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A quarter-symmetric metric connection on almost contect metric manifold

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Abstract

In this paper, we have studied some geometrical properties of quarter–symmetric metric connection in generalized cosymplectic manifold, nearly cosymplectic manifold, generalized nearly cosymplectic manifold and generalized quasi-Sasakian manifold of the second class.

Keywords- Almost contact manifold, quarter symmetric, induced connection

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1. Introduction

The idea of metric connection with torsion tensor in a Riemannian manifold was introduced by Hayden (1932). Further some properties of semi symmetric connection has been studied by Yano (1970). Agashe *et al*., (1992) defined and studied a semi symmetric metric connection in a Riemannian manifold. This was further developed by many geometers. In 1975 Golab (1975) defined and studied quarter symmetric connection in with an affine connection which generalizes the idea of semi symmetric connection. In the present paper, we have investigated some properties of semi symmetric metric connection on almost contect metric manifold which was defined by Mishra and Pandey (1980) and Mishra, 1981. The present paper is organized as follows. Section 2 is devoted to the preliminaries about the almost contact manifold and semi-symmetric metric connection. In section 3, we have discussed some geometrical properties of the quarter symmetric connection on almost contact manifolds such as generalized cosymplectic manifold, generalized nearly cosymplectic manifold, generalized quasi-Sasakian manifold of second class. Finally, we have studied induced metric connection and proved that the induced connection on submanifold of an almost contact manifold with quarter - symmetric metric connection is also a quarter - symmetric metric connection.

2. Preliminaries

Let M be an n (= 2m+1) dimensional C^∞ - manifold and let there exist in M^n a vector valued linear function F, a vector field T and a 1 – form A, such that

(a) $\overline{\overline{X}} + X = A(X)T$

(b) $\overline{X} \underset{=}{\text{def}} F(X)$

(c) $\overline{T} = 0$ (2.1)

for arbitrary vector field X, then M^n is called an almost contact manifold and the structure {F, T, A} is called an almost contact structure.

It follows from (2.1) that following hold in M^n:

(a) $A(\overline{X}) = 0$ (b) $A(T) = 1$ (2.2)

Let the almost contact manifold M^n be endowed with the non singular metric g satisfying

(a) $g(\overline{X}, \overline{Y}) = g(X, Y) - A(X) A(Y)$, (2.3)

(b) $g(X, T) = A(X)$,

then M^n is called an almost contact metric manifold with almost contact metric structure {F, T, A, g}.

If an almost contact metric manifold T satisfy

(a) $(D_X A)(\overline{Y}) = (D_{\overline{X}} A)(Y) = -(D_Y A)(\overline{X}) \Leftrightarrow$

(b) $(D_X A)(Y) = -(D_{\overline{X}} A)(\overline{Y}) = -(D_Y A)(X)$,

(c) $D_T F = 0$, (2.4)

then T is said to be of second class and the manifold is said to be of second class.

The almost contact metric manifold satisfying (Mishra, 1991)

$$(D_X 'F)(Y, Z) = A(Y)(D_X A)(\overline{Z}) + A(Z)(D_Y A)(\overline{X}) \quad (2.5)$$

$$(D_X 'F)(Y, Z) = (D_Y 'F)(Z, X) \quad (2.6)$$

$$(D_X 'F)(Y, Z) - (D_Y 'F)(Z, X) = A(Y)(D_{\overline{X}} A)(Z) - A(X)(D_Z A)(\overline{Y}) \quad (2.7)$$

$$(D_X 'F)(Y, Z) + (D_Y 'F)(Z, X) + (D_Z 'F)(X, Y)$$

$$= 2\left[A(X)(D_Z A)(\overline{Y}) + A(Y)(D_X A)(\overline{Z}) + A(Z)(D_Y A)(\overline{X})\right] \quad (2.8)$$

are called generalized cosymplectic manifold, nearly cosymplectic manifold, generalized nearly cosymplectic manifold and generalized quasi-Sasakian manifold of the second class respectively for vector fields X,Y,Z in M^n.

3. Quarter Symmetric Metric Connection

Let B be an affine connection in M^n defined by (Mishra and Pandey, 1980)

$$B_X Y = D_X Y + A(Y)\bar{X} - g(\bar{X}, Y)T = D_X Y + H(X, Y) \tag{3.1}$$

where

$$H(X, Y) = A(Y)\bar{X} - g(\bar{X}, Y)T.$$

The affine connection B is said to be metric if it satisfies

$$B_X g = 0. \tag{3.2}$$

The torsion tensor S of B is given by

$$S(X, Y) = A(Y)\bar{X} - A(X)\bar{Y} - 2\, g(\bar{X}, Y)T. \tag{3.3}$$

A metric connection is called quarter -symmetric connection whose torsion tensor is given in the form (3.3).Consequently

$$'S(X, Y, Z) \underset{\equiv}{\text{def}}\ g(S(X, Y), Z) = 'H(X, Y, Z) - 'H(Y, X, Z) \tag{3.4}$$

where

$$'H(X, Y, Z) \underset{\equiv}{\text{def}}\ g(H(X, Y), Z) \tag{3.5}$$

Further from above we have

(a) $\quad 'S(X, Y, Z) = A(Y)g(\bar{X}, Z) - A(X)\, g(\bar{Y}, Z) - 2A(Z)g(\bar{X}, Y)$

(b) $\quad 'H(X, Y, Z) = A(Y)\, g(\bar{X}, Z) - A(Z)\, g(\bar{X}, Y),$ (3.6)

and

$$(B_X A)(Y) = (D_X A)(Y) - 'F(X, Y). \tag{3.7}$$

Theorem 3.1- In an almost contact metric manifold with quarter- symmetric metric connection B, we have

(a) $\quad (D_X 'F)(Y, Z) = (B_X 'F)(Y, Z) + 'H(\bar{X}, Y, Z)$ (3.8)

(b) $\quad (B_X 'F)(Y, Z) = (D_X 'F)(Y, Z) + A(Y)g(X, Z) - A(Z)g(X, Y)$

(c) $\quad (B_X 'F)(Y, Z) = g((B_X F)Y, Z)$

(d) $\quad g((B_X F)Y, Z) = g((D_X F)Y, Z) + A(Y)\, g(X, Z) - A(Z)\, g(X, Y).$

Proof: We know that

$$X('F(Y, Z)) = (D_X 'F)(Y, Z) + 'F(D_X Y, Z) + 'F(Y, D_X Z)$$
$$= (B_X 'F)(Y, Z) + 'F(B_X Y, Z) + 'F(Y, B_X Z)$$

$$\Rightarrow \quad (D_X 'F)(Y, Z) = (B_X 'F)(Y, Z) + 'F(H(X, Y), Z) + 'F(Y, H(X, Z))$$
$$= (B_X 'F)(Y, Z) - 'H(X, Y, \bar{Z}) + 'H(X, Z, \bar{Y})$$

$$= (B_X{}'F)(Y,Z) + {}'H(X,\overline{Z},Y) + {}'H(X,Y,\overline{Z})$$

$$= (B_X{}'F)(Y,Z) + {}'H(\overline{X},Y,Z),$$

which gives ((3.8) a). From ((3.6) b) and (3.8) a), we get (3.8) b).

For (3.8) c), considering the relations ${}'F(X,Y) = g(\overline{X},Y)$, we obtain

$$X({}'F(Y,Z)) = (B_X{}'F)(Y,Z) + {}'F(B_X Y,Z) + {}'F(X,B_X Z)$$

$$= g(B_X \overline{Y},Z) + g(\overline{Y},B_X Z).$$

This implies that

$$(B_X{}'F)(Y,Z) = g(B_X \overline{Y},Z) - g(\overline{B_X Y},Z) = g((B_X F)Y,Z).$$

Finally to establish ((3.8) d), we have

$$g((B_X F)Y,Z) = g(B_X \overline{Y},Z) - g(\overline{B_X Y},Z)$$

$$= g(D_X \overline{Y} - g(\overline{X},\overline{Y})T,Z) - g(\overline{D_x Y} + A(Y)\overline{\overline{X}},Z)$$

$$= g((D_X F)Y,Z) + A(Y)g(X,Z) - A(Z)\,g(X,Y).$$

Theorem 3.2- On a generalized cosymplectic manifold with quarter -symmetric metric connection B, we have

(a) $(B_T{}'F)(Y,Z) = 0$ (3.9)

(b) $(B_X{}'F)(\overline{Y},\overline{Z}) = 0$

(c) $(B_X{}'F)(Y,Z) = -[A(Y)A(B_X\overline{Z}) + A(Z)A(B_Y\overline{X}) + 2A(Y)\,g(\overline{X},\overline{Z})]$

Proof: Replacing X with T in ((3.8) (a)), we have

$$(B_T{}'F)\ (Y,Z) = (D_T{}'F)\ (Y,Z).$$

Now, using (2.4) and (2.5) in the above equation, we get equation (3.9) (a).

From (3.8) (a), we have

$$(B_X{}'F)(Y,Z) = (D_X{}'F)(Y,Z) - {}'H\ (\overline{X},Y,Z). \tag{3.10}$$

Using (2.5) in above equation, we get (3.9) (b). Finally, to prove (3.9) (c), we have

$$(B_X{}'F)(Y,Z) = A(Y)(D_X A)(\overline{Z}) + A(Z)(D_Y A)(\overline{X}) + A(Z)\,g(X,Y) - A(Y)\,g(X,Z)$$

$$= -[A(Y)\,A(D_X\overline{Z}) + A(Z)A(D_Y\overline{X})] + A(Z)\,g(X,Y) - A(Y)\,g(X,Z).$$

From (3.1) (a) and above equation, we get

$$(B_X{}'F)(Y,Z) = -\left[A(Y)A\left(B_X\overline{Z} + g\left(\overline{X},\overline{Z}\right)T\right) + A(Z)A\left(B_Y\overline{X} + g\left(\overline{X},\overline{Z}\right)T\right)\right]$$

$$+ A(Z)\,g(X,Y) - A(Y)\,g(X,Z)$$

$$= -\left[A(Y)A\left(B_X\overline{Z}\right) + A(Z)A\left(B_Y\overline{X}\right) + 2A(Y)g\left(X,\overline{Z}\right)\right],$$

This completes the proof.

Theorem 3.3- On a generalized nearly cosymplectic manifold with quarter -symmetric metric connection B, we have $2\left(B_Z A\right)\left(\overline{Y}\right) = {}'F\left(B_Y T + \overline{Y}, Z\right)$.

Proof: We have

$$(D_X{}'F)(Y,Z) = (B_X{}'F)(Y,Z) + A(Z)g(X,Y) - A(Y)g(X,Z) \quad . \tag{3.11}$$

Using (3.11) and (3.7) in equation (2.7), we get

$$(B_X{}'F)(Y,Z) + A(Z)g(X,Y) - A(Y)g(X,Z) + A(X)\left[(B_Z A)\left(\overline{Y}\right) + {}'F\left(Z,\overline{Y}\right)\right]$$

$$= (B_Y{}'F)(Z,X) + A(X)g(Y,Z) - A(Z)g(Y,X)$$

$$+ A(Y)\left[\left(B_{\overline{X}}A\right)(Z) + {}'F\left(\overline{X},Z\right)\right].$$

Putting T for X in above equation, we get

$$\left(B_T{}'F\right)(Y,Z) + \left(B_Z A\right)\left(\overline{Y}\right) + {}'F\left(Y,\overline{Z}\right) = \left(B_Y{}'F\right)(Z,T) + {}'F\left(Y,\overline{Z}\right).$$

$$\Rightarrow \quad \left(B_T{}'F\right)(Y,Z) + \left(B_Z A\right)\left(\overline{Y}\right) = \left(B_Y{}'F\right)(Z,T). \tag{3.12}$$

Now, using (3.11) in equation (3.12), we obtain

$$2\left(B_Z A\right)\left(\overline{Y}\right) + {}'F\left(Z,\overline{Y}\right) = \left(B_Y{}'F\right)(Z,T).$$

$$\Rightarrow \quad 2\left(B_Z A\right)\left(\overline{Y}\right) = {}'F\left(B_Y T + \overline{Y}, Z\right).$$

Theorem 3.4- On a generalized quasi-Sasakian manifold, we have

$$\left(B_Z A\right)\left(\overline{\overline{Y}}\right) = {}'F\left(B_Y T + \overline{\overline{Y}}, Z\right) - {}'F\left(B_Z T, Y\right).$$

Proof: Inconsequence of (2.8), (3.7) and (3.8) (b), we have

$$(B_X{}'F)(Y,Z) + A(Z)g(X,Y) - A(Y)g(X,Z) + (B_Y{}'F)(Z,X) + A(X)g(Y,Z) - A(Z)g(Y,X)$$

$$+ (B_Z{}'F)(X,Y) + A(Y)g(Z,X) - A(X)g(Z,Y) = 2\left[A(X)(B_Z A)\left(\overline{Y}\right) + A(X){}'F\left(Z,\overline{Y}\right)\right.$$

$$\left. + A(Y)(B_X A)\left(\overline{Z}\right) + A(Y){}'F\left(X,\overline{Z}\right) + A(Z)(B_Y A)\left(\overline{X}\right) + A(Z){}'F\left(Y,\overline{X}\right)\right].$$

Putting T for X in the above equation, we get

$$(B_T{}'F)(Y,Z)+(B_Y{}'F)(Z,T)+(B_Z{}'F)(T,Y)$$

$$=2\left[(B_Z A)(\bar{Y})+{}'F(Z,\bar{Y})\right]+2A(Y)(D_X A)(\bar{Z}).$$

This gives

$$(B_Z A)(\bar{Y})+{}'F(Z,\bar{Y})+2A(Y)(B_X A)(\bar{Z})={}'F(B_Y T,Z)-{}'F(B_Z T,Y). \quad (3.13)$$

Barring Y in equation (3.13), we obtain

$$(B_Z A)(\bar{\bar{Y}})={}'F(B_Y T+\bar{\bar{Y}},Z)-{}'F(B_Z T,Y).$$

This proves the theorem 3.4

4. The induced Connection

Let V_{2m+1} be an odd dimensional differentiable manifold of class C^∞ and V_{2m-1} be a submanifold of V_{2m+1} and let $b: V_{2m-1} \to V_{2m+1}$ be the inclusion map such that $p \in V_{2m-1}$ and $bp \in V_{2m+1}$. The map b induces a linear transformation (Jacobian map) $J: T_{2m-1} \to T_{2m+1}$ where T_{2m-1} is a tangent space to V_{2m-1} at a point p and T_{2m+1} is a tangent space to V_{2m+1} at b p. The Riemannian metric G induced on V_{2m-1} from that of V_{2m+1} is given by

$$g(u,v)=G(Ju,Jv), \quad (4.1)$$

where u, v are arbitrary vector fields in V_{2m-1}. Let N_1 and N_2 be two mutually orthogonal unit normals to V_{2m-1} such that

(a) $G(Ju,N_1)=G(Ju,N_2)=G(N_1,N_2)=0$,

(b) $G(N_1,N_1)=G(N_2,N_2)=1$. (4.2)

Let us assume that the almost contact manifold V_{2m+1} admit a quarter-symmetric metric connection given by

$$B_X Y=D_X Y+A(Y)\bar{X}-g(\bar{X},Y)T. \quad (4.3)$$

We know that (Mishra, 1974)

(a) $F(Ju)=Jf(u)+\alpha(u)N_1+\gamma(u)N_2$, (4.4)

(b) $T=Jt+\sigma N_1+\delta N_2$,

where t is a vector field and $\alpha,\gamma,\sigma,\delta$ are c^∞ functions in V_{2m-1}

Let $\bar{B}$ be the induced connexion on the sub-manifold from B with respect to the unit normals N_1, N_2. Now the Gauss equation is

$$B_{Ju}Jv = J(\bar{B}_u v) + h_1(u,v)N_1 + h_2(u,v)N_2\ , \tag{4.5}$$

where $h_1$and h_2 are second fundamental tensors and u, v are the vector fields of the sub-manifold V_{2m-1}. Denoting $\overline{D}$ the connexion induced on the sub-manifold from D with respect to the unit normals N_1, N_2. The Gauss equation is given as

$$D_{Ju}Jv = J(\bar{D}_u v) + m_1(u,v)N_1 + m_2(u,v)N_2\ , \tag{4.6}$$

where m_1, m_2 are tensor fields of type (0, 2) of submanifold V_{2m-1}.

Theorem (4.1)- The induced connection on submanifold of an almost contact manifold with quarter - symmetric metric connection is again a quarter symmetric metric connection.

Proof: From (4.3), we have

$$B_{Ju}Jv = D_{Ju}Jv + A(Ju)F(Jv) - G(F(Ju), Jv))T\ .$$

By making use of (3.4), (3.5) and (3.6) the above equation can be written as

$$\begin{aligned} J(\bar{B}_u v) + h_1(u,v)N_1 + h_2(u,v)N_2 &= J(\bar{D}_u v) + m_1(u,v)N_1 + m_2(u,v)N_2 \\ &\quad + A(Ju)(Jf(v) + \alpha(v)N_1 + \gamma(v)N_2) \\ &\quad - G(Jf(u) + \alpha(u)N_1 + \gamma(u)N_2, Jv)(\,Jt + \sigma N_1 + \delta N_2). \end{aligned}$$

By the virtue of above equation, we obtain

$$(\bar{B}_u v) = (\bar{D}_u v) + a(u)f(v) - g(f(u),v)t, \tag{4.7}$$

where $(u) = A(Ju)$,

and

$$\begin{aligned} &\text{(a)} \quad h_1(u,v) = m_1(u,v) + a(v)\alpha(u) - g(f(u),v)\sigma, \\ &\text{(b)} \quad h_2(u,v) = m_2(u,v) + a(v)\alpha(u) - g(f(u),v)\delta\ . \end{aligned} \tag{4.8}$$

We know that

$$\begin{aligned} u\ g(v,w) &= (\bar{B}_u g)(v,w) + g(\bar{B}_u v, w) + g(v, \bar{B}_u w) \\ &= g(\bar{D}_u v, w) + g(v, \bar{D}_u w)\ . \end{aligned}$$

$$(\bar{B}_u g)(v,w) = g(\bar{D}_u v - \bar{B}_u v, w) + g(u, \bar{D}_v w - \bar{B}_v w)\ .$$

Using (4.7) in above equation, we get.

$$\begin{aligned} (\bar{B}_u g)(v,w) &= g\big(g(f(u),v)\big)t - a(v)f(u), w) \\ &\quad + g(v, g(f(u),w)t - a(w)f(u))\ . \end{aligned}$$

$$\Rightarrow \quad (\bar{B}_u g)(v,w) = 0\ .$$

Again in consequence of (4.7) we get

$$\bar{B}_u v - \bar{B}_v u - [u, v] = a(v)f(u) - f^{'}(u, v)t - a(u)f(v) - f^{'}(u, v)t$$
$$= a(v)f(u) - 2f^{'}(u, v)t - a(u)f(v) \, ,$$

where $g(f(u), v) = f^{'}(u, v)$.

This completes the proof.

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