ISSN:0976-4933
Journal of Progressive Science
A Peer-reviewed Research Journal
Vol.13, No.01 & 02, pp 28-37 (2022)

On nearly pseudo$-W_8$ $-$recurrent and Ricci recurrent generalized Sasakian space forms

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Abstract

The object of the present paper is to study pseudo W_8-flat generalized Sasakian-space-forms. We studied some properties of nearly ϕ $-$recurrent and extended nearly $\phi - \widetilde{W}_8$ $-$recurrent generalized Sasakian space forms. Also, we find some interesting results in nearly ϕ $-$Ricci recurrent and nearly $\phi - \widetilde{W}_8$ $-$Ricci recurrent generalized Sasakian-space-forms.

Key words and phrases- Generalized Sasakian space forms, nearly Ricci recurrent, pseudo W_8-flat, nearly $\phi -$recurrent, extended nearly $\phi - \widetilde{W}_8 -$recurrent, nearly $\phi -$Ricci recurrent and nearly $\phi - \widetilde{W}_8$ $-$Ricci recurrent.

1. Introduction

The curvature tensor of Riemannian in differential geometry plays an important role. As well as the sectional curvatures of a manifold determine the curvature tensor R completely. A Riemannian manifold with constant sectional curvature c is known as a real space form whose curvature tensor is given by

$$R(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y],$$

$\forall\, X, Y, Z \in$ TM.

A Sasakian manifold with constant ϕ $-$sectional curvature becomes a Sasakain space form and it has a specific form of its curvature tensor. In order to, Alegre, Blair and Carriazo in 2004, introduced the notion of generalized Sasakian space form.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is known as a generalized Sasakain space form whose curvature tensor R is given by

$$R(X,Y)Z = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

where f_1, f_2, f_3 are differential functions on M and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\,,$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

$\forall\, X, Y, Z \in$ TM. In 2004, the author gives several examples of generalized Sasakian space forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian space form becomes Sasakian space form. The geometry of generalized Sasakian space form have been developed by several authors as Alegre and Carriazo (2008), Sular and Özgür (2011), Nagaraja, Somashekhara and Shashidhar (2012), Sarkar and Akbar (2014), Prakasha and Chavan (2015), Hui and Chakraborty (2016) and many others.

Recurrence spaces have been of great importance and were studied by a large number of authors such as Ruse (1946), Walker (1950), Patterson (1952), Singh and Khan (1999), (2000) and Baishya and Chowdhury (2017) etc.

Recently Prasad and Yadav (2021) introduced a new type of non-flat recurrent Riemannian manifold whose curvature tensor R satisfies the condition:

$$DR = [A + B]R + B \otimes G, \tag{1.1}$$

where D denotes the operator of covariant differentiation with respect to metric tensor g and two non-zero 1-form defined as

$$A(X) = g(X, \rho_1) \text{ and } B(X) = g(X, \rho_2). \tag{1.2}$$

The tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.3}$$

$\forall\, X, Y, Z \in$ TM.

Such a manifold called as a nearly recurrent manifold and 1-form A and B shall be called its associated 1-forms and n-dimensional recurrent manifold of this kind were denoted by them as $(NR)_n$.

If in particular B = 0 in (1.4), then the space is reduced to a recurrent space according to Ruse (1946) and Walker (1950) which was denoted by K_n. Moreover, in particular, if A = B = 0 then (1.4) becomes $DR = 0$. That is, a Riemannian manifold is symmetric according to Kobayashi and Nomizu (1963) and Desai and Amur (1975). The name nearly recurrent Riemannian manifold was chosen because if B = 0 in (1.4) then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name "Nearly recurrent Riemannian manifold" for the manifold defined by (1.4) and the use of the symbol $(NR)_n$ for it.

Further, Prasad and Yadav (2021) introduced a new type of non-flat Ricci recurrent Riemannian manifold whose Ricci tensor S satisfies the condition:

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z), \tag{1.4}$$

$\forall\, X, Y, Z \in$ TM where D, A and B defined as above.

Such a manifold called as a nearly Ricci recurrent manifold and 1-forms A and B be its associated 1-form. Nearly Ricci recurrent manifold of this kind were denoted by him as a $N\{R(R_n)\}$. The name nearly Ricci recurrent Riemannian manifold was chosen because if $B = 0$ in (1.3) then the manifold reduces to a Ricci recurrent manifold which is very close to Ricci recurrent space. This justified the name "Nearly Ricci recurrent manifold" for the manifold defined by (1.3) and the use of the symbol $N\{R(R_n)\}$ for it.

Let $(M^n, g), n > 3$ be a connected Riemannian manifold of C^∞ and D be its Riemannian connection. The pseudo W_8 curvature tensor $\widetilde{W}_8$ Prasad Yadav and Pandey (2018) of (M^{2n+1}, g) are defined as

$$\widetilde{W}_8(X, Y)Z = aR(X, Y)Z + b[S(X, Y)Z - S(Y, Z)X] - \frac{r}{2n+1}\left(\frac{a}{2n} - b\right)[g(X, Y)Z - g(Y, Z)X], \tag{1.5}$$

$\forall\, X, Y, Z \in$ TM where r is the scalar curvature tensor and if $= 1$, $b = \frac{1}{2n}$, then the pseudo W_8 curvature tensor $\widetilde{W}_8$ reduces to W_8 curvature tensor Pokhariyal (1982).

In (1969), Tanno introduced the notion of ϕ-recurrent Sasakian manifold and then Takahashi (1977) investigated the notion of a locally $\phi-$symmetric Sasakian manifold and studied its various properties. This was further deduced by many authors such as Venkatesha, Sumangala and Bagewadi (2012), Kishor and Singh, Peghan and Tayeba (2013) and many others. Further in (2017), Prasad and Yadav studied the notion of semi-generalized $\phi-$recurrent LP-Sasakain manifold. Recently Hui, Gowda and Chavan (2017) studied generalized $\phi-$recurrent generalized Sasakian space forms. He proved a generalized $\phi-$recurrent generalized Sasakian space form is generalized Ricci recurrent if and only if $f_1 - f_3$ is constant. Also, he proved some geometric properties of generalized Sasakian space form which are dependent on the nature of differentiable functions f_1, f_2 and f_3.

The motivation of the above ideas, the object of the present paper is the study of nearly $\phi-$ recurrent, extended nearly $\phi - \widetilde{W}_8 -$ recurrent, nearly $\phi-$ Ricci recurrent and nearly $\phi - \widetilde{W}_8 -$Ricci recurrent generalized Sasakian space form. The paper is organized after the introduction and preliminaries in section 3, $\xi - \widetilde{W}_8$ flat generalized Sasakian space form studied. Section 4 is devoted to the study of nearly $\phi-$recurrent. In section 5, we show that an extended nearly $\phi - \widetilde{W}_8 -$ recurrent generalized Sasakian space form $M(f_1, f_2, f_3)$ is Einstein manifold, provided $f_1 - f_3 \neq 0$. We investigate in nearly $\phi-$Ricci recurrent generalized Sasakain space-form $M(f_1, f_2, f_3)$, 1-forms A and B are in opposite directions. Finally, in the last section, we show that nearly $\phi - \widetilde{W}_8 -$Ricci recurrent generalized Sasakian space form is an Einstein manifold, provided $a - 2nb \neq 0$ and $f_1 - f_3 \neq 0$.

2. Preliminaries

A (2n+1) dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if it admits a tensor ϕ of type (1,1), ξ is a vector fields of type (0,1) and 1-form η is a tensor of the type (1,0) satisfying (Blair, 1976, 2000):

$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \phi\xi = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(\phi X, X) = 0, \tag{2.4}$$

for all $X, Y \in TM$.

Again for a (2n+1) dimensional generalized Sasakian space form, the following relation holds (Alegre, Blair and Carriazo, 2004):

$$\begin{aligned} R(X,Y)Z &= f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad +2g(X,\phi Y)\phi Z] + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\ &\quad g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{2.5}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.6}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - [3f_2 + (2n-1)f_3]\eta(X)\eta(Y), \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - [3f_2 + (2n-1)f_3]\eta(X)\xi, \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y), \tag{2.9}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.12}$$

$$S(\xi,\xi) = 2n(f_1 - f_3), \tag{2.13}$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3] \tag{2.14}$$

and

$$D_X\xi = -(f_1 - f_3)\phi X,\ (D_X\eta)(Y) = -(f_1 - f_3)g(\phi X, Y), \tag{2.15}$$

for all $X, Y \in TM$.

In view of (1.5), we have

$$'\widetilde{W}_8(X,Y,Z,U) = a\,'R(X,Y,Z,U) + b[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)]$$
$$-\frac{r}{2n+1}\left(\frac{a}{2n} - b\right)[g(X,Y)g(Z,U) - g(Y,Z)g(X,U)], \tag{2.16}$$

$\forall\ X, Y, Z \in$ TM, where

$$'R(X,Y,Z,U) = g(R(X,Y)Z,U) \text{ and } '\widetilde{W}_8(X,Y,Z,U) = g(\widetilde{W}_8(X,Y)Z,U).$$

Let $X = U = e_i, i = 1,2,3,\ldots,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then from (2.16), we have

$$\widetilde{W}_8(Y,Z) = (a - 2nb)\left[S(Y,Z) + \frac{r}{2n+1}g(Y,Z)\right], \tag{2.17}$$

where

$$\widetilde{W}_8(Y,Z) = \sum_i^{2n+1}{}'\widetilde{W}_8(e_i,Y,Z,e_i).$$

In view of (2.17), we get

$$Q_{\widetilde{W}}Y = (a - 2nb)\left[QY + \frac{r}{2n+1}Y\right], \tag{2.18}$$

where $Q_{\widetilde{W}}Y$ and Q are called pseudo W_8 Ricci and Ricci operators respectively.

3. $\xi - \widetilde{W}_8$ flat generalized Sasakian space form

A generalized Sasakian space form is $\xi - \widetilde{W}_8$ flat if

$$\widetilde{W}_8(X,Y)\xi = 0. \tag{3.1}$$

Taking ξ for Z in (1.5) and using (2.2), (2.10) and (2.12), we get

$$\widetilde{W}_8(X,Y)\xi = a(f_1 - f_3)[\eta(Y)X - \eta(X)Y] + [S(X,Y)\xi - 2n(f_1 - f_3)\eta(Y)X]$$
$$-\frac{r}{2n+1}\left(\frac{a}{2n} - b\right)[g(X,Y)\xi - \eta(Y)X]. \tag{3.2}$$

From (3.1) and (3.2), we have

$$a(f_1 - f_3)[\eta(Y)X - \eta(X)Y] + [S(X,Y)\xi - 2n(f_1 - f_3)\eta(Y)X]$$
$$-\frac{r}{2n+1}\left(\frac{a}{2n} - b\right)[g(X,Y)\xi - \eta(Y)X] = 0. \tag{3.3}$$

Putting ξ for Y in (3.3) and using (2.2) and (2.12), we get

$$r = -2n(2n+1)(f_1 - f_3), \text{ provided } a - 2nb \neq 0. \tag{3.4}$$

Hence we have the following theorem:

Theorem (3.1): A (2n+1) dimensional generalized Sasakain space-form $M(f_1, f_2, f_3)$ is $\xi - \widetilde{W}_8$ flat if the scalar curvature tensor $r = -2n(2n+1)(f_1 - f_3)$, Provided $a - 2nb \neq 0$.

4. Nearly ϕ −recurrent generalized Sasakian space form

Definition (4.1). A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called nearly ϕ −Ricci recurrent generalized Sasakian space form if its Ricci tensor S satisfies the condition:

$$\phi^2\big((D_UR)(X,Y)Z\big) = [A(U) + B(U)]R(X,Y)Z + B(U)G(X,Y)Z. \tag{4.1}$$

Using (2.1) in (4.1) we get

$$-(D_UR)(X,Y)Z+\eta\big((D_UR)(X,Y)Z\big)\xi=[A(U)+B(U)]R(X,Y)Z$$
$$+B(U)G(X,Y)Z. \qquad (4.2)$$

Contracting (4.2) with respect to X, we get

$$-(D_US)(Y,Z)+\eta\big((D_UR)(\xi,Y)Z\big)=[A(U)+B(U)]S(Y,Z)$$
$$+2nB(U)g(Y,Z). \qquad (4.3)$$

In view of (2.11), we get

$$(D_UR)(\xi,Y)Z=U(f_1-f_3)[g(Y,Z)\xi-\eta(Z)Y]-$$
$$(f_1-f_3)[g(Y,Z)\phi U-g(\phi U,Z)Y-R(\phi U,Y)Z],$$

which gives

$$\eta\big((D_UR)(\xi,Y)Z\big)=U(f_1-f_3)[g(Y,Z)-\eta(Y)\eta(Z)]+$$
$$(f_1-f_3)[g(\phi U,Z)\eta(Y)+\eta(R(\phi U,Y)Z)]. \qquad (4.4)$$

Using (2.6) in (4.4), we have

$$\eta\big((D_UR)(\xi,Y)Z\big)=U(f_1-f_3)[g(Y,Z)-\eta(Y)\eta(Z)]. \qquad (4.5)$$

Using (4.5) in (4.3), we obtain

$$(D_US)(Y,Z)=-[A(U)+B(U)]S(Y,Z)+[U(f_1-f_3)-2nB(U)]g(Y,Z)$$
$$-U(f_1-f_3)\eta(Y)\eta(Z). \qquad (4.6)$$

which can be written as

$$DS=-[A+B]\otimes S+C\otimes g+D\otimes\eta\otimes\eta, \qquad (4.7)$$

where $C=U(f_1-f_3)-2nB(U)$ and $D(U)=-U(f_1-f_3)\,\eta(Y)\eta(Z)$.

Hence, we have the following theorem:

Theorem (4.1): A nearly ϕ −recurrent generalized Sasakian space form is nearly Ricci recurrent if and only if f_1-f_3 is constant.

If $f_1-f_3=1$ then the generalized Sasakian space form reduces to Sasakian space form. Hence due to Alegre, Blair and Carriazo (2004) and Theorem (4.1) can be restated as follows:

Corollary (4.1): A nearly ϕ −recurrent generalized Sasakian space form is nearly Ricci recurrent.

Putting $Z=\xi$ in (4.7) and using (2.1), (2.2) and (2.12), we get

$$(D_US)(Y,\xi)=-2n(f_1-f_3)[A(U)+B(U)]\eta(Y)+2nB(U)\eta(Y). \qquad (4.8)$$

We have

$$(D_US)(Y,\xi)=US(Y,\xi)-S(D_UY,\xi)-S(Y,D_U\xi). \qquad (4.9)$$

Using (2.12) and (2.15), we get

$$(D_US)(Y,\xi)=2nU(f_1-f_3)\eta(Y)+(f_1-f_3)[S(Y,\phi U)-$$
$$2nU(f_1-f_3)g(Y,\phi U). \qquad (4.10)$$

From (4.8) and (4.10), we get

$$2nU(f_1-f_3)\eta(Y)+(f_1-f_3)[S(Y,\phi U)-2n(f_1-f_3)g(Y,\phi U)$$
$$=-2n(f_1-f_3)[A(U)+B(U)]\eta(Y)+2nB(U)\eta(Y). \qquad (4.11)$$

Y is replaced by ϕY in (4.11) and using (2.10), we get

$$S(\phi Y,\phi U)=2n(f_1-f_3)g(\phi Y,\phi U),\ \text{provided}\ f_1-f_3\neq 0. \qquad (4.12)$$

From (2.3), (2.9) and (4.12), we get

$$S(Y,U)=2n(f_1-f_3)g(Y,U).$$

Hence, we have the following theorem:

Theorem (4.3): A nearly ϕ-recurrent generalized Sasakian space form is an Einstein manifold provided $f_1 - f_3 \neq 0$.

Now, putting ξ for Z in (4.2), we get

$$-(D_U R)(X,Y)\xi + \eta\big((D_U R)(X,Y)\xi\big)\xi = [A(U) + B(U)]R(X,Y)\xi + B(U)G(X,Y)\xi. \quad (4.13)$$

We have

$$(D_U R)(X,Y)\xi = D_U R(X,Y)\xi - R(D_U X, Y)\xi - R(X, D_U Y)\xi - R(X,Y)D_U\xi. \quad (4.14)$$

From (2.10), (2.15) and (4.14), we have

$$(D_U R)(X,Y)\xi = U(f_1 - f_3)[\eta(Y)X - \eta(X)Y] - (f_1 - f_3)^2. [g(X,\phi U)Y - g(Y,\phi U)X] + (f_1 - f_3)R(X,Y)\phi U, \quad (4.15)$$

which gives

$$\eta\big((D_U R)(X,Y)\xi\big) = (f_1 - f_3)^2[g(X,\phi U)\eta(Y) - g(Y,\phi U)\eta(X)] + (f_1 - f_3)\eta(R(X,Y)\phi U). \quad (4.16)$$

Using (2.6) in (4.16), we get

$$\eta\big((D_U R)(X,Y)\xi\big) = 0. \quad (4.17)$$

With the help of (4.15) and (4.17), the equation (4.13) reduces to

$$(f_1 - f_3)R(X,Y)\phi U = (f_1 - f_3)^2[g(Y,\phi U)X - g(X,\phi U)Y] + [(f_1 - f_3)\{A(U) + B(U)\} + B(U) + U(f_1 - f_3)]. [\eta(Y)X - \eta(X)Y]. \quad (4.18)$$

If X, Y and U are orthogonal to ξ then (4.18) becomes

$$R(X,Y)\phi U = (f_1 - f_3)[g(Y,\phi U)X - g(X,\phi U)Y], \text{ provided } f_1 - f_3 \neq 0. \quad (4.19)$$

$Now\ U$ is replaced by ϕU in (4.19) and using (2.1), we get

$$R(X,Y)U = (f_1 - f_3)[g(Y,U)X - g(X,U)Y],$$

$\forall\, X, Y, U \in$ TM.

Hence, we have the following theorem:

Theorem (4.4): A locally nearly ϕ-recurrent generalized Sasakian space form is a constant curvature provided $f_1 - f_3 \neq 0$.

5. Extended nearly $\boldsymbol{\phi - \widetilde{W}_8}$ −recurrent generalized Sasakian space form

Definition (5.1). A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called extended nearly $\phi - \widetilde{W}_8 -$ recurrent generalized Sasakian space form if pseudo $W_8 -$ curvature tensor $\widetilde{W}_8$ satisfies the condition:

$$\phi^2\left(\big(D_U \widetilde{W}_8\big)(X,Y)Z\right) = [A(U) + B(U)]\phi^2\big(\widetilde{W}_8(X,Y)Z\big) + B(U)\phi^2(G(X,Y)Z), \quad (5.1)$$

$\forall\, X, Y, U \in$ TM.

Using (2.1) in (5.1) we get

$$-\big(D_U \widetilde{W}_8\big)(X,Y)Z + \eta\left(\big(D_U \widetilde{W}_8\big)(X,Y)Z\right)\xi = -[A(U) + B(U)]\widetilde{W}_8(X,Y)Z$$

$$+ [A(U) + B(U)]\eta(\widetilde{W}_8(X,Y)Z)\xi + B(U)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi]. \tag{5.2}$$

In view of (1.5) and (5.2), we get

$$-\Big[a(D_U R)(X,Y)Z + \{(D_U S)(X,Y)Z - (D_U S)(Y,Z)X\} - \frac{Ur}{2n+1}.$$
$$\left(\frac{a}{2n} - b\right)\{g(X,Y)Z - g(Y,Z)X\}\Big] + [a\eta((D_U R)(X,Y)Z) + b\{(D_U S)(X,Y)\eta(Z)$$
$$-(D_U S)(Y,Z)\eta(X)\} - \frac{Ur}{2n+1}\left(\frac{a}{2n} - b\right)\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\}]\xi =$$
$$-[A(U) + B(U)]\Big[aR(X,Y)Z + b\{S(X,Y)Z - S(Y,Z)X\} - \frac{r}{2n+1}.$$
$$\left(\frac{a}{2n} - b\right)\{g(X,Y)Z - g(Y,Z)X\}\Big] + [A(U) + B(U)][a\eta(R(X,Y)Z) +$$
$$b\{S(X,Y)\eta(Z) - S(Y,Z)\eta(X)\} - \frac{r}{2n+1}\left(\frac{a}{2n} - b\right)\{g(X,Y)\eta(Z) -$$
$$g(Y,Z)\eta(X)\}]\xi + B(U)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi]. \tag{5.3}$$

Contracting (5.3) with respect to X, we get

$$-(a - 2nb)\left[(D_U S)(Y,Z) - \frac{Ur}{2n+1}g(Y,Z)\right] + [a\eta((D_U R)(\xi,Y)Z) +$$
$$b\{(D_U S)(\xi,Y)\eta(Z) - (D_U S)(Y,Z)\} - \frac{Ur}{2n+1}\left(\frac{a}{2n} - b\right)\{\eta(Y)\eta(Z) - g(Y,Z)]$$
$$= -(a - 2nb)[A(U) + B(U)]\left[S(Y,Z) - \frac{r}{2n+1}g(Y,Z)\right] + [A(U) + B(U)].$$
$$[a\eta(R(\xi,Y)Z) + b\{S(\xi,Y)\eta(Z) - S(Y,Z)\} - \frac{r}{2n+1}\left(\frac{a}{2n} - b\right)\{\eta(Y)\eta(Z) -$$
$$g(Y,Z)\}] + B(U)[\eta(Y)\eta(Z) - (2n-1)g(Y,Z)]. \tag{5.4}$$

Using (2.11), (2.12), (4.6) and (4.10) in (5.4), we get

$$-(a - 2nb)\left[(D_U S)(Y,Z) - \frac{Ur}{2n+1}g(Y,Z)\right] + [a.U(f_1 - f_3)\{g(Y,Z)$$
$$-\eta(Y)\eta(Z)\}] + b[2nU(f_1 - f_3)\eta(Y)\eta(Z) - 2n(f_1 - f_3)^2 g(\phi U,Y)\eta(Z)$$
$$+2n(f_1 - f_3)g(\phi U,Y)\eta(Z) - (D_U S)(Y,Z)] - \frac{Ur}{2n+1}\left(\frac{a}{2n} - b\right)\{\eta(Y)\eta(Z)$$
$$-g(Y,Z)] = -(a - 2nb)[A(U) + B(U)]\left[S(Y,Z) - \frac{r}{2n+1}g(Y,Z)\right] +$$
$$[A(U) + B(U)][a.(f_1 - f_3)\{g(Y,Z) - \eta(Y)\eta(Z)\} +$$
$$b\{2n(f_1 - f_3)\eta(Y)\eta(Z) - S(Y,Z)\} - \frac{r}{2n+1}\left(\frac{a}{2n} - b\right)\{\eta(Y)\eta(Z) -$$
$$g(Y,Z)\}] + B(U)[\eta(Y)\eta(Z) - (2n-1)g(Y,Z)]. \tag{5.5}$$

Putting ξ for Z in (5.5), we have

$$-(a - 2nb)[2nU(f_1 - f_3)\eta(Y) - 2n(f_1 - f_3)^2 g(\phi U,Y)$$
$$+2n(f_1 - f_3)S(\phi U,Y) - \frac{Ur}{2n+1}\eta(Y)] =$$
$$-(a - 2nb)[A(U) + B(U)]\left[2n(f_1 - f_3) - \frac{r}{2n+1}\right]\eta(Y)$$
$$-2(n-1)B(U)\eta(Y). \tag{5.6}$$

Y is replaced by ϕY in (5.6), we get

$$S(\phi U,\phi Y) = 2n(f_1 - f_3)g(\phi U,\phi Y), \text{ provided } f_1 - f_3 \neq 0. \tag{5.7}$$

Using (2.3) and (2.9) in (5.7), we obtain

$$S(U,Y) = 2n(f_1 - f_3)g(U,Y). \tag{5.8}$$

Hence, we have the following theorem:

Theorem (5.1): An extended nearly $\phi - \widetilde{W}_8 -$ recurrent generalized Sasakian space form $M(f_1, f_2, f_3)$ is Einstein manifold provided $f_1 - f_3 \neq 0$.

6. Nearly ϕ −Ricci recurrent generalized Sasakian space form

Definition (6.1). A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called nearly ϕ −Ricci recurrent generalized Sasakian space form if its Ricci tensor S satisfies the condition:

$$\phi^2\big((D_X Q)(Y)\big) = [A(X) + B(X)]QY + B(X)Y. \tag{6.1}$$

Using (2.1) in (6.1) we get `

$$-(D_X Q)(Y) + \eta\big((D_X Q)(Y)\big)\xi = [A(X) + B(X)]QY + B(X)Y,$$

which gives

$$-g(D_X QY, Z) + S(D_X Y, Z) + \eta\big((D_X Q)(Y)\big)\eta(Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z). \tag{6.2}$$

Putting ξ for Y in (6.2) and using (2.8) and (2.15), we have

$$(f_1 - f_3)S(\phi X, Z) - 2n(f_1 - f_3)^2 g(\phi X, Z) + 2n(f_1 - f_3)[A(X) + B(X)]\eta(Z) + B(X)\eta(Z) = 0. \tag{6.3}$$

Z is replaced by ϕZ, we get

$$S(\phi X, \phi Z) = 2n(f_1 - f_3)g(\phi X, \phi Z),$$

Provided $f_1 - f_3 \neq 0$. (6.4)

Using (2.3) and (2.9) in (6.4), we obtain

$$S(X, Z) = 2n(f_1 - f_3)g(X, Z). \tag{6.5}$$

Hence we have the following theorem:

Theorem (6.1): A nearly ϕ −Ricci recurrent generalized Sasakain space-form $M(f_1, f_2, f_3)$ is Einstein manifold provided $f_1 - f_3 \neq 0$.

Again putting $Z = \xi$ in (6.3) and using (2.1), (2.2) and (2.12), we get

$$2n(f_1 - f_3)A(X) + [2n(f_1 - f_3) + 1]B(X) = 0. \tag{6.6}$$

Hence we have the following theorem:

Theorem (6.2): In nearly ϕ −Ricci recurrent generalized Sasakain space-form $M(f_1, f_2, f_3)$, 1-forms A and B are in opposite directions.

7. Nearly $\phi - \widetilde{W}_8$ −Ricci recurrent generalized Sasakian space form

Definition (7.1). A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called nearly $\phi - \widetilde{W}_8$ −Ricci recurrent generalized Sasakian space form if pseudo W_8 curvature tensor $\widetilde{W}_8$ satisfies the condition:

$$\phi^2\big((D_X Q_{\widetilde{W}})(Y)\big) = [A(X) + B(X)]Q_{\widetilde{W}}Y + B(X)Y. \tag{7.1}$$

Using (2.1) in (7.1) we get

$$-(D_X Q_{\widetilde{W}})(Y) + \eta\big((D_X Q_{\widetilde{W}})(Y)\big)\xi = [A(X) + B(X)]Q_{\widetilde{W}}Y + B(X)Y,$$

which gives

$$-g(D_X Q_{\widetilde{W}}Y, Z) + \widetilde{W}_8(D_X Y, Z) + \eta\big((D_X Q_{\widetilde{W}})(Y)\big)\eta(Z) =$$

$$[A(X)+B(X)]\widetilde{W}_8(Y,Z)+B(X)g(Y,Z). \quad (7.2)$$

Putting ξ for Y in (7.2) and using (2.8), (2.12), (2.15), (2.17) and (2.18), we have

$$2n(f_1-f_3)^2(a-2nb)S(\phi X,Z)=$$
$$-(a-2nb)\left[2n(f_1-f_3)+(f_1-f_3+1)\frac{r}{2n+1}\right]g(\phi X,Z)+$$
$$-(a-2nb)[A(X)+B(X)]\left[2n(f_1-f_3)+\frac{r}{2n+1}\right]\eta(Z)+B(X)\eta(Z)=0. \quad (7.3)$$

Z is replaced by ϕZ in (7.3), we get

$$S(\phi X,\phi Z)=\lambda g(\phi X,\phi Z), \text{ provided } a-2nb\neq 0, \quad (7.4)$$

where $\lambda=-\frac{2n(f_1-f_3)+(f_1-f_3+1)\frac{r}{2n+1}}{2n(f_1-f_3)}$

Using (2.3) and (2.9) in (7.4), we obtain

$$S(X,Z)=\lambda g(X,Z).$$

Hence we have the following theorem:

Theorem (7.1): A nearly $\phi-\widetilde{W}_8$ −Ricci recurrent generalized Sasakain space-form $M(f_1,f_2,f_3)$ is Einstein manifold provided $a-2nb\neq 0$ and $f_1-f_3\neq 0$.

Again putting $Z=\xi$ in (7.3) and using (2.1), (2.2) and (2.12), we get

$$r=-(2n+1)\left[2n(f_1-f_3)+\frac{B(X)}{\{A(X)+B(X)\}(a-2nb)}\right]. \quad (7.7)$$

Hence we have the following theorem:

Theorem (7.2): The scalar curvature of nearly $\phi-\widetilde{W}_8$ −Ricci recurrent generalized Sasakain space-form $M(f_1,f_2,f_3)$ is given by (4.7), provided $a-2nb\neq 0$ and $A+B\neq 0$.

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Received on 22.05.2022 and accepted on 18.09.2022