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A special type of semi-symmetric non-metric connection on a Riemannian manifold

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Abstract

The aim of the present paper is to introduce a special type of connection on a Riemannian manifold. At first we prove the existence of such a connection and then we study some properties of curvature tensor and Weyl projective curvature tensor with respect to special type of semi-symmetric non-metric connection. Finally, a non-trivial example of this connection have been constructed.

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Key words- Riemannian manifold, semi-symmetric non-metric connection, curvature tensor, Ricci tensor, Einstein manifold, Killing vector field, Weyl projective curvature tensor.

Introduction

Let $\bar{D}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor $\bar{T}$ and the curvature $\bar{R}$ of $\bar{D}$ are given by $\bar{T}(X,Y) = \bar{D}_X Y - \bar{D}_Y X - [X,Y]$ and $R\,(X,Y)Z = \bar{D}_X\bar{D}_Y Z - \bar{D}_Y\bar{D}_X Z - \bar{D}_{[X,Y]}Z$. The connection $\bar{D}$ is symmetric if its torsion tensor $\bar{T}(X,Y) = 0$, otherwise it is non-symmetric. The connection $\bar{D}$ is metric if $(\bar{D}_X\, g)(Y,Z) = 0$, otherwise non-metric. It is well known that linear connection is symmetric and metric if and only if it is the Riemannian connection. In Hayden (1932) introduced a metric connection $\bar{D}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called a Hayden connection. On the other hand, in a Riemannian manifold, Weyl connection defined by Folland (1970) was a symmetric non-metric connection. Another symmetric non-metric connection is projectively related to the Levi-Civita connection studied by Yano (1970) and Smaranda (1981). In 1924 Friedmann and Schouten introduced the concept of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor $\bar{T}$ is of the form $\bar{T}X,Y) = \eta(Y)X - \eta(X)Y$, where η is a 1-form defined by $\eta(X) = g(X,\xi)$. A connection with the torsion tensor of the above form is a semi-symmetric metric connection, which appeared in a study of Pok (1969). A systematic study of the semi-symmetric metric connection $\bar{D}$ on a Riemannian manifold was initiated by Yano (1970). He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is group manifold (Eisenhart, 1961). Some different kind of semi-symmetric non-metric connection was studied by De and Kamilya (1995), Chaubey and Ojha (2012), Agashe and Chafle (1992), Sengupta etal (2000), Prasad and Verma

(2004), Prasad and Singh (2006) and many others. In this paper we proved the existence of a new type of special connection. Then we find the relation of curvature tensors between the Levi-Civita connection D and semi-symmetric non-metric connection $\overline{D}$ and proved some basic properties of the curvature tensor of $\overline{D}$. We also defined Weyl projective tensor with respect to special connection. In last we list a number of connection as particular cases.

2. Special linear connection $\overline{D}$: A type of special linear connection given by

$$\overline{D}_X Y = D_X Y + a.\eta(X)Y + b.\eta(Y)X + c.g(X,Y)\xi \tag{2.1}$$

where a, b and c are non-zero real numbers and ξ is a vector field defined by

$$\eta(X) = g(X,\xi) \tag{2.2}$$

for all X and $Y \epsilon M$.

Using (2.1), the torsion $\overline{T}$ of M with respect to the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y] = (a-b)[\eta(X)Y - \eta(Y)X]. \tag{2.3}$$

A linear connection satisfying (2.3) is called a semi-symmetric connection. Further from (2.1), we have

$$(\overline{D}_X g)(Y,Z) = -2a\eta(X)g(Y,Z) - (b+c)[\eta(Y)g(X,Z) + \eta(Z)g(X,Y)]. \tag{2.4}$$

Hence a linear connection $\overline{D}$ defined by 2.1) satisfies (2.3) and (2.4) and hence we call $\overline{D}$ as semi-symmetric non-metric connection.

Now we prove the existence of such a connection on an n-dimensional Riemannian manifold in the following way:

We consider a linear connection $\overline{D}$ and the Riemannian connection D of a Riemannian manifold M such that

$$\overline{D}_X Y = D_X Y + H(X,Y) \tag{2.5}$$

where H is a tensor of type (1,2). For $\overline{D}$ to be a semi-symmetric non-metric connection in M, we have

$$H(X,Y) = \frac{1}{2}[\overline{T}(X,Y) + \overline{T}'(X,Y) + \overline{T}'(Y,X)] + a.[\eta(X)Y + \eta(Y)X] + [a-(b+c)]g(X,Y)\xi, \tag{2.6}$$

where $g(\overline{T}'(X,Y),Z) = g(\overline{T}(Z,X),Y)$.

From (2.3) and (2.6), we get

$$\overline{T}'(X,Y) = (a-b)[g(X,Y)\xi - \eta(X)Y] \tag{2.7}$$

In view of (2.3), (2.6) and (2.7), we get

$$H(X,Y) = a.\eta(X)Y + b.\eta(Y)X + c.g(X,Y)\xi. \tag{2.8}$$

Hence from (2.5) and (2.8), we get

$$\overline{D}_X Y = D_X Y + a.\eta(X)Y + b.\eta(Y)X + c.g(X,Y)\xi.$$

Conversely, we prove that a linear connection $\overline{D}$ such that $\overline{D}_X Y = D_X Y + a.\eta(X)Y + b.\eta(Y)X + c.g(X,Y)\xi$ is semi-symmetric non-metric connection on a Riemannian manifold.

The torsion tensor $\overline{T}$ of the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y] = (a-b)[\eta(X)Y - \eta(Y)X]$$

From the above equation, we obtain that the connection $\overline{D}$ is a semi-symmetric connection. Also we have

$$(\overline{D}_X g)(Y,Z) = -2a\eta(X)g(Y,Z) - (b+c)[\eta(Y)g(X,Z) + \eta(Z)g(X,Y)] \neq 0.$$

Therefore, we are in a position to conclude that the connection $\overline{D}$ is a semi-symmetric non-metric connection. Since this connection includes known symmetric, semi-symmetric metric and non-metric connection hence we call it as a special type of semi-symmetric non-metric connection on a Riemannian manifold.

3. Definition (3.1) The 1-form η is closed w.r.t. Levi-Civita connection if

$$(D_X \eta)(Y) - (D_Y \eta(X) = 0,$$

where ξ is a vector field defined by $\eta(X) = g(X,\xi), D$ denotes the operator of covariant differentiate with respect to metric tensor g and X, Y be the arbitrary vector field on M.
The vector field ξ is irrotational if $g(Y, D_X \xi) = g(X, D_Y \xi)$ and integral curves of the vector field ξ are geodesic if $D_\xi \xi = 0$. From (2.1), we get

$$(\bar{D}_X \eta)(Y) - (\bar{D}_Y \eta)(X) = (D_X \eta)(Y) - (D_Y \eta)(X), \tag{3.1}$$

this show that 1-form η is closed with respect to Levi-Civita connection D if and only if η is closed with respect to special type of semi-symmetric non-metric connection $\bar{D}$.
Putting ξ for Y in (2.1), we get

$$\bar{D}_X \xi = D_X \xi + a.\eta(X)\xi + b.\eta(\xi)X + c.g(X,\xi)\xi \tag{3.2}$$

The above equation yields

$$g(Y, \bar{D}_X \xi) - g(X, \bar{D}_Y \xi) = g(Y, D_X \xi) - g(X, D_Y \xi)$$

which shows that the vector field ξ is irrotational with respect to D if and only if ξ is irrotational with respect to $\bar{D}$.
Again putting xi for X in (3.2), we get

$$\bar{D}_\xi \xi = D_\xi \xi + (\mathrm{a} + \mathrm{b} + \mathrm{c})\eta(\xi)\xi. \tag{3.3}$$

If $\mathrm{a} + \mathrm{b} = -\mathrm{c}$, then from equation (3.3), we get

$$\bar{D}_\xi \xi = D_\xi \xi \tag{3.4}$$

From (3.4), we can say that the integral curves of the unit vector field ξ are geodesic with respect to D iff integral curves of the unit vector field ξ is geodesic with respect to $\bar{D}$.
From the above discussion we can state that the following theorem:
Theorem (3.1): If a Riemannian manifold admits a special type of semi-symmetric non-metric connection. Then the integral curves of the unit vector field ξ are geodesic with respect to D iff integral curves of the unit vector field ξ is geodesic with respect to $\bar{D}$.

4. Expression of the curvature tensor of special semi-symmetric non-metric connection $\bar{D}$:

In this section we obtain the expressions of the curvature tensor and Ricci tensor of M with respect to semi-symmetric non-metric connection defined by (2.1).
Analogous to the definition of the curvature tensor R of M with respect to Levi-Civita connection D, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric connection $\bar{D}$ given by

$$\bar{R}(X,Y)Z = \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X,Y]} Z \tag{4.1}$$

and

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

is the curvature tensor of a Riemannian manifold M with respect to Riemannian connection D.
From (2.1) and (4.1), we get

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + a[D_X \eta)(Y) - (D_Y \eta)(X)]Z + \\ & b[(D_X\eta)(Z)Y - (D_Y \eta))(Z)X] + c[g(Y,Z)(D_X \xi) - g(X,Z)(D_Y \xi)] \\ & + b^2[\eta(Y)X - \eta(X)Y]\eta(Z) + c^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ & + bc[g(Y,Z)X - g(X,Z)Y]\eta(\xi).\end{aligned} \tag{4.2}$$

Rearranging expression (4.2), we get

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + a.d\eta(X,Y)Z - \\ & \left[b(D_Y \eta)(Z)) - b^2\eta(Y)\eta(Z) - \tfrac{1}{2}bc.g(Y,Z)\eta(\xi)\right]X + \\ & \left[b(D_X \eta)(Z)) - b^2\eta(X)\eta(Z) - \tfrac{1}{2}bc.g(X,Z)\eta(\xi)\right]Y + \\ & \left[c(D_X \ \xi) + c^2\eta(X)\xi + \tfrac{1}{2}bc.\eta(\xi)X\right]g(Y,Z) - \\ & \left[c(D_Y \ \xi) + c^2\eta(Y)\xi + \tfrac{1}{2}bc.\eta(\xi)Y\right]g(X,Z).\end{aligned} \tag{4.3}$$

Equation (4.3) can be written as

$$\bar{R}(X,Y)Z = R(X,Y)Z + a.d\eta(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y + g(Y,Z)QX - g(X,Z)QY, \quad (4.4)$$

where

$$\alpha(Y,Z) = g(QY,Z) = b(D_Y\,\eta)(Z)) - b^2\eta(Y)\eta(Z) - \frac{1}{2}bc.g(Y,Z)\eta(\xi) \quad (4.5)$$

and

$$QX = c(D_X\ \xi) + c^2\eta(X)\xi + \frac{1}{2}bc.\eta(\xi)X$$

A relation between the curvature tensor of M with respect to the Riemannian manifold D and semi-symmetric non-metric connection $\bar{D}$ is given by (4.4).

From (4.5), we have

$$\alpha(X,Y) - \alpha(Y,X) = b.d\eta(X,Y) \quad (4.6)$$

Thus a tensor α is symmetric iff 1-form η is closed.

Let

$$'R(X,Y,Z,W) = g(R(X,Y)Z,W)$$
$$'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W) \quad (4.7)$$

for the arbitrary vector fields X,Y,Z and W on M. From (4.4) and (4.7), we get $'\bar{R}(X,Y,Z,W) =$

$$'R(X,Y,Z,W) + a.d\eta(X,Y)g(Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) + g(Y,Z)\,\alpha(X,W) - g(X,Z)\alpha(Y,W) \quad (4.8)$$

In view of (4.8), we have

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,X,Z,W) = 0 \quad (4.9)$$

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(X,Y,W,Z) = -2\alpha(Y,W)g(X,Z) + 2\alpha(X,W)g(Y,Z) + 2\alpha(X,Z)g(Y,W) - 2\alpha(Y,Z)g(X,W)$$

(4.10)

$$'\bar{R}(X,Y,Z,W) - '\bar{R}(Z,W,X,Y) = a[d\eta(X,Y)g(Z,W) + d\eta(Z,W)g(X,Y)] + b[d\eta(X,Z)g(Y,W) + d\eta(W,Y)g(X,Z)] + [\alpha(X,W) + \alpha(W,X)]g(Y,Z) - [\alpha(Y,Z) + \alpha(Z,Y)]g(X,W) \quad (4.11)$$

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,Z,X,W) + '\bar{R}(Z,X,Y,W) = (a-b)[d\eta(X,Y)g(Z,W) + d\eta(Y,Z)g(X,W)] + d\eta(Z,X)g(Y,W)]$$

Analogous to the definition of Ricci tensor of Riemannian manifold M with respect to the Riemannian connection$D$, we define Ricci tensor of M with respect to semi-symmetric non-metric connection $\bar{D}$ by

$$\bar{R}ic(Y,Z) = \sum_{i=1}^{n} {}'\bar{R}\,(E_i\,,Y,Z,E_i) \quad (4.12)$$

where $E_i's, 1 \leq i \leq n$, orthonormal vector fields on M.

From (4.4) and (4.12), we get

$$\bar{R}ic(Y,Z) = Ric(Y,Z) + a.d\alpha(Z,Y) - \alpha(Y,Z)n + g(Y,Z)p, \quad (4.13)$$

where $p = trace\alpha$ and $Ric(Y,Z) = \sum_{i=1}^{n} R\,(E_i\,,Y,Z,E_i)$ is the Ricci tensor of M with respect to the Riemannian connection. By virtue of (4.13), we get

$$\bar{R}ic(Y,Z) - \bar{R}ic(Z,Z) = Ric(Y,Z) - Ric(Z,Y) - (a+nb)d\eta(Y,Z) + a.d\eta(Z,Y). \quad (4.14)$$

Thus we see that $\bar{R}ic(Y,Z)$ is symmetric in Y and Z iff $(a+nb)d\eta(Y,Z) = a.d\eta(Z,Y)$.

From (4.13), we obtain relation between the scalar curvature of (M,g) with respect to Riemannian connection D and semi-symmetric connection $\bar{D}$ which is given by

$$\bar{r} = r, \quad (4.15)$$

where $r = \sum_{i=1}^{n} R\,ic(E_i,E_i)$ is the scalar curvature of M with respect to the Riemannian connection.

Summing up, we state the following theorem.

Theorem (4.1): If a Riemannian manifold admits a special semi-symmetric non-metric connection $\bar{D}$, then we have

(i) $'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,X,Z,W) = 0$

(ii) $'\bar{R}(X,Y,Z,W) + '\bar{R}(X,Y,W,Z) = -2\alpha(Y,W)g(X,Z) + 2\alpha(X,W)g(Y,Z) + 2\alpha(X,Z)g(Y,W) - 2\alpha(Y,Z)g(X,W)$

(iii) $'\bar{R}(X,Y,Z,W) - '\bar{R}(Z,W,X,Y) = a[d\eta(X,Y)g(Z,W) + d\eta(Z,W)g(X,Y)] + b[d\eta(X,Z)g(Y,W) + d\eta(W,Y)g(X,Z)] + [\alpha(X,W) + \alpha(W,X)]g(Y,Z) - [\alpha(Y,Z) + \alpha(Z,Y)]g(X,W)$

(iv) $'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,Z,X,W) + '\bar{R}(Z,X,Y,W) = (a-b)[d\eta(X,Y)g(Z,W) + d\eta(Y,Z)g(X,W)] + d\eta(Z,X)g(Y,W)]$

(v) Ricci tensor $\bar{R}ic$ is symmetric if and only if $(a+nb)d\eta(Y,Z) = a.\,d\eta(Z,Y)$.

5. Einstein manifold with respect to special semi-symmetric non-metric connection $\bar{D}$:

A Riemannian manifold M is called an Einstein manifold with respect to Riemannian connection if

$$Ric\,(X,Y) = \frac{r}{n}g(X,Y) \tag{5.1}$$

Analogues of this definition, we define that a Riemannian manifold M with respect to special semi-symmetric non-metric connection is called an Einstein manifold if

$$\bar{R}ic\,(X,Y) = \frac{\bar{r}}{n}\,g(X,Y) \tag{5.2}$$

From (4.13) and (4.15), we have

$$\bar{R}ic(X,Y) - \frac{\bar{r}}{n}g(X,Y) = Ric(X,Y) + a.\,d\eta(Y,X) - n.\,\alpha(X,Y) + p.\,g(X,Y) - \frac{r}{n}g(X,Y) \tag{5.3}$$

If

$$a.\,d\eta(Y,X) - n.\,\alpha(X,Y) + p.\,g(X,Y) = 0, \tag{5.4}$$

then from (5.3), we get

$$\bar{R}ic(X,Y) - \frac{\bar{r}}{n}g(X,Y) = Ric\,(X,Y) - \frac{r}{n}g(X,Y)$$

From (5.4), we can state the following theorem:

Theorem (5.1): If a Riemannian manifold M admits a special semi-symmetric non-metric connection satisfying the condition (5.4), then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\bar{D}$.

6.Killing Vector Field: In this section, we shall consider

$$(D_X\,\eta)(Y) + (D_Y\,\eta)(X) = 0 \tag{6.1}$$

Then ξ is called a killing vector field.

Taking covariant differentiation (2.2), we get

$$g(X, D_Y\,\xi) = (D_Y\,\xi)(X). \tag{6.2}$$

From (6.2), we have

$$g(Y, D_X\,\xi) = (D_X\,\eta)(Y). \tag{6.3}$$

Hence from (6.1), (6.2) and (6.3), we get

$$g(X, D_Y\,\xi) + g(Y, D_X\,\xi) = 0. \tag{6.4}$$

Thus the killing vector field ξ satisfies (6.4).

From (4.5) and (4.3), we get

$$\bar{R}ic\,(Y,Z) = Ric(Y,Z) + a.[(D_Y\eta)(Z) - (D_Z\eta)(Y)] - n\,[b(D_Y\eta)(Z) - b^2\,\eta(Y)\eta(Z) - \frac{1}{2}bc.\,g(Y,Z)\eta(\xi)] + p.\,g(Y,Z) \tag{6.5}$$

From (6.5), we get

$$\bar{R}ic\,(Y,Z) + \bar{R}ic\,(Z,Y) = 2Ric(Y,Z) + n.\,b\,[2b\,\eta(Y)\eta(Z) + c.\,g(Y,Z)\eta(\xi)]. \tag{6.6}$$

If $\bar{R}ic\ (Y,Z) + \bar{R}ic\ (Z,Y) = 0$, then from (6.6), we get

$$Ric(Y,Z) = -n.b\left[b\ \eta(Y)\eta(Z) + \frac{c}{2}.g(Y,Z)\eta(\xi)\right] \tag{6.7}$$

If (6.7) exist then from (6.6), we get

$$\bar{R}ic\ (Y,Z) + \bar{R}ic\ (Z,Y) = 0$$

Hence, we can state the following theorem:

Theorem (6.1): If a Riemannian Manifold M admits a special semi-symmetric non-metric connection $\bar{D}$ with ξ as a killing vector field, then a necessary and sufficient condition for the Ricci tensor of $\bar{D}$ be skew-symmetric is that the Ricci tensor of Levi-Civita connection D is given by (6.7).

Also if $Ric(Y,Z) = 0$, then (6.7), we get

$$n.b\left[b\ \eta(Y)\eta(Z) + \frac{c}{2}.g(Y,Z)\eta(\xi)\right] = 0,$$

which gives on contraction

$$b\left[b + \frac{c}{2}\right]\eta(\xi) = 0$$

which is not possible.

Hence, we can state the following theorem:

Theorem (6.2): If a Riemannian manifold M admits a special semi-symmetric non-metric connection $\bar{D}$ whose Ricci tensor is skew-symmetric and ξ is killing vector field, then the manifold can not be Ricci-flat.

7. Weyl projective curvature tensor of a Riemannian manifold admitting a special type of the semi-symmetric non-metric connection $\bar{D}$:

The Weyl projective curvature tensor $\bar{P}$ with respect to semi-symmetric non-metric is given by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}[\bar{R}ic(Y,Z)X - \bar{R}ic(X,Z)Y] \tag{7.1}$$

From (7.1), it follows that

$$'\bar{P}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{1}{n-1}[\bar{R}ic(Y,Z)g(X,W) - \bar{R}ic(X,Z)g(Y,W)] - \tag{7.2}$$

where $'\bar{P}(X,Y,Z,W) = g(\bar{P}(X,Y)Z,W)$ for all $X,Y,Z,W \in M$.

In view of (4.8), (4.3) and (7.2), we get

$$\begin{aligned} '\bar{P}(X,Y,Z,W) = {}&'P(X,Y,Z,W) + a.d\eta(X,Y)g(Z,W) + \frac{a}{n-1}[d\eta(Z,Y)g(X,W) \\ &-d\eta(Z,X)g(Y,W)] + \frac{1}{n-1}[\alpha(Y,Z)g(X,W) - \alpha(X,Z)g(Y,W)] \\ &+ g(Y,Z)\left[\alpha(X,W) - \frac{p}{n-1}g(X,W)\right] \\ &- g(X,Z)\left[\alpha(Y,W) - \frac{p}{n-1}g(Y,W)\right] \end{aligned} \tag{7.3}$$

where

$$'P(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{n-1}[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W)]$$

is the Weyl projective curvature tensor with respect to Levi-Civita connection D (Yano,1970).

Contracting (7.3) with respect to Z and, we get

$$d\eta(X,Y) = 0,\ a(n^2 - n + 2) + nb \neq 0.$$

Hence, we can state the following theorem:

Theorem (7.1): The necessary condition for the Weyl projective curvature tensor $'P$ of the manifold M with respect to Levi-Civita connection and Weyl projective curvature tensor $'\bar{P}$ of the manifold M with respect to special semi-symmetric non-metric connection to be equal that $d\eta(X,Y) = 0,\ a(n^2 - n + 2) + nb \neq 0$.

8. Particular caces:

In this section, we list the following twelve particular cases.

(i) Semi-symmetric metric connection:

If $a = 0, b = 1$ and $c = -1$, then the equation (2.1), (2.3) and (2.4) becomes

$$\bar{D}_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi \tag{8.1}$$

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = 0.$$

Connection (8.1) is introduced by Yano in 1970 and known as semi-symmetric metric connection. From (4.3) and (4.5), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)QX + g(X,Z)QY,$$

where

$$\alpha(Y,Z) = g(QY,Z) = (\bar{D}_Y \backslash \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z)\eta(\xi).$$

Yano in (1970) prove that if a Riemannian manifold admits a semi-symmetric metric connection where Ricci tensor vanishes, then the curvature tensor of the semi-symmetric metric connection is equal to Weyl Conformal curvature tensor of manifold.

(ii) If $a = 1, b = 1$ and $c = 0$, then the equation (2.1), (2.3) and (2.4) becomes

$$\bar{D}_X Y = D_X Y + \eta(X)Y + \eta(Y)X \tag{8.2}$$

$$\bar{T}(X,Y) = 0$$

and

$$(\bar{D}_X g)(Y,Z) = -2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$

This connection was introduced by (Yanno 1970, Smaranda 1981) and called by them as semi-symmetric non-metric connection. This connection is projectively related to the Levi-Civita connection D. Then in view of (4.3) and (4.5), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y + [\alpha(X,Y) - \alpha(Y,X)]Z,$$

where

$$\alpha(X,Y) = (\bar{D}_X\eta)(Y) - \eta(X)\eta(Y).$$

(iii) If $a = 1, b = 1$ and $c = -1$, then the equation (2.1), (2.3) and (2.4) becomes

$$\bar{D}_X Y = D_X Y + \eta(X)Y + \eta(Y)X - g(X,Y)\xi$$

$$\bar{T}(X,Y) = 0$$

and

$$(\bar{D}_X g)(Y,Z) = -2\eta(X)g(Y,Z).$$

This connection is semi-symmetric non-metric connection due to (Yano 1970). This connection $\bar{D}$ is conformaly related to Levi-Civita connection D.

(iv) If $a = -\frac{1}{2}, b = -\frac{1}{2}$ and $c = \frac{1}{2}$, then in view of (2.1), (2.3) and (2.4), we get

$$\bar{D}_X Y = D_X Y - \frac{1}{2}\eta(X)Y - \frac{1}{2}\eta(Y)X + \frac{1}{2} g(X,Y)\xi$$

$$\bar{T}(X,Y) = 0$$

and

$$(\bar{D}_X g)(Y,Z) = \eta(X)g(Y,Z).$$

Connection is constructed by 1-form η and vector field ξ is Weyl connection according to (Folland, 1970). This equation is a symmetric recurrent-metric connection (symmetric but not-metric).

(v) If $a = 0, b = 1$ and $c = 0$, then we obtained a semi-symmetric non-metric connection $\bar{D}$ given by (Agashe and Chafle, 1992)

$$\bar{D}_X Y = D_X Y + \eta(Y)X$$

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$

Authors proved that Weyl projective curvature tensor of the manifold with respect to D is equal to Weyl projective curvature tensor with respect to $\bar{D}$.

(vi) If $a = -1, b = 0$ and $c = 0$, then we get

$$\bar{D}_X Y = D_X Y - \eta(X)Y$$

This connection satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = 2\eta(X)g(Y,Z).$$

Actually this is semi-symmetric non-metric connection but Liang (1994) called this a semi-symmetric recurrent metric connection and obtained some geometrical properties.

(vii) If $a = 1, b = 0$ and $c = 0$, then we get

$$\bar{D}_X Y = D_X Y + \eta(X)Y$$

This semi-symmetric non-metric connection was due to Melhotra and Prasad (2014) and he proved that under certain condition Weyl projective curvature of semi-symmetric non-metric connection $\bar{D}$ is equal to weyl projective curvature of connection D.

(viii) If $a = 0, b = 1$ and $c = 1$, then we obtain another type of semi-symmetric non-metric connection given by (De and Biswas, 1996/1997)

$$\bar{D}_X Y = D_X Y + \eta(Y)X + g(X,Y)\xi$$

This connection satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = -2\eta(Y)g(X,Z) - 2\eta(Z)g(X,Y).$$

Later on, this connection was studied by Han, Yun and Zhao (2013) under projective transformation.

(ix) $a = -1, b = 0$ and $c = 1$ gives Barua and Mukhopadhya (1992)

$$\bar{D}_X Y = D_X Y - \eta(X)Y + g(X,Y)\xi$$

This connection satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$

(x) If $a = 0, b = -1$ and $c = -1$, then we get

$$\bar{D}_X Y = D_X Y - \eta(Y)X - g(X,Y)\xi$$

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = 2\eta(Y)g(X,Z) + 2\eta(Z)g(X,Y).$$

This semi-symmetric non-metric connection in a generalized co-symplectic manifold investigated by Kumar and Chaubey in 2010.

(xi) If $c = 0$ in (2.1), (2.3) and (2.4), we get

$$\bar{D}_X Y = D_X Y + a.\eta(X)Y + b.\eta(Y)X. \quad .$$

$$\bar{T}(X,Y) = (a-b)[\eta(X)Y - \eta(Y)X]$$

and

$$(\bar{D}_X g)(Y,Z) = -2a\eta(X)g(Y,Z) - b.\eta(Y)g(X,Z) - b\eta(Z)g(X,Y)$$

This semi-symmetric non-metric connection was introduced by Chaturvedi and Pandey (2009) in a Riemannian manifold whose torsion tensor $\bar{T}$ and curvature tensor $\bar{R}$ satisfy the conditions

$$(D_X\bar{T})(Y,Z) = \eta(X)\bar{T}(Y,Z)$$

and

$$\bar{R}(X,Y)Z = 0$$

and obtained various geometrical properties.
Recently, in 2016, De, Han Zhao studied this connection in Riemannian manifold where torsion tensor is pseudo symmetric.
(xii) If $a = -\frac{1}{2}, b = \frac{1}{2}$ and $c = 0$ then from (2.1), (2.3) and (2.4), we get

$$\bar{D}_X Y = D_X Y - \frac{1}{2}\eta(X)Y + \frac{1}{2}\eta(Y)X$$
$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$(\bar{D}_X g)(Y,Z) = \frac{1}{2}[2\,\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y)]$$

This new connection was introduced by Chaubey and Yieldiz in 2019 under the title "Riemannian manifolds admitting a new type of semi-symmetric non-metric connection".
Finally, if we take $a = -1, b = 1$ and $c = -1$, then from (2.1), (2.3) and (2.4), we get

$$\bar{D}_X Y = D_X Y - \eta(X)Y + \eta(Y)X - g(X,Y)\xi$$
$$\bar{T}(X,Y) = 2[\eta(Y)X - \eta(X)Y].$$

and

$$(\bar{D}_X g)(Y,Z) = 2\,\eta(X)g(Y,Z)$$

We call it as a "semi-symmetric recurrent non-metric connection" whose curvature tensor $\bar{R}(X,Y)Z$, Ricci tensor $\bar{R}ic(Y,Z)$ and scalar curvature $\bar{r}$ are given by respectively

$$\bar{R}(X,Y)Z = R(X,Y)Z - [\alpha(X,Y) - \alpha(Y,X)] - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)QX + g(X,Z)QY,$$

where

$$\alpha(Y,Z) = g(QY,Z) = (D_Y\eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z)\eta(\xi).$$
$$QX = D_X\xi - \eta(X)\xi + \frac{1}{2}\eta(\xi)X.$$
$$\bar{R}ic(Y,Z) = Ric(Y,Z) - d\eta(Z,Y) - (n-2)\alpha(Y,Z) - g(Y,Z)p. \text{ and}$$
$$\bar{r} = r - 2(n-1)p.$$

This is a new connection for future scope.
9. Example: Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard co-ordinate of $\mathbb{R}^3$.
We choose the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = e^{ix}\frac{\partial}{\partial y}, \quad e_3 = e^{-ix}\frac{\partial}{\partial z} \tag{9.1}$$

which is linearly independently at each point of M.
Let g be the Riemannian metric denoted by

$$g(e_i,e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{9.2}$$

Then from equation (9.1), we have

$$[e_1,e_2] = ie_2, \quad [e_1,e_3] = -ie_3, \quad [e_2,e_3] = 0. \tag{9.3}$$

By Koszul's formula

$$2g(D_XY,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \tag{9.4}$$

Using (9.2) and (9.3) in (9.4), we get

$$\left.\begin{aligned} D_{e_1}e_1 &= 0, & D_{e_1}e_2 &= 0, & D_{e_1}e_3 &= 0, \\ D_{e_2}e_1 &= -ie_2, & D_{e_2}e_2 &= ie_1, & D_{e_2}e_3 &= 0, \\ D_{e_3}e_1 &= ie_3, & D_{e_3}e_2 &= 0, & D_{e_3}e_3 &= -ie_3. \end{aligned}\right\} \tag{9.5}$$

The curvature tensor is given by

$$R(X,Y)Z = D_XD_YZ - D_Y D_X Z - D_{[X,Y]}Z \tag{9.6}$$

Using (9.3) and (9.5) in (9.6), we get

$$\left.\begin{array}{l} R(e_1,e_2)e_1 = -e_2, \quad R(e_1,e_2)e_2 = -e_1, \quad R(e_1,e_2)e_3 = 0, \\ R(e_2,e_3)e_1 = -e_3, \quad R(e_2,e_3)e_2 = e_3, \quad R(e_2,e_3)e_3 = 0, \\ R(e_1,e_3)e_1 = -e_3, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = e_3, \\ R(e_1,e_1)e_1 = R(e_1,e_1)e_2 = R(e_1,e_1)e_3 = 0, \\ R(e_2,e_2)e_1 = R(e_2,e_2)e_2 = R(e_2,e_2)e_3 = 0, \\ R(e_3,e_3)e_1 = R(e_3,e_3)e_2 = R(e_3,e_3)e_3 = 0. \end{array}\right\} \qquad (9.7)$$

The Ricci tensor can be calculated by the formulae

$Ric(X,Y) = \sum_{i=1}^{3} g\,(R(X,e_i)e_i, Y)$ as

$Ric(e_1,e_1) = -1, Ric(e_2,e_2) = 1 \;\; Ric(e_3,e_3) = 0, \;\; Ric(e_1,e_3) = 1, \qquad Ric(e_2,e_3) = 0.$

It is obvious that the scalar curvature of (M^3, g) is $r = 1$.

In consequences of the above discussion and equation (2.1), we obtain

$$\left.\begin{array}{l} \bar{D}_{e_1}e_1 == (a+b+c)e_1, \quad \bar{D}_{e_1}e_2 = ae_2, \quad \bar{D}_{e_1}e_3 = ae_3, \\ \bar{D}_{e_2}e_1 = (-i+b)e_2, \quad \bar{D}_{e_2}e_1 = (i+c)e_1, \quad \bar{D}_{e_2}e_3 = 0, \\ \bar{D}_{e_3}e_1 = (i+b)e_3, \quad \bar{D}_{e_3}e_2 = 0, \; \bar{D}_{e_3}e_3 = (-i+c)e_3. \end{array}\right\} \qquad (9.8)$$

In view of (9.8) we can easily prove that equation (2.3) holds for all vector fields $e_i, (i = 1,2,3)$ e.g. $\bar{T}(e_1, e_2) = (a-b)e_2$ and $(a-b)[\eta(e_1)e_2 - \eta(e_2)e_1] = (a-b)e_2$. This show that the linear connection $\bar{D}$ defined as (2.1) is a semi-symmetric connection on (M^3, g). Also

$$(\bar{D}_{e_1}g)(e_2, e_2) = -2a \neq 0.$$

Similarly, we can verify this for other components. Hence, the semi-symmetric connection $\bar{D}$ is non-metric on (M^3, g).

The curvature tensor and Ricci tensor can be calculated by the formulae

$\bar{R}(X,Y)Z = \bar{D}_X\bar{D}_Y Z - \bar{D}_Y\,\bar{D}_X\,Z - \bar{D}_{[X,Y]}Z$

and

$\bar{R}ic(X,Y) = \sum_{i=1}^{3} g\,(\bar{R}(X,e_i)e_i, Y)$ as

$\bar{R}(e_1,e_2)e_1 = (i-b)(b+c+i)e_2, \bar{R}(e_1,e_2)e_2 = (c+i)(b+c-i)e_1, \bar{R}(e_1,e_2)e_3 = 0$

$\bar{R}(e_2,e_3)e_1 = 0, \qquad \bar{R}(e_2,e_3)e_2 = -(c+i)(b+i)e_3, \qquad R(e_2,e_3)e_3 = 0$

$\bar{R}(e_1,e_3)e_1 = (b+i)(-b-c+i)e_3, \; \bar{R}(e_1,e_3)e_2 = 0, \; \bar{R}(e_1,e_3)e_3 = (1+ic)e_3$.

$\bar{R}ic(e_1,e_1) = (c+i)(b+c-i), \quad \bar{R}ic(e_2,e_2) = (b-i)(b+c-i),$

$\bar{R}ic(e_3,e_3) = (b+i)(2b+c).$

and scalar curvature is $\bar{r} = 2(c+i)(b+c-i)(b-i)(b+c-i)$.

Hence the curvature tensor and the Ricci tensor with respect to semi-symmetric non-metric connection exit.

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