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# Quasi Conharmonic Curvature tensor on a Riemannian manifold

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## Abstract

*(Yano and Sawaki,1972) introduced quasi conformal curvature tensor in a Riemannian manifold. Recently one of the author (Prasad,2002) investgated pseudo projective curvature tensore tensor in a Riemannian manifold. In this paper, we defined quasi conharmonic curvature tensor on a Riemannian manifold and obtained its several properties. Finally a particular case has been investigated.*

## 1. Introduction

Let conformal curvature tensor C, projective curvature tensor P, concircular curvature tensor V, conhormonic curvature tensor H, be respectively given by (Mishra, 1984)

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature, provided

$$'C(X;Y;Z;W) = g(C(X;Y)Z;W),$$
$$'R(X;Y;Z;W) = g(R(X;Y)Z;W)$$

and

$$S(X,Y)=g(QX,Y).$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{1.2}$$

$$V(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.3}$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{1.4}$$

These satisfy the symmetric and skew symmetric property as well as cyclic property possessed by the curvature tensor R(X,Y)Z.

In 1970-72, (Mishra and Pokhariyal, 1970, 1971 and 1972) defined following curvature tensors on a Riemannian manifolds

$$H_1(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

$$H_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{1.6}$$

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$$H_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \tag{1.7}$$

$$H_4(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ]. \tag{1.8}$$

Some properties of these curvature tensors have been studied by many authors such as (De and Ghosh, 1994, 1995 & 1996), (De and Yieldiz,2010), (Pokhariyal,1982), (Prasad, 1997, 2002 & 2003), (Prakash, 2010) and many others.

(Yano and Sawaki, 1972) introduced quasi conformal curvature tensor in Remannian manifold as follows

$$C(X,Y)Z = aR(X,Y,Z) + b\,[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \tag{1.9}$$

where a and b are constants such that a, b $\neq$ 0.

Some properties of quasi conformal curvature tensor have been studied by (Amur and Maralabhavi, 1977), (Chaki and Ghosh, 1997), (De and Shaikh, 1997), (De and Cihan, 2006), (De and Jun and Gazi, 2008), (Kumar, Prasad and Verma, 2009), (Prakash and Singh, 2009) and other workers. Recently one of the author (Prasad, 2002) introduced pseudo projective curvature tensor in a Riemannian manifold (n>2) by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b\,[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.10}$$

where a and b are constants such that a,b $\neq$ 0.

In recent papers, (Bagewadi, Prakash and Venkatesha, 2007), (Bagewadi, Basavarajappa and Venkatesha, 2008), (Narain, Prakash and Prasad, 2009), (Sreenivasa, Bagewadi and Venkatesha, 2009), (Jaisaval and Ojha, 2010), (Prakasha, Bagewadi and Prasad, 2010) explored various geometrical properties by using this curvature tensor (1.10) on LP-Sasakian manifold, K-contact and trans-sasakian manifold, $(LCS)_{2n+1}$ manifold, weakly symmetric manifold and contact metric manifold with $\xi \in N(K)$. Further in 2007, (Prasad and Maurya, 2007) investigated another curvature tensor on a Riemannian manifold (n>2),

$$\tilde{V}(X,Y)Z = aR(X,Y)Z + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.11}$$

and named as quasi concircular curvature tensor.

Narain *et al.,* (2009) and Kumar *et al.,* (2009) extended this notation to LP-Sasakian manifold and P-Sasakian manifold. In continuation of above study, we define another new curvature tensor which we call it quasi-conharmonic curvature tensor $\tilde{H}$ (n>3) of the type (1,3) as follows

$$\tilde{H}(X,Y)Z = aR(X,Y)Z + b\,[S(Y,Z)X - S(X,Z)Y] + c[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{2a}{n-2} + b + c\right)[g(Y,Z)X - g(X,Z)Y]. \tag{1.12}$$

where a and b are constants such that a,b ,c $\neq$ 0.

This paper deals with Riemannian manifold $(M^n, g)$ (n> 2) for which $\tilde{H}$ is conservative (Hicks, 1969). A manifold $(M^n, g)$ (n>3) shall be called quasi conharmonically at or quasi conharmonically conservative according as $\tilde{H}$ =0 or div $\tilde{H}$ =0. In this paper it is proved that quasi conharmonically manifold is of zero scalar curvature provided that a (3n-4)+2(n-1)(n-2)(b+c) $\neq$ 0. Further a necessary and sufficient condition for an $(M^n, g)$ to be quasi conharmonically conservative is obtained.It can be easily verifed that

$$'\tilde{H}(X,Y,Z,W) = -'\tilde{H}(Y,X,Z,W)$$

$$'\tilde{H}(X,Y,Z,W) = -'\tilde{H}(X,Y,W,Z)$$
$$'\tilde{H}(X,Y,Z,W) = -'\tilde{H}(Z,W,X,Y)$$

and

$$'\tilde{H}(X,Y,Z,W) + '\tilde{H}(Y,Z,X,W) + '\tilde{H}(Z,X,Y,W) = 0.$$

If a=1 and b = c = $-\frac{1}{n-2}$, then (1.12) takes the form

$$\tilde{H}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\left[S(Y,Z)X - S(X,Z)Y\right] + [g(Y,Z)QX - g(X,Z)QY] = H(X,Y)Z$$

where H(X, Y ) Z is conharmonic curvature tensor. Thus the conharmonic curvature tensor is a particular case of the tensor $\tilde{H}(X,Y)Z$. For this reason $\tilde{H}$ is called the quasi conharmonic curvature tensor.

## 2 Quasi conharmonically at manifold

In this case assume that $\tilde{H}(X,Y)Z = 0$.

Then from (1.12), we get

$$aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] + c[g(Y,Z)QX - g(X,Z)QY]$$
$$-\frac{r}{n}\left(\frac{2a}{n-2} + b + c\right)[g(Y,Z)X - g(X,Z)Y] = 0. \tag{2.1}$$

From (2.1), we have

$$a\,'R(X,Y,Z,W) + b\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\right] + c[g(Y,Z)S(X,W) -$$
$$g(X,Z)S(Y,W)] - \frac{r}{n}\left(\frac{2a}{n-2} + b + c\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \tag{2.2}$$

Putting $X = W = e_i$ in (2.2), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over $e_i$, $1 \le i \le n$, we get

$$[a+(n-1)b - c]S(Y,Z) = \frac{r}{n(n-2)}[\{a + (n-2)b\}(n-1) + c(2n-1)(n-2)]g(Y,Z),$$

which gives on further contraction

$$[a + (3n-4) + 2(n-1)(n-2)(b+c)]r = 0 \tag{2.3}$$

If $[a+(3n-4)+2(n-1)(n-2)(b+c)] \neq 0$, then from (2.3), we have r = 0.

Hence we have the following theorem:

**Theorem 2.1** A quasi conharmonically flat manifold is of zero scalar curvature provided that a (3n-4)+2(n-1)(n-2)(b+c) ≠ 0.

## 3 Quasi Conharmonically conservative ($M^n$, g)(n >3)

In this section we assumet tha

$$\mathrm{div}\tilde{H} = 0. \tag{3.1}$$

Now differentiating (1.12) covariantly, we get

$$(D_U\tilde{H})(X,Y)Z = a(D_UR)(X,Y)Z + b\left[(D_US)(Y,Z)X - (D_US)(X,Z)Y\right] +$$
$$c[g(Y,Z)(D_UQ)X - g(X,Z)(D_UQ)Y]$$
$$-\ \frac{D_Ur}{n}\left(\frac{2a}{n-2} + b + c\right)[g(Y,Z)X - g(X,Z)Y],$$

which gives on contraction

$$(\mathrm{div}\widetilde{H})(X,Y)Z = a(\mathrm{div}R)(X,Y)Z + b\left[(D_XS)(Y,Z) - (D_YS)(X,Z)\right] + c[g(Y,Z)dr(X) - g(X,Z)dr(Y)] - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad (3.2)$$

From (Eisenhat,1926), we have

$$(\mathrm{div}\ R)(X,Y)Z = (D_XS)(Y,Z) - (D_Y S)(X,Z).$$

Hence (3.2) gives

$$(\mathrm{div}\widetilde{H})(X,Y)Z = (a+b)[(D_XS)(Y\text{-}Z) - (D_Y S)(X;Z)] - \left[\frac{2a + (n-2)b + (n+1)(n-2)c}{n(n-1)}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad (3.3)$$

Suppose the Ricci tensor S(X,Y) is of Codazzi type i.e.

$$(D_XS)(Y,Z) = (D_Y S)(X,Z)$$

Then from (3.3), we get

$$(\mathrm{div}\widetilde{H})(X,Y)Z = \left[\frac{2a + (n-2)b + (n+1)(n-2)c}{n(n-1)}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad (3.4)$$

Hence if $(\mathrm{div}\widetilde{H})=0$ then from (3.4), we get

$$[2a + (n-2)b + (n+1)(n-2)c][g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0;$$

Since $2a+(n-2)b+(n+1)(n-2)c \neq 0$ and hence

$$g(Y,Z)dr(X) - g(X,Z)dr(Y) = 0;$$

which shows that r is constant. Again if r is constant then from (3.4), we get

$$(\mathrm{div}\widetilde{H})(X,Y)Z = 0.$$

Hence we can state the following theorem

**Theorem 3.1** If in a Riemannian manifold ($M^n$, g(n > 3)), the Ricci tensoris Codazzi type then the manifold quasi conharmonically conservative if and only if the scalar curvature is constant provided that $2a+(n-2)b+(n+1)(n-2)c\neq 0$.

According to as assumption, we have $a+b \neq 0$, then from (3.3) we have

$$\frac{(\mathrm{div}\widetilde{H})(X,Y)Z}{a+b} = [(D_XS)(Y\text{-}Z) - (D_Y S)(X;Z)] - \left[\frac{2a + (n-2)b + (n+1)(n-2)c}{n(n-1)(a+b)}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad (3.5)$$

Hence from (3.5), we can state theorem as follows

**Theorem 3.2** If in a Riemannian manifold ($M^n$, g(n > 3)), the quasi conharrmonic curvature tensor is such that $a+b\neq 0$, the manifold is quasi conharrmonically conservative if and only if

$$[(D_XS)(Y\text{-}Z) - (D_Y S)(X;Z)] = -\left[\frac{2a + (n-2)b + (n+1)(n-2)c}{n(n-1)(a+b)}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)].$$

# 4 Particular case

If the manifold becomes an Eienstein manifold (1.12) takes the form

$$\widetilde{H}(X,Y)Z = aR(X,Y)Z - \frac{r}{n}\left(\frac{2a}{n-2} + 2b + 2c\right)[g(Y,Z)X\text{-}g(X,Z)Y]. \text{ (a new curvature tensor)} \qquad (4.1)$$

If a=1, b= $-\frac{1}{2(n-1)}$ and c= $-\frac{1}{n-2}$, then (4.1) takes the form

$$\widetilde{H}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X\text{-}g(X,Z)Y] = V(X,Y)Z \qquad (4.2)$$

where V is the concircular curvature tensor.
Thus we see that concircular curvature tensor V is a special case of the tensor $\widetilde{H}$ (X,Y)Z, which seemes to be generalization of concircular curvature tensor.
Hence we call it as generalized concircular curvature tensor.

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