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A study of η-Ricci soliton on 3-dimensional f-Kenmotsu manifolds with semi-symmetric non-metric connection

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Abstract
In this paper we have studied 3-dimensional f-Kenmotsu manifolds with semi-symmetric non-metric connection satisfying the curvature conditions $\tilde{R}.\tilde{S} = 0$ and $\tilde{S}.\tilde{R} = 0$. Next, we have η-Ricci solitons on 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection and obtained some results.

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Key words- 3-dimensional f-Kenmotsu manifold, η-Ricci solitons, Semi-symmetric non-metric connection.

1. Introduction
Kenmotsu (1972) studied a class of contact Riemannian manifold satisfying some special conditions and named this manifold as Kenmotsu manifold.

The manifold M, with the structure (ϕ, ξ, η, g) is said to be:

(A) normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$,

(B) almost cosymplectic if $d\eta = 0$ and $d\phi = 0$,

(C) cosympectic if it is normal and almost cosymplectic (equivalently $\nabla\ \phi$=0), ∇ being covarient differentiation with respect to the Levi-Civita connection.

Olszak and Rosca (1991) studied geometrical aspect of f-Kenmotsu manifolds and gave some curvature conditions. Also the other mathematicians proved that a Ricci-symmetric f-Kenmotsu Manifold is an Einstein Manifold. An f-Kenmotsu manifold means an almost contact metric manifold which is normal and locally conformal almost cosympectic.

The notion of Ricci (1982) flow to find a canonical metric on a smooth manifold was first introduced by Hamilton (1988) and (2009). Later Ricci flow has become important for the study of Riemannian manifolds, especially for the manifolds with positive curvature. Perelman (2017) proved Poincare conjecture with the help of Ricci flow. A Ricci soliton appears as the limit of the solutions of Ricci flow, an evolution equation for metrics on a Riemannian manifolds defined as $\frac{\partial g}{\partial t} = -2S$.

A Ricci soliton (g, V, λ) represent a natural generalization of Einstein metric on a Riemannian manifolds. In a Riemannian manifold (M, g), the metric g is Ricci soliton (Tripathi, 2018), (De, 2010) if

$$£_V g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \quad (1)$$

£ being a lie derivative, V a complete vector field, S the Ricci tensor and λ a constant. If $V = 0$, Ricci solitons reduces to Einstein manifold. The Ricci solitons is said to be shrinking, steady or expanding if λ is negative, zero or positive respectively.

The more general notion of η-Ricci solitons as the generalization of Ricci soliton was introduced by Cho and Kimura (2009). They, after a study of Ricci soliton of real hypersurfaces in a non-flat complex spaces form Yildize *et al.* (2013), defined η-Ricci soliton, satisfying

$$£_V g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \quad (2)$$

where S is Ricci tensor associated to g. In this connection we have studied the works of Blaga (2016), (2014) with η-Ricci solitons.

A linear connection $\bar{\nabla}$ is said to be a semi-symmetric connection on M if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y, \quad (3)$$

where $\bar{T} \neq 0$ and η is a 1-form.

If moreover $\bar{\nabla} g = 0$, then the connection is called semi-symmetric metric connection.

If $\bar{\nabla} g \neq 0$, the connection is called semi-symmetric non-metric connection .

Agashe and Chafle (1292), Pravonovic and Sengupta (2000) and other mathematicians studied semi-symmetric non-metric connection in different ways.

The paper is organised as follows: Section 2 is preliminaries giving an idea of f-Kenmotsu manifold. We then obtained the relation between Riemannian connection and semi-symmetric non-metric connection in 3-dimensional f-Kenmotsu manifold in section 3. Section 4 and 5 deals with 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connetion satisfying the curvature conditions $\tilde{R}.\tilde{S} = 0$ and $\tilde{S}.\tilde{R} = 0$ respectively. In the last section, we discuss η-Ricci soliton on 3-dimensional f-Kenmotsu manifold and obtained some results.

2. Preliminaries

An odd-dimensional smooth manifold endowed with almost contact structure (ϕ, ξ, η, g) , where ϕ is (1,1)-type tensor field, ξ is vector field, η is 1-form and g is metric tensor of M, satisfying

$$\phi^2 X = -X + \eta(X)\xi,$$

$$\eta \circ \phi = 0,$$

$$\phi\xi = 0,$$

$$\eta(\xi) = 1, \quad (4)$$

$g(X,\xi) = \eta(X),$

$g(X,\phi Y) = -g(\phi X, Y),$

$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$

for any vector fields $X, Y \in \chi(M)$, is called an almost contact metric manifold. An almost contact metric manifold is an f-Kenmotsu manifold (Bagewadi et.al, 2013) if the covarient derivative of ϕ satisfies

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{5}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$.

If $f = \alpha \neq 0 =$ constant, then manifold is called α-Kenmotsu manifold , if $f = 1$, f-Kenmotsu manifold reduces to Kenmotsu manifold . Also for $f = 0$, the manifold is cosymplectic manifold . For an f-Kenmotsu manifold from equation (5), we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{6}$$

In 3-dimensional f-Kenmotsu manifold, we have

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z$$

$$-(\frac{r}{2} + 3f^2 + 3f')[\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z], \tag{7}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y). \tag{8}$$

where r is scaler curvature of M. Thus from (7) and (8), we get

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{9}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{10}$$

$$R(\xi,X)\xi = -(f^2 + f')(\eta(X)\xi - X), \tag{11}$$

$$S(\xi,\xi) = -2(f^2 + f'), \tag{12}$$

$$QX = (\frac{\tau}{2} + f^2 + f')X - (\frac{\tau}{2} + 3f^2 + 3f')\eta(X)\xi. \tag{13}$$

3. 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection

Let $\overline{\nabla}$ be linear connection and ∇ be Riemann connection of a 3-dimensional f-Kenmotsu manifold M. This linear connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{14}$$

where η is 1-form and $X, Y \in \chi(M)$, denotes the semi-symmetric non-metric connection (Ingalahalli and Bagewadi, 2012).

Now using (14), we have

$$\overline{\nabla}_X \xi = f(X - \eta(X)\xi) + X \tag{15}$$

In a 3-dimensional f-Kenmotsu manifold M let $\bar{R}$ be curvature tensor with respect to semi-symmetric non-metric connection $\overline{\nabla}$, then

$$\bar{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z. \quad (16)$$

Now using (14), (4), (5) in (16) we have

$$\bar{R}(X,Y)Z = R(X,Y)Z - f(g(Y,Z)X - g(X,Z)Y)$$
$$+(f+1)\eta(Z)[\eta(Y)X - \eta(X)Y]. \quad (17)$$

Replacing $Z=\xi$ in (17) and using (4) and (5), we have

$$\bar{R}(X,Y)\xi = -(f^2 + f' - 1)[\eta(Y)X - \eta(X)Y]. \quad (18)$$

Taking inner product of equation (17) with W and using equation (5), we have

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - f[g(Y,Z)g(X,W) - g(X,Z)g(Y,W]$$
$$+(1+f)[\eta(Y)g(X,W) - \eta(X)g(Y,W)]\eta(Z) \quad (19)$$

Let $\{e_1, \phi e_1, \xi\}$ be a local orthonormal basis of vector fields on 3-dimensional manifold M. Then using (8), we have

$$\bar{S}(X,Y) = (\tfrac{r}{2} + f^2 + f' - 2f)g(X,Y) - (\tfrac{r}{2} + 3f^2 + 3f' - 2f - 2)\eta(X)\eta(Y). \quad (20)$$

Replacing $Z = \xi$ in equation (19) and using (4) and (5)

$$\bar{S}(Y,\xi) = -2(f^2 + f' - 1)\eta(Y). (21)$$

and replacing $X = \xi$ in equation (18) and using (5)

$$\bar{R}(\xi,Y)\xi = (f^2 + f' - 1)(\eta(Y)\xi - Y). \quad (22)$$

Again replacing $X = \xi$ in equation (17) and using (5) we have

$$\bar{R}(\xi,Y)Z = -2(f^2 + f' - 1)[g(Y,Z) - \eta(Z)Y]. \quad (23)$$

4. 3-dimensional f-Kenmotsu manifolds with semi-symmetric non-metric connection satisfying $\bar{R}.\bar{S} = 0$

Considering a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection $\overline{\nabla}$ satisfying the condition

$$\bar{R}(X,Y).\bar{S}(U,V) = 0, \quad (24)$$

then we have

$$\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) = 0, \quad (25)$$

for any vector fields $X,Y,U,V \in \chi(M)$. Now put $X = \xi$ in (25), we have

$$\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U,\bar{R}(\xi,Y)V) = 0. \quad (26)$$

From (21) and (23) in (26), we have

$$-(f^2+f'-1)[g(Y,U)\bar{S}(\xi,V)-\eta(U)\bar{S}(Y,V))+g(Y,V)\bar{S}(U,\xi)+\eta(V)\bar{S}(U,Y)]=0. \quad (27)$$

Now putting $U=\xi$ in (27) and by using (4) and (5), we have

$$\bar{S}(Y,V)=-2(f^2+f'-1)g(Y,V). \quad (28)$$

Thus we have a following theorem:

Theorem 1. For a 3-dimensional f-Kenmotsu manifold M with the semi-symmetric non-metric connection satisfying the condition $\bar{R}.\bar{S}=0$, the manifold M is an Einstein manifold and the Ricci tensor $\tilde{S}$ is given by

$$\bar{S}(Y,V)=-2(f^2+f'-1)g(Y,V).$$

5. 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying $\bar{S}.\bar{R}=0$

Considering a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection $\bar{\nabla}$ satisfying the condition

$$(\bar{S}(X,Y).\bar{R})(U,V)Z=0, \quad (29)$$

for any vector fields $X,Y,U,V\in\chi(M)$.

Then we have

$$0=(X\wedge_{\bar{S}}Y)\bar{R}(U,V)Z+\bar{R}((X\wedge_{\bar{S}}Y)U,V)Z+\bar{R}(U,(X\wedge_{\bar{S}}Y)V)Z$$
$$+\bar{R}(U,V)(X\wedge_{\bar{S}}Y)Z, \quad (30)$$

where the endomorphism $X\wedge_{\bar{S}}Y$ is defined as

$$(X\wedge_{\bar{S}}Y)Z=\bar{S}(Y,Z)X-\tilde{S}(X,Z)Y. \quad (31)$$

By taking $Y=\xi$ in (31), we get

$$(X\wedge_{\bar{S}}\xi)\bar{R}(U,V)Z+\bar{R}((X\wedge_{\bar{S}}\xi)U,V)Z+\bar{R}(U,(X\wedge_{\bar{S}}\xi)V)Z+\bar{R}(U,V)(X\wedge_{\bar{S}}\xi)Z=0. \quad (32)$$

Using (20), (21) in (31) and (32), we have

$$0=-2(f^2+f'-1)[\eta(\bar{R}(U,V)Z)X+\eta(U)\bar{R}(X,V)Z+\eta(V)\bar{R}(U,X)Z$$
$$+\eta(Z)\bar{R}(U,V)X]-\bar{S}(X,\bar{R}(U,V)Z)\xi-\bar{R}(\xi,V)Z\bar{S}(X,U)$$
$$-\bar{R}(U,\xi)Z\bar{S}(X,V)-\bar{R}(U,V)\bar{S}(X,Z)\xi. \quad (33)$$

Taking inner product with ξ in equation (33), we have

$$0=-2(f^2+f-1')[\eta(\bar{R}(U,V)Z)\eta(X)+\eta(U)\eta(\bar{R}(X,V)Z)$$
$$+\eta(V)\eta(\bar{R}(U,X)Z)+\eta(Z)\eta(\bar{R}(U,V)X)]-\bar{S}(X,\bar{R}(U,V)Z)$$
$$-\bar{S}(X,U)\eta(\bar{R}(\xi,V)Z)-\bar{S}(X,V)\eta(\bar{R}(U,\xi)Z)$$

$$-\bar{S}(X,Z)\eta(\bar{R}(U,V)\xi). \tag{34}$$

By taking $U = Z = \xi$ in (34) and using , (21), (22), (23), we have

$$\bar{S}(X,V) = -2(f^2 + f' - 1)g(V,X) + 4(f^2 + f' - 1)\eta(V)\eta(X). \tag{35}$$

Thus we have a following theorem:

Theorem 2 Let M be a 3-dimensional regular, non-cosympectic f-Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfying the condition $\bar{S}.\bar{R} = 0$, Then M is an η-Einstein manifold given by the equation

$$\bar{S}(X,Y) = -2(f^2 + f' - 1)g(X,Y) + 4(f^2 + f' - 1)\eta(X)\eta(Y).$$

6. η-Ricci soliton on 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection

Suppose a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection admits an η-Ricci soliton (g,ξ,λ,μ), then equation (2) holds and we have

$$£_\xi g + 2\bar{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{36}$$

where $£_\xi$ is the Lie derivative operator along the vector field ξ, $\bar{S}$ is the Ricci curvature tensor field with semi-symmetric non-metric connection of the metric g and λ and μ are real constants. The (g,ξ,λ,μ), satisfying the equation (36) is called η-Ricci soliton on M (Hamilton,1988).

Now writing $£_\xi g$ in terms of the semi-symmetric non-metric connection $\overline{\nabla}$, we get

$$2\bar{S}(X,Y) = -g(\overline{\nabla}_X \xi, Y) - g(X, \overline{\nabla}_Y \xi) - 2(\lambda - 1)g(X,Y) - 2(\mu + 1)\eta(X)\eta(Y), \tag{37}$$

for any $X, Y \in \chi(M)$

If $\mu = 0$, the (g,ξ,λ) is a Ricci soliton (Perelman, 2003)), which is called expanding, steady or shrinking as λ is positive, zero or negative respectively.

On a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection using equation (15), we get

$$£_\xi g = 2fg(X,Y) - 2(f - 1)\eta(X)\eta(Y),$$

so the equation (37) becomes

$$\bar{S}(X,Y) = -(f + \lambda)g(X,Y) + (f - 1 - \mu)\eta(X)\eta(Y). \tag{38}$$

Particularly, for $Y = \xi$, we have

$$\bar{S}(X,\xi) = -(\lambda + \mu + 1)\eta(X). \tag{39}$$

In the case given above, $\bar{Q}$, the Ricci operator, given by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$ has the expression

$$\bar{Q}X = -(f + \lambda)X + (f - 1 - \mu)\eta(X)\xi. \tag{40}$$

From above discussion we have following proposition

Proposition 6.1 *The existence of an η-Ricci soliton in a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection results that ξ, the characteristic vector field is an eigen vector of Ricci operator $\bar{Q}$ corresponding to the eigen value $-(\lambda + \mu + 1)$.*

Theorem 3. If (M, ϕ, ξ, η, g) is a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection admitting η-Ricci soliton, then

1. $\bar{Q} \circ \phi = \phi \circ \bar{Q}$

2. $\bar{Q}$ and $\bar{S}$ are parallel along ξ.

***Proof*.** From equation (40) by direct computation we have the proof of first statement and the second statement is verified by replacement of $\bar{Q}$ and $\bar{S}$ from the equations (39) and (40) in

$$(\bar{\nabla}_{\xi} \bar{Q})X = \bar{\nabla}_{\xi} \bar{Q}X - \bar{Q}(\bar{\nabla}_{\xi} Y) \tag{41}$$

and

$$(\bar{\nabla}_{\xi} \bar{S})(X,Y) = \xi(\bar{S}(X,Y)) - \bar{S}(\bar{\nabla}_{\xi} X, Y) - \bar{S}(X, \bar{\nabla}_{\xi} Y). \tag{42}$$

A particular case arises when the manifold is ϕ-Ricci symmetric , which means that $\phi^2 \circ \bar{\nabla} \bar{Q} = 0$, that is stated in next theorem.

Theorem 4. Let (M, ϕ, ξ, η, g) be a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection. If M is ϕ-Ricci symmetric and equation (36) defines an η-Ricci soliton on M, then $\mu = f - 1$ and (M, g) is Einstein manifold.

Proof. Replacing $\bar{Q}$ from (40) in (41) we get

$$(\bar{\nabla}_X \bar{Q})Y = (f - \mu - 1)[X\eta(Y)\xi + \eta(Y)\, \bar{\nabla}_X \xi - \eta(\bar{\nabla}_X Y\xi)], \tag{43}$$

and applying ϕ^2, we have

$$(f - \mu - 1)\eta(Y)[-X + \eta(X)\xi] = 0. \tag{44}$$

for any $X, Y \in \chi(M)$. It yields $\mu = f - 1$ and $\bar{S}(X,Y) = -(f + \lambda)g(X,Y)$.

7. η-Ricci solitons on a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection satisfyng $\bar{R}(\xi, X).\bar{S} = 0$

We consi der a 3-dimaensional f-Kenmotsu manifold with semi-symmetric non-metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{S}(\bar{R}(\xi,X)Y,Z) + \bar{S}(Y, \bar{R}(\xi,X)Z) = 0, \tag{45}$$

for any $X, Y, Z \in \chi(M)$. By using equation (38) and using symmetries of $\bar{R}$, we have

$$(\mu - f + 1)(f^2 + f' - 1)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{46}$$

for any $X, Y \in \chi(M)$.

Now for $Z = \xi$, we have

$$(\mu - f + 1)(f^2 + f' - 1)g(\phi X, \phi Y) = 0. \tag{47}$$

This leads the following theorem.

Theorem 5. If a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection $\bar{\nabla}$, is an η-Ricci soliton on M satisfying $\bar{R}(\xi, X).\bar{S} = 0$, then the manifold is an η-Einstein manifold.

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