Sinha *et al.,* 1971)

**Definition (1.2).** A non-flat Finsler space $F_n$ in which Berwald's curvature tensor field satisfy the relation

$$H^i_{jkh(l)(m)} = b_{lm} H^i_{jkh} , \tag{1.11}$$

where $b_{lm}$ means a non-zero covariant tensor, then the space is called Bi-recurrent Finsler space or BR $F_n$ space (Pandey,1985). We denote such a space by $F^*_n$

Let us consider an infinitesimal point transformation:

$$\bar{x}^i = x^i + v^i(x)dt, \tag{1.12}$$

where $v^i(x)$ is a contravariant vector field and ***dt*** is an infinitesimal point constant.

The Lie-derivative of a tensor $T^i_j$ and the connection coefficients are characterized by (Yano, 1978)

$$L_v T^i_j = v^k T^i_{j(k)} - T^h_j v^i_{(h)} + T^i_h v^h_{(j)} + (\dot{\partial}_h T^i_j) v^h_{(s)} \dot{x}^s \tag{1.13}$$

and

$$L_v G^i_{jk} = v^i_{(j)(k)} + H^i_{jkh} v^h + G^i_{jkh} v^h_{(s)} \dot{x}^s \tag{1.14}$$

respectively. The Lie-derivative of the curvature tensor is given by

$$L_v H^i_{jkh} = v^l H^i_{jkh(l)} - H^l_{jkh} v^i_{(l)} + H^i_{lkh} v^l_{(j)} + H^i_{jlh} v^l_{(k)} + H^i_{jkl} v^l_{(h)} + (\dot{\partial}_l H^i_{jkh}) v^l_{(m)} \dot{x}^m. \tag{1.15}$$

The processes of Lie-differentiation and other differentiations are connected by

$$(L_v T^i_{jk(l)}) - (L_v T^i_{jk})_{(l)} = T^s_{jk} L_v G^i_{sl} - T^i_{sk} L_v G^s_{jl} - T^i_{js} L_v G^s_{kl} \tag{1.16}$$

$$(L_v G^i_{jh})_{(k)} - (L_v G^i_{kh})_{(j)} = L_v H^i_{hjk} + (L_v G^r_{kl}) \dot{x}^l G^i_{rjh} \tag{1.17}$$

$$L_v(\dot{\partial}_l T^i_j) - \dot{\partial}_l (L_v T^i_j) = 0. \tag{1.18}$$

Let us consider an infinitesimal transformation similar to that of Okumura (1962) of the form

$$\bar{x}^i = x^i + v^i(x)dt, \quad v^i_{(k)} = \lambda \delta^i_k, \tag{1.19}$$

where $\lambda(x, \dot{x})$ is a scalar function. Such a transformation is called a special concircular transformation.

## 2. Special concircular h-curvature collineation

**Definition (2.1).** In a Bi-recurrent Finsler space $F^*_n$, if the curvature tensor field $H^i_{jkh}$ satisfies the relation.

$$L_v H^i_{jkh} = 0 \tag{2.1}$$

where $L_V$ represents Lie-derivative defined by transformation (1.19), then the transformation (1.19) is called the Special concircular H-curvature collineation.

If special con-circular transformation (1.19) defines H-curvature collineation, then equation (1.15) takes the following form:

$$v^l H^i_{jkh(l)} - H^l_{jkh} v^i_{(l)} + H^i_{lkh} v^l_{(j)} + H^i_{jlh} v^l_{(k)} + H^i_{jkl} v^l_{(h)} + (\dot{\partial}_l H^i_{jkh}) v^l_{(m)} \dot{x}^m = 0,$$

which in view of transformation (1.19) and homogeneity property of curvature tensor $H^i_{jkh}$ reduce to

$$v^l H^i_{jkh(l)} = -2\lambda H^i_{jkh}. \tag{2.2}$$

Taking covariant derivative of (2.2) with respect to $x^m$, we get

$$v^l H^i_{jkh(l)(m)} + v^l_{(m)} H^i_{jkh(l)} = -2\lambda_{(m)} H^i_{jkh} - 2\lambda H^i_{jkh(m)}. \tag{2.3}$$

Introducing result (1.11) in (2.3), we find

$$v^l b_{lm} H^i_{jkh} + \lambda \delta^l_m H^i_{jkh(l)} = -2\lambda_{(m)} H^i_{jkh} - 2\lambda H^i_{jkh(m)}. \tag{2.4}$$

Transvecting (2.4) by $v^m$, we get

$$v^l v^m b_{lm} H^i_{jkh} + \lambda v^l H^i_{jkh(l)} = -2\lambda_{(m)} v^m H^i_{jkh} - 2\lambda v^m H^i_{jkh(m)}. \tag{2.5}$$

In view of result (2.2), (2.5) become

$$(6\lambda^2 - 2\lambda_m v^m - b_{lm} v^l v^m) H^i_{jkh} = 0. \tag{2.6}$$

For non-flat space, we have $H^i_{jkh} \neq 0.$

Therefore (2.6) becomes

$$3\lambda^2 = \lambda_m v^m + \frac{1}{2} b_{lm} v^l v^m. \tag{2.7}$$

Taking covariant derivative of (1.19) with respect to $x^m$, we get

$$v^i_{(k)(m)} = \lambda_m \delta^i_k.$$

Contracting $i$ and $k$ in above, we get

$$v^i_{(i)(m)} = \lambda_m. \tag{2.8}$$

Putting the, value of $\lambda_m$ in (2.7), we get

$$3\lambda^2 = v^i_{(i)(m)} v^m + \frac{1}{2} b_{lm} v^l v^m. \tag{2.9}$$

Thus, we have the following theorum

**Theorem (2.1).** The necessary condition for existence of special concircular H-curvature collineation in Bi-recurrent Finsler space is given by (Okumura, 1962 and Singh, 2001).

Applying commutation formula (1.4) for curvature tensor and introducing condition (1.11), we find

$$(b_{lm} - b_{ml}) H^i_{jkh} = -\dot{\partial}_r H^i_{jkh} H^r_{lm} + H^r_{jkh} H^i_{rlm} - H^i_{rkh} H^r_{jlm} - H^i_{jrh} H^r_{klm} - H^i_{jkr} H^r_{hlm}. \tag{2.10}$$

Since $H^i_{jkh} \neq 0,$ therefore (2.10) gives $b_{lm} - b_{ml} \neq 0 \quad \Rightarrow \quad b_{lm} \neq b_{ml}.$

Thus, we state following lemma

**Lemma (2.1)**. In Bi-recurrent Finsler space, the tensor $b_{lm}$ is non-symmetric. Particularly, if we take $b_{lm}$ as skew-symmetric, then

$$b_{lm} = -b_{ml}.$$

Therefore $\quad b_{lm} v^l v^m = b_{ml} v^l v^m = -b_{lm} v^l v^m$

or $$b_{lm} v^l v^m = 0. \tag{2.11}$$

In view of (2.11), (2.9) becomes

$$3\lambda^2 = v^k_{(k)(m)} v^m. \tag{2.12}$$

Therefore, under this condition, theorem (2.1) may be restated in following way

**Corollary (2.1).** The necessary condition for existence of special con-circular H-curvature collineation in Finsler space is given by (2.12) if the recurrence tensor $b_{lm}$ is skew symmetric.

## 3. Discussion

Contracting *i* and *h* in (2.1), we get

$$L_v H^i_{jkl} = 0,$$

which in view of result (1. 8) reduces to

$$L_v H_{jk} = 0. \tag{3.1}$$

Thus, we have

**Theorem (3.1).** Every special concircular H-curvature collineation in Bi-recurrent Finsler space is Ricci H-curvature collineation.

Contracting (2.2) for indices *i* and h, and using (1.8), we get (3.2)

$$v^l H^i_{jk(l)} = -2\lambda H_{jk}. \tag{3.2}$$

Hence in view of (2.2) and (3.2), we have

**Theorem (3.2).** The curvature tensor $H^i_{jkh}$ and Ricci tensor are not recurrent in Bi-recurrent Finsler space admitting special concircular H-curvature collineation.

The commutation formula (1.18) for Berwald curvatur is given by :

$$\dot{\partial}_l (L_v H^i_{jkh}) - L_v (\dot{\partial}_l H^i_{jkh}) = 0. \tag{3.3}$$

In Bi-recurrent Finsler space admitting special con-circular H-curvature collineation, commutation formula (3.3) reduces to :

$$L_v (\dot{\partial}_l H^i_{jkh}) = 0. \tag{3.4}$$

Thus, we can state the following theorum

**Theorem (3.3).** In Bi-recurrent Finsler space admitting special concircular H-curvature collineation, the partial derivative of curvature tensor $H^i_{jkh}$ is Lie-invariant.

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