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3-Contact CR – sub-manifolds of a manifold with LP–Sasakian 3-structure

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Abstract

This paper has been divided into three sections starting with an introductory section. In second section we have established the necessary and sufficient conditions for a submanifold to be a 3-contact CR-submanifold. In the third section along with the integrability of distributions D and D^1, we have also discussed the cases when the submanifold is mixed totally geodesic or the leafes of the distributions are totally geodesic. Finally we have established some theorems by taking the canonical structures, defined for the 3-contact LP-submanifold to be parallel.

Keywords: Lorentzian para-Sasakian 3-structure, 3 contact CR-Submanifold

1. Introduction

Let $\bar{M}$ be an n-dimensional differentiable manifold equipped with a Lorentzian para – contact metric 3-structure (ϕ_i, ξ_i, n_i, g) (i = 1, 2, 3) defined by Kuo (1970)

$$\eta_i (\xi_j) = - \delta_{i\,j} \tag{1.1}$$

$$\phi_i \, \xi_j = - \phi_j \, \xi_i = \xi_k \tag{1.2}$$

$$\eta_i \circ \phi_j = - \eta_j \circ \phi_i = \eta_k \tag{1.3}$$

$$\phi_i \;\; \phi_j - \eta_j \otimes \xi_i = - \phi_j \, \phi_i + \eta_i \otimes \xi_j = \phi_k \tag{1.4}$$

$$\phi_i^2 = I + \eta_i \otimes \xi_I \tag{1.5}$$

$$g (\phi_i \, x, \phi_i \, y) = g (x, y) + \eta_i (x) \, \eta_i (y) \tag{1.6}$$

$$\eta_i (x) = g (x, \xi_i) \tag{1.7}$$

for any cyclic permutation (i, j, k) of (1,2,3) where x, y are vector fields on $\bar{M}$. $\bar{M}$ is called a manifold with a Lorentzian para – Sasakian (LP-Sasakian) 3-structure if

$$(\bar{\nabla}_x \, \phi_i) (y) = (g(x, y) + \eta_i (x) \, \eta_i (y))\xi_i + (x + \eta_i (x) \, \xi_i) \, \eta_i (y) \tag{1.8}$$

$$\bar{\nabla}_x \, \xi_i = \phi_i \, x \tag{1.9}$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to metric g on $\bar{M}$.

Let M be an m-dimensional submanifold isometrically immersed in $\tilde{M}$ with LP-Sasakian 3-structure. We assume that M is a tangent to $\{\xi_1, \xi_2, \xi_3\}$ which is subspace spanned by ξ_1, ξ_2, ξ_3.

The Gauss and weingarten formula are respectively given by

$$\tilde{\nabla}_x y = \nabla_x y + h(x, y) \quad (1.10)$$

$$\tilde{\nabla}_x N = - A_N x + D_x N. \quad (1.11)$$

where ∇ and D respectively denote the connection in the tangent and the normal bundle of M and the second fundamental forms A and h are related by

$$g(h(x, y), N) = g(A_N x, y). \quad (1.12)$$

Definition (Kobayashi, 1983)

A submanifold M of a manifold $\tilde{M}$ with a 3-structure is said to be 3-contact CR-submanifold if there exists a diferentiable distribution D on M such that

$$\phi_i D \subset D \quad (1.13)$$

$$\phi_i D^{\perp} \subset T(M)^{\perp} \ (i = 1, 2, 3) \quad (1.14)$$

where $D^{\perp}$ denote the orthogonal complementary distribution of D. We call D a horizontal distribution and $D^{\perp}$ a vertical distribution.

2. 3-Contact CR-Submanifolds

Lemma (2.1) for a 3-contact CR-submanifold M of $\tilde{M}$ with LP-Sasakian 3-structure, $\xi_i \in D$ (i = 1, 2, 3)

Proof Let $Z \in \Gamma(D)^{\perp}$, then (1.2) gives

$$g(\xi_k, z) = g(\phi_i \xi_j, z)$$
$$= g(\xi_j, \phi_i z)$$
$$= o$$

showing that $\xi_i \in D$ for i = 1, 2, 3.

For $x \in \Gamma T(M)$, we put

$$\phi_i x = T_i x + F_i x \quad (2.1)$$

where $T_i x$ are the tangent part and $F_i x$ are the normal part of $\phi_i x$.

For $N \in \Gamma T(M)^{\perp}$, we put

$$\phi_i N = t_i N + f_i N \quad (2.2)$$

where $t_i N$ are the tangent part and $f_i N$ are the normal part of $\phi_i N$.

The relation $g(\phi_i x, y) = g(x, \phi_i y)$ (i = 1, 2, 3) together with (2.1) and (2.2) , we have

$$g(T_i x, y) = g(x, T_i y) \quad (2.3)$$

$$g(f_i N_1, N_2) = g(N_1, f_i N_2) \quad (2.4)$$

$$g(F_i x, N) = g(x, t_i N) \quad (2.5)$$

where x and y are tangent vector fields on M and N_1 and N_2 are normal vector fields on M.

Putting $x = \xi_i$ in (2.1), we get

$$F_i \xi_i = O, \ T_i \xi_i = O \quad (2.6)$$

Putting $x = \xi_j$ in (2.1), we get

$$T_i \xi_j = \xi_k, \ F_i \xi_j = O \quad (2.7)$$

Now operating (2.1) by ϕ_i and separating the tangent and the normal parts, we get

$$T^2_i\, x = x + \eta_i(x)\, \xi_i - t_i\, F_i\, x \tag{2.8}$$

$$F_i\, T_i\, x + f_i\, F_i\, x = O \tag{2.9}$$

Again operating (2.2) by ϕ_i and separating the tangent and the normal parts, we get.

$$T_i\, t_i\, N + t_i\, f_i\, N = O \tag{2.10}$$

$$f^2_i\, N + F_i\, t_i\, N = N \tag{2.11}$$

Now, operating (2.1) by ϕ_j and separating the tangent and the normal parts we get.

$$T_i\, T_j\, x + t_i\, F_i\, x - \eta_i(x)\, \xi_j = T_k\, x \tag{2.12}$$

$$F_i\, T_j\, x + f_i\, F_j\, x = F_k\, x \tag{2.13}$$

Lastly, operating (2.2) by ϕ_j and separating the tangent and the normal parts, we get

$$T_i\, t_j\, N + t_i\, f_j\, N = t_k\, N \tag{2.14}$$

$$F_i\, t_j\, N + f_i\, f_j\, N = f_k\, N. \tag{2.15}$$

Lemma (2.2) Let M be a 3-contact CR submanifold of $\tilde{M}$ with LP-Sasakian 3-structure, then

$$F_i\, T_i = O,\ f_i\, F_i = O \tag{2.16}$$

$$t_i\, f_i = O,\ T_i\, t_i = O \tag{2.17}$$

$$F_i\, T_j = O,\ f_i\, F_j = F_k \tag{2.18}$$

Proof Let us denote by l and m, the projection operators corresponding to D and $D^{\perp}$ respectively. Then we have,

$$1 + m = I,\ lm = ml = 0,\ l^2 = l,\ m^2 = m \tag{2.19}$$

Then from (2.1), we have

$$\phi_i\, lx = T_i\, lx + F_i\, lx \tag{2.20}$$

$$\phi_i\, mx = T_i\, mx + F_i\, mx \tag{2.21}$$

Giving $mT_i\, l = 0$, $F_i\, l = 0$ and $T_i\, m = 0$. Hence we have

$$T_i\, l = T_i\,(I-m) = T_i - T_i\, m = T_i \tag{2.22}$$

Putting lx for x in (2.9) and using (2.22), we get (2.16). Next putting $f_i\, N$ for N in (2.5) and using (2.4) and (2.10), we get $t_i\, f_i = 0$ and hence (2.10) gives $T_i\, t_i = 0$. Putting lx for x in (2.13), we get $F_i\, T_j = 0$ and hence $f_i\, F_j = F_k$.

With the help of Lemma (2.2) we can easily prove the following

Theorem (2.1) Let M be a submanifold of manifold $\tilde{M}$ with LP-Sasakian 3-structure. Then M is a 3-contact CR-submanifold if and only if

$$F_i\, T_j = 0\ (i, j = 1, 2, 3) \tag{2.23}$$

or

$$f_i\, F_i = 0 \text{ and } f_i\, F_j = F_k. \tag{2.24}$$

Theorem (2.2) Let M be a 3-contact CR-submanifold of M with LP-Sasakian 3-structure. Then T_i and f_i (i = 1, 2, 3) are para f-structures in M and its normal bundle respectively.

Proof: Applying ϕ_i to (2.1) and using (1.5), (2.1), (2.2), (2.16) and (2.17), we get

Putting $x = T_i\, x$ and using (2.6) we get

$$T_i^3 - T_i = 0 \tag{2.25}$$

Again applying ϕ_i to (2.2) and using (1.5), (2.1), (2.2), (2.16) and (2.17)

$N = f^2_i N$

Putting $f_i N$ for N, we get $f^3_i - f_i = 0$.

3. Integrability of Distributions D and $D^\perp$

In the equation of Gauss, putting $y = \xi_i$, we get

$$\nabla_x \xi_i = T_i x, \; h(x, \xi_i) = F_i x \tag{3.1}$$

Lemma (3.1)- In a 3-contact CR-submanifold of $\bar{M}$ with LP-Sasakian 3-structure. We have

$$(\nabla_x T_i)(y) = g(\phi_i x, \phi_i y)\xi_i + \{x + \eta_i(x)\xi_i\}\eta_i(y) + t_i h(x, y) + A_{F_i y} x \tag{3.2}$$

$$(\nabla_x F_i)(y) = f_i h(x, y) - h(x, T_i y) \tag{3.3}$$

$$(\nabla_x t_i)(N) = A_{f_i N} x - T_i A_N x \tag{3.4}$$

$$(\nabla_x f_i)(N) = -h(x, t_i N) - F_i A_N x \tag{3.5}$$

for any $x, y \in T(TM)$ and $N \in \Gamma(TM)^\perp$

Proof Differentiating (2.1) along x and using (1.8), and separating the tangent and the normal parts,

We respectively get (3.2) and (3.3).

Similarly differentiating (2.2) along x and using (1.8), we get (3.4) and (3.5)

Lemma (3.2) For $z, W \in D^\perp$, we have

$$A_{F_i z} w + A_{F_i w} z = 0 \tag{3.6}$$

$$A_N \phi_i y = A_{\phi_i N} y \text{ for } y \in \Gamma(D) \text{ and } z \in \Gamma(D^\perp). \tag{3.7}$$

Proof For any $x \in \Gamma(TM)$, we have

$$\nabla_x \phi_i z = -A_{\phi_i z} x + D_x \phi_i z \text{ for } z \in \Gamma(D^\perp).$$

We also have

$$\bar{\nabla}_x \phi_i z = (\bar{\nabla}_x \phi_i)(z) + \phi_i \bar{\nabla}_x z$$

Using (1.1) and equating the above two equations, we get

$$g(\phi_i h(x, z), W) = -g(A_{\phi_i z} x, W), \; W \in \Gamma(D^\perp)$$

i.e. $g(A_{\phi_i W} z + A_{\phi_i z} W, x) = 0$ which gives (3.6).

Now, for $x \in \Gamma(TM)$ and $y \in \Gamma(D)$, we have

$$g(\bar{\nabla}_x \phi_i y, N) = g(h(x, \phi_i y), N)$$

and

$$g(\bar{\nabla}_x \phi_i y, N) = g((\bar{\nabla}_x \phi_i)(y) + \phi_i \bar{\nabla}_x y, N)$$

Hence giving

$$g(A_N \phi_i y, x) = g(A_{\phi_i N} y, x)$$

That is (3.7) hold good

Theorem (3.1) Let M be a 3-contract CR-submanifold of $\bar{M}$ with LP-Sasakian 3-structure, then the horizontal distribution D is integrable if and only if

$h(x, T_i y) = h(y, T_i x)$ for $x, y \in \Gamma(D)$. (3.8)

Proof For $x, y \in \Gamma(D)$, using (3.3), we have.

$\phi_i [x, y] = T_i [x, y] + F_i [x, y]$

$= T_i [x, y] + h(x, T_i y) - h(y, T_i x)$

From which we have our assertion.

Theorem (3.2) Let M be a 3-contract CR-submanifold of $\bar{M}$ with LP-Sasakian 3-structure. Then $D^{\perp}$ is integrable if and only if $g(h(y, z), F_i x) = 0$ (3.9)

for any $x, y \in \Gamma(D^{\perp})$ and $z \in \Gamma(D)$

Proof : Taking $x, y \in \Gamma(D^{\perp})$ in (3.2), we obtain

$- T_i \nabla_x y = g(x, y)\, \xi_i + t_i h(x, y) + A_{F_i y} x.$ (3.10)

from (3.10) follows

$A_{F_i x} y - A_{F_i y} x = T_i [x, y]$ (3.11)

Using (3.6), we get

$2 A_{F_i x} y = T_i [x, y]$ (3.12)

For $z \in \Gamma(D)$, (3.12) gives

$2g(h(y, z), F_i x) = g([x, y], T_i z).$ (3.13)

From (3.13), it follows that $D^{\perp}$ is integrable if and only if (3.9) holds.

Theorem (3.3) Let M be a 3-contract CR-submanifold of $\bar{M}$ with LP-Sasakian 3-structure. If $D^{\perp}$ is integrable then each leaf of $D^{\perp}$ is totally geoderic immersed in M.

Proof For $x \in \Gamma(D^{\perp})$ and $y \in \Gamma(D)$, (3.3) gives.

$F_i \nabla_x y = f_i h(x, y) - h(x, T_i y)$ (3.14)

For any $z \in \Gamma(D^{\perp})$, (3.18) gives

$g(F_i \nabla_x y, F_i z) = g(f_i h(x, y), F_i z) - g(h(x, T_i y), F_i z).$ (3.15)

Which with the help of (1.6), (2.4) and (3.9) gives

$g(y, \nabla_x z) = - g(\nabla_x y, z)$

$= - g(h(x, y), f_i F_i z)$

$= 0$

Hence we have the assertion.

Theorem (3.4) Let M be a 3-contact CR-submanifold of $\bar{M}$ with LP-Sasakian 3-structure. If t_i (i = 1, 2, 3) are paralle, then M is an invariant submanifold.

Proof By Theorem (3.1), we have

$F_j T_i = 0$ (3.16)

For any $S \in \Gamma(TM)$, (2.3) and (2.5) gives

$g(T_i t_j N, S) = g(t_j N, T_i S)$

$= g(N, F_j T_i S) = 0$

Implying that $T_i t_j N = 0$ and hence by (2.14), we get

$$t_i f_j = t_k \tag{3.17}$$

Since t_i are parallel, putting $x = \xi_j$ in (3.4), we get

$$A_{f_i N} \xi_j - T_i A_N \xi_j = 0 \tag{3.18}$$

For any $U \in \Gamma(TM)$, (3.18) gives

$$0 = g(h(\xi_j, U), f_i N) - g(h(\xi_j, T_i U), N)$$

$$= g(F_j U, f_i N) \text{ [since } h(\xi_j, T_i U) = F_j T_i U = 0]$$

$$= g(U, t_j f_i N)$$

$$= -g(U, t_k N) = g(F_k U, N)$$

from which we have $F_k U = 0$ showing that M is an invariant submanifold of $\tilde{M}$.

Theorem (3.5) Let M be a 3-contact CR-submanifold of $\tilde{M}$ with LP-Sasakian 3-structure. Then T_i can not be parallel.

Proof If T_i are paralle, then putting $y = \xi_i$ in (3.2) and using (1.1) and (3.1), we get

$$x + \eta_i(x)\xi_i + t_i F_i x = 0 \tag{3.19}$$

Now putting $x = \xi_j$ $(j \neq i)$ in (3.19) and using (1.1), we get $\xi_j = 0$. Hence we get a contradiction i.e. T_i can not be parallel.

Theorem (3.6) Let M be a 3-contact CR-submanifold of $\tilde{M}$ with LP-Sasakian 3-structure. Then F_i are parallel if and only if t_i are parallel.

Proof For any $y \in \Gamma(TM)$, (3.3) and (3.4) gives

$$g((\nabla_x t_i)(N), y) = g(A_{f_i N}x - T_i A_N x, y) \tag{3.20}$$

$$= g(h(x, y), f_i N) - g(h(x, T_i y), N)$$

$$= g(f_i h(x, y) - h(x, T_i y), N)$$

$$= g((\nabla_x F_i)(y), N)$$

Showing that F_i are parallel if and only if t_i are parallel

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