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Riemannian manifold admitting a new type of semi-symmetric metric connection

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Abstract

We define a new type of semi-symmetric metric connection on a Riemannian manifold and established its existence. Further, we find some basic results of curvature tensor and Ricci tensor. It is proved that such connection on a Riemannian manifold is conformally invariant under certain conditions. We also studied some properties of concircular curvature tensor of the Riemannian manifolds with respect to the semi-symmetric metric connection $\overline{\mathrm{D}}$. To validate our findings, we construct a non-trivial example of 3-dimensional Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\mathrm{D}}$.

Keywords and Phrases: Riemannian manifold, semi-symmetric metric connection, curvature tensor, Ricci tensor, conformal curvature tensor, concircular curvature tensor.

1. Introduction

Let M^n be an $n-$dimensional Riemannian manifold and let D denote the Levi-Civita connection corresponding to the Riemannian metric g on M^n. A linear connection $\overline{\mathrm{D}}$ defined on M^n is said to be symmetric if its torsion tensor $\bar{T}$ on $\overline{\mathrm{D}}$ defined by

$$\bar{T}(X,Y) = \overline{\mathrm{D}}_{\mathrm{X}}Y - \overline{\mathrm{D}}_{\mathrm{Y}}X - [X,Y],$$

is zero for all X and Y on M^n; otherwise, it is non-symmetric. In 1924, Friedmann and Schouten considered a differentiable manifold and introduced the idea of a semi-symmetric linear connection on it. A linear connection on M^n is said to be semi-symmetric if

$$\bar{T}(X,Y) = [\eta_1(Y)X - \eta_1(X)Y] - [\eta_2(Y)X - \eta_2(X)Y] \neq 0, \tag{1.1}$$

holds for all vector fields X, Y on M^n, where η_1 and η_2 are two non-zero 1- forms associated with the vector fields U and V such that

$$\eta_1(X) = g(X,U) \text{ and } \eta_2(X) = g(X,V). \tag{1.2}$$

In 1932, Hayden gave the idea of a metric connection $\bar{D}$ on a Riemannian manifold and later named such connection as a Hayden connection. A linear connection $\bar{D}$ is said to be metric on M^n, if $\bar{D}g = 0$; otherwise, it is non-metric. A systematic study of the semi-symmetric metric connection $\bar{D}$ on Riemannian manifold was initiated by Yano (1970). Various properties of such connection have been studied by Imai (1972), Nakao (1976), Smaranda (1977), Amur and Pujar (1978), Barua and Ray (1985), Hit (1974), De and Biswas (1997), and many authors. In 1992, Agashe and Chafle introduced a new class of connection, called the semi-symmetric non-metric connection on a Riemannian manifold and obtained its various geometric properties. This was further developed by Agashe and Chafle (1994), Prasad (1994), De and Kamilya (1995), Tripathi and Kakkar (2001) and several geometers. Binh, De and Sengupta (2000), Verma and Prasad (2004) defined new types of semi-symmetric non-metric connections on Riemannian manifold in which they generalized the Yano's (1970) and Agashe and Chafle's (1994) connections and studied some properties of curvature tensor, Ricci tensor and Projective curvature tensor with respect to such connections. In 2008, Prasad, Verma and De introduced the most general form of the semi-symmetric metric and non-metric connections on a Riemannian manifold which includes the known semi-symmetric metric and non- metric connection. Recently, Prasad, Dubey and Yadav (2011) and Prasad, Kumar and Singh (2021) defined and studied a new type of semi-symmetric metric connection on Riemannian manifolds. Motivated by the above studies, in the present paper, we define a new type of semi-symmetric metric connection on a Riemannian manifold and then prove its existence.

We organize our present work as follows: After an introduction in Section 1, we define a new type of semi-symmetric metric connection on a Riemannian manifold and prove its existence in Section 2. In Section 3, we established the relation between curvature tensors of the Levi-Civita D and semi-symmetric metric connections $\overline{D}$ and prove some algebraic properties of the curvature tensors and Ricci tensor of connection $\overline{D}$. The necessary and sufficient conditions for conformally invariant curvature tensors are proved in Section 4. Section 5, deals with the concircular curvature tensor and its relation with Riemannian manifold. In the last Section 6, we construct a non-trivial example of 3-dimensional Riemannian manifold with a semi-symmetric metric connection.

2. Semi-symmetric metric connection $\overline{D}$

Let (M^n, g) be a Riemannian manifold of dimension n endowed with a Levi-Civita connection D corresponding to the Riemannian metric g. A linear connection $\overline{D}$ on (M^n, g) defined by

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X - g(X,Y)U - \eta_2(Y)X + g(X,Y)V, \tag{2.1}$$

for arbitrary vector fields X and Y on M^n is said to be a semi-symmetric connection if the torsion tensor $\overline{T}$ on M^n with respect to $\overline{D}$ satisfies equation (1.1) and (1.2). In view of equation (2.1), the metric g holds the relation

$$(\overline{D}_X g)(X,Y) = 0, \tag{2.2}$$

for all vector fields X, Y, Z on M^n and called semi-symmetric metric connection. Now, we prove the existence of such connection on an $n-$dimensional Riemannian manifold.

Let us suppose that (M^n, g) is a Riemannian manifold of dimension n and equipped with a linear connection. Then the Levi-Civita connection D are connected by the relation

$$\overline{D}_X Y = D_X Y + H(X,Y), \tag{2.3}$$

for arbitrary vector fields X and Y on M^n, where H is a tensor of type (1,2). By definition of the torsion $\overline{T}$ and equation (2.3), we conclude that

$$\overline{T}(X,Y) = H(X,Y) - H(Y,X), \tag{2.4}$$

which gives

$$g(\bar{T}(X,Y),Z) = g(H(X,Y),Z) - g(H(Y,X),Z). \tag{2.5}$$

From (1.1) and (2.5), we have

$$g(H(X,Y),Z) - g(H(Y,X),Z) =$$
$$[\eta_1(Y)g(X,Z) - \eta_1(X)g(Y,Z)] - [\eta_2(Y)g(X,Z) - \eta_2(X)g(Y,Z)]. \tag{2.6}$$

In the view of equation (2.1), we conclude that

$$g(H(X,Y),Z) + g(H(X,Z),Y) = (\bar{D}_X\, g)(Y,Z),$$
$$\Rightarrow \quad g(H(X,Y),Z) + g(H(X,Z),Y) = (\bar{D}_X\, g)(Y,Z) = 0,$$
$$H'(X,Y,Z) = (\bar{D}_X\, g)(Y,Z) = 0, \tag{2.7}$$

where

$$H^{'}(X,Y,Z) = g(H(X,Y),Z) + g(H(X,Z),Y).$$

Further, we have

$$g(\bar{T}(X,Y),Z) + g(\bar{T}(Z,X),Y) + g(\bar{T}(Z,Y),X)$$
$$= g(H(X,Y),Z) - H^{'}(X,Y,Z) + H^{'}(Z,X,Y) - H^{'}(Y,X,Z), \tag{2.8}$$

where equations (2.4), (2.5) and (2.7) are used. In consequences of equations (2.2) and (2.7), the equation **(**2.8**)** assumes the form

$$2g(H(X,Y),Z) = g(\bar{T}(X,Y),Z) + g(\bar{T}'(X,Y),Z) + g(\bar{T}'(Y,X),Z),$$
$$\Rightarrow \quad g(H(X,Y),Z) = \frac{1}{2}\{g(\bar{T}(X,Y),Z) + g(\bar{T}'(X,Y),Z) + g(\bar{T}'(Y,X),Z)\},$$
$$\Rightarrow \quad H(X,Y) = \tfrac{1}{2}\{\bar{T}(X,Y) + \bar{T}'(X,Y) + \bar{T}'(Y,X)\}, \tag{2.9}$$

where

$$g(\bar{T}'(X,Y),Z) = g(\bar{T}(Z,X),Y),$$
$$\Rightarrow \quad \bar{T}'(X,Y) = \eta_1(X)Y - g(X,Y)U - \eta_2(X)Y + g(X,Y)V, \tag{2.10}$$

and

$$g(\bar{T}'(Y,X),Z) = g(\bar{T}(Z,Y),X),$$
$$\Rightarrow \quad \bar{T}'(X,Y) = \eta_1(X)Y - g(X,Y)U - \eta_2(X)Y + g(X,Y)V, \tag{2.11}$$

for all vector fields X, Y and Z on M^n. By using equation (1.1), (2.10) and (2.11) in equation (2.9), we have

$$H(X,Y) = \eta_1(Y)X - g(X,Y)U - \eta_2(Y)X + g(X,Y)V. \tag{2.12}$$

Thus equations (2.3) and (2.12) give (2.1). This proves the existence of such connection.

Thus, we can state the following theorem:

Theorem 2.1: Let (M^n, g) be an $n-$dimensional Riemannian manifold endowed with the Levi-Civita connection D. Then there exist a unique linear connection on M^n called a semi-symmetric metric connection given by (2.1) and it satisfies equation (1.1) and (2.2).

Remark.1: From (2.1), we see that if $\eta_1 = 0$ or $\eta_2 = 0$, then connection remains semi-symmetric metric connection, but if we take $\eta_1 = \eta_2$ then it is trivial. So, it must said that $\eta_1 \neq \eta_2$.

Theorem 2.2: On an $n-$dimensional Riemannian manifold (M^n, g) endowed with a semi-symmetric metric connection $\bar{D}$, the torsion tensor $\bar{T}$ satisfies the following algebraic properties:

$$'\bar{T}(X,Y,Z) + {}'\bar{T}(Y,X,Z) = 0,$$

$$'\bar{T}(X,Y,Z) + {}'\bar{T}(Y,Z,X) + {}'\bar{T}(Z,X,Y) = 0.$$

Proof: We define $'\bar{T}(X,Y,Z) = g(\bar{T}(X,Y),Z)$ on (M^n, g). Therefore, equation (1.1) gives

$$'\bar{T}(X,Y,Z) = \eta_1(Y)g(X,Z) - \eta_1(X)g(Y,Z) - \eta_2(Y)g(X,Z) + \eta_2(X)g(Y,Z), \tag{2.13}$$

with the help of equation (2.13), we can easily prove the statement of theorem 2.2.

Theorem 2.3: If $(M^n, g)$is an $n-$dimensional Riemannian manifold equipped with a semi-symmetric metric connection $\bar{D}$, then $\bar{T}$ is cyclically parallel if and only if 1- forms η_1 and η_2 are closed.

Proof: Taking the covariant derivative of (1.1) with respect to the semi-symmetric metric connection $\bar{D}$, we find that

$$(\bar{D}_X\bar{T})(Y,Z) = [(\bar{D}_X\bar{T})\eta_1(Z)Y - (\bar{D}_X\bar{T})\eta_1(Y)Z] - [(\bar{D}_X\bar{T})\eta_2(Z)Y - (\bar{D}_X\bar{T})\eta_2(Y)Z]. \tag{2.14}$$

The cyclic sum of (2.14) for vector fields X, Y and Z, we get

$$\begin{aligned}(\bar{D}_X\bar{T})(Y,Z) + (\bar{D}_Y\bar{T})(Z,X) + (\bar{D}_Z\bar{T})(X,Y) \\ = [(\bar{D}_Y\eta_1)(X) - (\bar{D}_X\eta_1)(Y) - (\bar{D}_Y\eta_2)(X) + (\bar{D}_X\eta_2)(Y)]Z \\ -[(\bar{D}_Z\eta_1)(X) - (\bar{D}_X\eta_1)(Z) - (\bar{D}_Z\eta_2)(X) + (\bar{D}_X\eta_2)(Z)]Y \\ -[(\bar{D}_Y\eta_1)(Z) - (\bar{D}_Z\eta_1)(Y) - (\bar{D}_Y\eta_2)(Z) + (\bar{D}_Z\eta_2)(Y)]X.\end{aligned} \tag{2.15}$$

From equation (2.15), we can easily show that

$(\bar{D}_X\bar{T})(Y,Z) + (\bar{D}_Y\bar{T})(Z,X) + (\bar{D}_Z\bar{T})(X,Y) = 0$, if and only if 1-form η_1 and η_2 are closed.

Hence theorem 2.3 is proved.

Theorem 2.4: If an $n-$dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric connection $\bar{D}$, then the Lie derivatives along the vector fields U and V corresponding to $\bar{D}$ is equal to Lie derivative along the vector fields U and V with respect to D if and only if η_1 and η_2 are closed.

Proof: It is well known that

$$(L_U g)(X,Y) = g(D_X U, Y) + g(X, D_Y U) \text{ and } (L_V g)(X,Y) = g(D_X V, Y) + g(X, D_Y V), \tag{2.16}$$

holds for arbitrary vector fields X and Y on M^n, where L_U and L_V denote the Lie derivatives along the vector fields U and V corresponding to D respectively.

Analogous to the above definition of Lie derivative, we define

$$(\bar{L}_U g)(X,Y) = g(\bar{D}_X U, Y) + g(X, \bar{D}_Y U) \text{ and } (\bar{L}_V g)(X,Y) = g(\bar{D}_X V, Y) + g(X, \bar{D}_Y V), \tag{2.17}$$

holds for arbitrary vector fields X and Y on M^n, where $\bar{L}_U$ and $\bar{L}_V$ denote the Lie derivatives along the vector fields U and V corresponding to $\bar{D}$ respectively.

On addition of (2.16) and (2.17) and using equation (2.1), we find

$$(\bar{L}_U g)(X,Y) + (\bar{L}_V g)(X,Y) = (L_U g)(X,Y) + (L_V g)(X,Y) - 2[\{\eta_1(X)\eta_1(Y) - \eta_2(X)\eta_2(Y)\} - \{\eta_1(U) - \eta_2(U) + \eta_1(V) - \eta_2(V)\}g(X,Y)],$$

Hence, the statement of theorem 2.4 is proved.

If the vector fields U and V are killing on (M^n, g), then $(L_U g)(X,Y) = 0$ and $(L_V g)(X,Y) = 0$. Thus, we can state the following proposition:

Proposition 2.5: If an n −dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric connection and U and V are killing vector fields with respect $\bar{D}$ then Lie derivative with respo D is also killing if and only if η_1 and η_2 are closed

3. Curvature tensor, Ricci tensor and scalar curvature tensor with respect to the semi-symmetric metric connection $\bar{D}$

Let (M^n, g) be an n −dimensional Riemannian manifold admitting a semi-symmetric metric connection $\bar{D}$. The curvature tensor $\bar{R}$ corresponding to $\bar{D}$ is defined by

$$\bar{R}(X,Y)Z = \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X,Y]} Z, \tag{3.1}$$

for arbitrary vector fields X, Y and Z on (M^n, g) and the Riemannian curvature R of the Levi-civita connection D defined by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \tag{3.2}$$

for all vector fields X, Y and Z on (M^n, g). By using equation (2.1) and (2.2), equation (3.1) becomes

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)AX + g(X,Z)AY + \beta(Y,Z)X - \beta(X,Z)Y + g(Y,Z)BX - g(X,Z)BY - \gamma(Y,Z)X + \gamma(X,Z)Y - g(Y,Z)CX + g(X,Z)CY, \tag{3.3}$$

where α, β and γ are tensor field of type (0,2) and given by

$$\alpha(Y,Z) = (D_Y \eta_1)(Z) - \eta_1(Y)\eta_1(Z) + \frac{1}{2} g(Y,Z)\eta_1(U),$$

$$AX = D_X U - \eta_1(X)U + \frac{1}{2}\eta_1(U)X,$$

$$\beta(Y,Z) = (D_Y \eta_2)(Z) - \eta_2(Y)\eta_2(Z) + \frac{1}{2} g(Y,Z)\eta_2(V),$$

$$BX = D_X V - \eta_2(X)V + \frac{1}{2}\eta_2(V)X,$$

$$\gamma(Y,Z) = \eta_1(Y)\eta_2(Z) + \eta_1(Z)\eta_2(Y) - \frac{1}{2}g(Y,Z)\eta_1(V) - \frac{1}{2}g(Y,Z)\eta_2(U),$$

$$CX = \eta_1(X)V + \eta_2(X)U - \frac{1}{2}\eta_1(V)X - \frac{1}{2}\eta_2(U). \tag{3.4}$$

Remark (1): If in particular $\eta_2 = 0$, then our connection becomes

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X - g(X,Y)U,$$

which is a Yano's connection (1970). From (3.3), the expression for the curvature $\bar{R}$ of the manifold with respect to this connection can be written as follows:

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)AX + g(X,Z)AY,$$

where

$$\alpha(Y,Z) = (D_Y\eta_1)(Z) - \eta_1(Y)\eta_1(Z) + \frac{1}{2}g(Y,Z)\eta_1(U),$$

$$AX = D_X U - \eta_1(X)U + \frac{1}{2}\eta_1(U)X.$$

On contracting (3.3) with respect to X, we have

$$\overline{Ric}(Y,Z) = Ric(Y,Z) - (n-2)\{\alpha(Y,Z) - \beta(Y,Z) + \gamma(Y,Z)\} - g(Y,Z)(p-q+s), \tag{3.5}$$

where $p = trace\,\alpha$, $q = trace\,\beta$ and $s = trace\,\gamma$. (3.6)

Equation (3.5) can be put as

$$g(\overline{QY},Z) = g(QY,Z) - (n-2)\{g(AY,Z) - g(BY,Z) + g(CY,Z) - g(Y,Z)(p-q+s),$$

i.e., $\overline{QY} = QY - (n-2)(AY - BY + CY) - Y(p-q+s)$.

Contracting above equation with respect to Y, we get

$$\bar{r} = r - 2(n-1)(p-q+s). \tag{3.7}$$

where $\bar{r}$ and r are scalar curvature with respect to $\overline{D}$ and D respectively.

Theorem 3.1: Let (M^n, g) denote an $n-$ dimensional Riemannian manifold endowed with a semi-symmetric metric connection $\overline{D}$. Then the necessary and sufficient condition for the scalar curvatures $\bar{r}$ of $\overline{D}$ is equal to scalar curvature r of D if and only if $p - q + s = 0$.

Interchanging Y and Z in (3.5), we have

$$\overline{Ric}(Z,Y) = Ric(Z,Y) - (n-2)\{\alpha(Z,Y) - \beta(Z,Y) + \gamma(Z,Y)\} - g(Z,Y)(p-q+s). \tag{3.8}$$

Subtracting (3.8) from the equation (3.6) and then using the symmetric property of the Ricci tensor in it, we conclude that

$$\overline{Ric}(Y,Z) - \overline{Ric}(Z,Y) = -(n-2)\{d\eta_1(Y,Z) - d\eta_2(Y,Z)\}. \tag{3.9}$$

Hence in view of (3.9), we are in a position to state the following theorem.

Theorem 3.2: If an $n-$dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric connection $\overline{D}$, then the Ricci tensor $\overline{Ric}$ corresponding to the connection $\overline{D}$ is symmetric if and only if $d\eta_1(Y,Z) = d\eta_2(Y,Z)$.

Remark (1): If in particular $\eta_2 = 0$, then equation (3.5) becomes

$$\overline{Ric}(Y,Z) = Ric(Y,Z) - (n-2)\alpha(Y,Z) - g(Y,Z)p,$$

where $p = trace\ \alpha$.

Remark (2): If $\eta_2 = 0$, then equation (3.9) becomes,

$$\overline{Ric}(Y,Z) - \overline{Ric}(Z,Y) = -(n-2)d\eta_1(Y,Z).$$

Therefore, we can state that Ricci tensor is symmetric if and only if $d\eta_1 = 0$.

Theorem 3.3: Let (M^n, g) be a $n-$dimensional Riemannian manifold for equipped with a semi-symmetric metric connection $\overline{D}$ then the following relations hold for all the vector fields $X,\ Y,\ Z$ and W on M^n.

(i) $\bar{R}(X,Y)Z + \bar{R}(Y,X)Z = 0,$

(ii) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$if and only if $d\eta_1(X,Y) = d\eta_2(X,Y),$

(iii) $(\overline{D}_X\bar{R})(Y,Z)W + (\overline{D}_Y\bar{R})(Z,X)W + (\overline{D}_Z\bar{R})(X,Y)W = 0,$ if and only if $\eta_1(X) = \eta_2(X),$

(iv) $'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = 0,$

(v) $'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) = 0,$

(vi) $'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) = 0,$ if and only if η_1 and η_2 are closed.

Proof: Interchanging X and Y in equation (3.3) and then adding with (3.3), we obtain (i). On interchanging X, Y and Z in equation (3.3) in a cyclic order and then adding we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = [d\eta_1(X,Y) - d\eta_2(X,Y)]Z - [d\eta_1(Y,Z) - d\eta_2(Y)]X - [d\eta_1(Z,X) - d\eta_2(Z,X)]Y.$$

This expression shows that the Riemannian manifold (M^n, g) equipped with a semi-symmetric metric connection $\overline{D}$ satisfies Bianchi's first identity if and only if $d\eta_1(X,Y) = d\eta_2(X,Y)$, result (ii). Bianchi's second identity for a semi-symmetric metric connection $\overline{D}$ is given by the expression

$$(\overline{D}_X\bar{R})(Y,Z)W + (\overline{D}_Y\bar{R})(Z,X)W + (\overline{D}_Z\bar{R})(X,Y)W = -\bar{R}(\bar{T}(X,Y),Z)W - \bar{R}(\bar{T}(Y,Z),X)W - \bar{R}(\bar{T}(Z,X),Y)W,$$

for arbitrary vector fields X, Y, Z and Won M^n. Using (1.1) in above equation, we have

$$(\overline{D}_X\bar{R})(Y,Z)W + (\overline{D}_Y\bar{R})(Z,X)W + (\overline{D}_Z\bar{R})(X,Y)W = 2[\{\eta_1(X) - \eta_2(X)\}\bar{R}(Y,Z)W + \{\eta_1(Y) - \eta_2(Y)\}\bar{R}(Z,X)W + \{\eta_1(Z) - \eta_2(Z)\}\bar{R}(X,Y)W], \quad (3.10)$$

satisfies, result (iii) if and only if $\eta_1(X) = \eta_2(X)$.

If we define$'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$, then equation (3.3) becomes

$$'\bar{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - g(Y,Z)\alpha(X,Z) + g(X,Z)\alpha(X,W) + \beta(Y,Z)g(X,W) - \beta(X,Z)g(Y,W) + g(Y,Z)\beta(X,W) -$$

$$g(X,Z)\beta(Y,W) - \gamma(Y,Z)g(X,W) + \gamma(X,Z)g(Y,W) - g(Y,Z)\gamma(X,W) + g(X,Z)\gamma(Y,W), \qquad (3.11)$$

where

$$g(AX,Y) = \alpha(X,Y),\ g(BX,Y) = \beta(X,Y), \text{ and } g(CX,Y) = \gamma(X,Y), \qquad (3.12)$$

for all vector fields X, Y, Z and W on M^n. Expressions (iv) and (v) are obvious from equations (3.11) and (3.12) and the symmetric properties of the curvature tensor. Now, from equation (3.11), we can show that the curvature tensor corresponding to $\bar{D}$ is symmetric in pair of slots if and only if 1- forms η_1 and η_2 are closed. This proves (vi). Hence, the proof is complete.

4. Conformal Curvature tensor with respect to the semi-symmetric metric connection $\bar{D}$

The Conformal curvature tensor C with respect to D is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \qquad (4.1)$$

for arbitrary vector field X, Y, Z on M^n. The inner product of equation (4.1) with W is given by

$$'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{n-2}[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) + g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \qquad (4.2)$$

The Conformal curvature tensor with respect to $\bar{D}$ of equation (4.2) is given by

$$'\bar{C}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{1}{n-2}[\overline{Ric}(Y,Z)g(X,W) - \overline{Ric}(X,Z)g(Y,W) + g(Y,Z)\overline{Ric}(X,W) - g(X,Z)\overline{Ric}(Y,W)] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \qquad (4.3)$$

where

$$'C(X,Y,Z,W) = g(C(X,Y)Z,W) \text{ and } '\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W).$$

Using equations (3.5), (3.7), (3.11) and (4.2) in equation (4.3), we have

$$'\bar{C}(X,Y,Z,W) = {}'C(X,Y,Z,W). \qquad (4.4)$$

Thus, we can state the following theorem:

Theorem 4.1: If an $(n > 2)$ −dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric connection $\bar{D}$, then the conformal curvature tensor $'\bar{C}$ with respect to $\bar{D}$ is invariant to that of conformal curvature tensor with respect to D , that is ,the Weyl Conformal curvature tensor of the manifold M^n with respect to the Levi-Civita connection and the semi-symmetric metric connection are equal.

Let $\bar{R}(X,Y,Z) = 0$, then

$$\Rightarrow \overline{Ric}(Y,Z) = 0 \text{ and } \bar{r} = 0,$$

then from equation (4.3), we have

$$'\bar{C}(X,Y,Z,W) = 0,$$

and from equation (4.4), we have

$$'C(X,Y,Z,W) = 0.$$

Thus, we can state the following theorem:

Theorem 4.2: If in a Riemannian manifold, the curvature tensor of a semi-symmetric metric connection $\bar{D}$ vanishes then manifold is conformally flat.

If $\overline{Ric}(Y,Z) = 0$ and $\bar{r} = 0$, then from equation (4.3), we have

$$'\bar{C}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W).$$

Using above result in equation (4.4), we have

$$'\bar{R}(X,Y,Z,W) = {}'C(X,Y,Z,W).$$

Thus, we can state the following theorem:

Theorem 4.3: If a Riemannian manifold admits a semi-symmetric metric connection $\bar{D}$ whose Ricci tensor vanishes then the curvature tensor of the connection $\bar{D}$ is equal to the Conformal curvature tensor of the manifold.

Theorem 4.4: The Conformal curvature tensor $'\bar{C}$ with respect to semi-symmetric metric connection satisfies the following algebraic properties:

(i) $'\bar{C}(X,Y,Z,W) + {}'\bar{C}(Y,X,Z,W) = 0,$

(ii) $'\bar{C}(X,Y,Z,W) + {}'\bar{C}(X,Y,W,Z) = 0,$

(iii) $'\bar{C}(X,Y,Z,W) - {}'\bar{C}(Z,W,X,Y,) = 0,$

(iv) $'\bar{C}(X,Y,Z,W) + {}'\bar{C}(Y,Z,X,W) + {}'\bar{C}(Z,X,Y,W) = 0.$

5. Concircular curvature tensor with respect to the semi-symmetric metric connection $\bar{D}$

The Concircular curvature tensor L with respect to D is defined by

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{5.1}$$

for arbitrary vector field X,Y,Z on M^n. The inner product of equation (5.1) with W is given by

$$'L(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{r}{(n-1)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \tag{5.2}$$

The Concircular curvature tensor with respect to $\bar{D}$ of equation (5.2) is given by

$$'\bar{L}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{\bar{r}}{(n-1)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \tag{5.3}$$

where

$$'L(X,Y,Z,W) = g(L(X,Y)Z,W) \text{ and } '\bar{L}(X,Y,Z,W) = g(\bar{L}(X,Y)Z,W).$$

Using equations (3.7), (3.11) and (5.2) in equation (5.3), we have

$$'\bar{L}(X,Y,Z,W) - '\bar{R}(X,Y,Z,W) = \quad 'L(X,Y,Z,W) - 'R(X,Y,Z,W)$$
$$-2\left(\frac{p-q+s}{n-1}\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \qquad (5.4)$$

Thus, we can state the following theorem:

Theorem 5.1: The difference of the concircular curvature tensor and Riemannian curvature tensor with respect to $\bar{D}$ is equal to the difference of the concircular curvature tensor and Riemannian curvature tensor with respect to D if and only if $p - q + s = 0$.

Theorem 5.2: The Concircular curvature tensor $'\bar{L}$ with respect to semi-symmetric metric connection satisfies the following algebraic properties:

(i) $'\bar{L}(X,Y,Z,W) + '\bar{L}(Y,X,Z,W) = 0,$

(ii) $'\bar{L}(X,Y,Z,W) + '\bar{L}(X,Y,W,Z) = 0,$

(iii) $'\bar{L}(X,Y,Z,W) - '\bar{L}(Z,W,X,Y) = 0,$ if $p - q + s = 0,$

(iv) $'\bar{L}(X,Y,Z,W) + '\bar{L}(Y,Z,X,W) + '\bar{L}(Z,X,Y,W) = 0.$

6. Example

Let us consider a 3-dimensional manifold $M^3 = \{(x,y,z) \in R^3, z \neq 0\}$,where $\{x,y,z\}$ are standard co-ordinate in R^3.

We choose the vector fields

$$e_1 = \frac{1}{4}\frac{\partial}{\partial y}, e_2 = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y} \text{ and } e_3 = \frac{\partial}{\partial z}, \qquad (6.1)$$

which are linearly independent at each point of M^3 and therefore it forms a basis for the tangent space $T(M^3)$.

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{Bmatrix} 1, i = j \\ 0, i \neq j \end{Bmatrix}. \qquad (6.2)$$

Let D be the Levi-Civita connection with respect to metric g. Then from (6.2), we have

$$[e_1, e_2] = 0, [e_1, e_3] = 0, [e_2, e_3] = 4e_1. \qquad (6.3)$$

The Riemannian connection D of the metric g is given by

$$2g(D_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)$$
$$-g(X,[Y,Z]) - g(Y,[X,Z)] + g(Z,[X,Y]), \qquad (6.4)$$

which is known as Kaszul's formula.

Thus we obtain

$$D_{e_1}e_1 = 0, \quad D_{e_1}e_2 = -2e_3, \quad D_{e_1}e_3 = 2e_2,$$
$$D_{e_2}e_1 = -2e_3, \quad D_{e_2}e_2 = 0, \quad D_{e_2}e_3 = 2e_1,$$
$$D_{e_3}e_1 = 2e_2, \quad D_{e_3}e_2 = -2e_1, \quad D_{e_3}e_3 = 0, \qquad (6.5)$$

where D denotes the Levi-Civita connection corresponding to the metric g. The non-vanishing components of the Riemannian curvature tensor can be calculated by the using the equation (3.2)and (6.5), we have

$$R(e_1,e_2)e_1 = 0, \quad R(e_1,e_2)e_2 = 4e_1, \quad R(e_1,e_2)e_3 = 0,$$
$$R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = 8e_3, \quad R(e_2,e_3)e_3 = 0,$$
$$R(e_1,e_3)e_1 = 0, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = 0,$$
$$R(e_1,e_1)e_1 = R(e_1,e_1)e_2 = R(e_1,e_1)e_3 = 0,$$
$$R(e_2,e_2)e_1 = R(e_2,e_2)e_2 = R(e_2,e_2)e_3 = 0,$$
$$R(e_3,e_3)e_1 = R(e_3,e_3)e_2 = R(e_3,e_3)e_3 = 0. \tag{6.6}$$

The Ricci tensor can be calculated by the following expression

$$Ric(X,Y) = \sum_{i=1}^{3} g(R(e_i,X)Y,e_i). \tag{6.7}$$

From (6.6) and (6.7), we get

$$Ric(e_1,e_1) = 0, \quad Ric(e_1,e_2) = 0, \quad Ric(e_1,e_3) = 0,$$
$$Ric(e_2,e_1) = 0, \quad Ric(e_2,e_2) = -4, \quad Ric(e_2,e_3) = 0,$$
$$Ric(e_3,e_1) = 0, \quad Ric(e_3,e_2) = 0, \quad Ric(e_3,e_3) = 0. \tag{6.8}$$

It is obvious that the scalar curvature is $r = -4$.

Taking $U = e_1$ and $V = e_3$,

In consequences of the above discussion and equation (2.1), we have

$$\bar{D}_{e_1}e_1 = e_3, \quad \bar{D}_{e_1}e_2 = -2e_{3,}, \quad \bar{D}_{e_1}e_3 = 2e_2 - e_1,$$
$$\bar{D}_{e_2}e_1 = -2e_3 + e_{2,}, \quad \bar{D}_{e_2}e_2 = -e_1 - e_3, \quad \bar{D}_{e_2}e_3 = 2e_1 + e_2,$$
$$\bar{D}_{e_3}e_1 = 2e_2 + e_{3,}, \quad \bar{D}_{e_3}e_2 = -e_1, \quad \bar{D}_{e_3}e_3 = -e_1. \tag{6.9}$$

In view of (6.9), we can easily prove that equation (1.1) holds for all vector fields $e_i (i = 1,2,3), e.g.$,

$$\bar{T}(e_1,e_1) = \bar{T}(e_2,e_2) = \bar{T}(e_3,e_3) = 0.$$
$$\bar{T}(e_1,e_2) = -e_2, \bar{T}(e_1,e_3) = -e_3 - e_1, \bar{T}(e_2,e_3) = -e_2. \tag{6.10}$$

This shows that the linear connection $\bar{D}$ defined as (2.1) is a semi-symmetric connection on (M^3, g), also

$$\left(\bar{D}_{e_i}g\right)\left(e_j,e_k\right) = 0, \text{ for every } i,j,k = 1,2,3.$$

Let X, Y and Z be vector fields on M^3. Then it can be expressed as a linear combination of e_1, e_2 and e_3, that is,

$$X = X^1e_1 + X^2e_2 + X^3e_3,$$
$$Y = Y^1e_1 + Y^2e_2 + Y^3e_3,$$
$$Z = Z^1e_1 + Z^2e_2 + Z^3e_3,$$

where X^i, Y^i and Z^i, $i = 1,2,3$ are real constants, we have

$$'\bar{T}(X,Y,Z) = g(\bar{T}(X,Y),Z),$$

$$'\bar{T}(X,Y,Z) = (X^3Y^1 - X^1Y^3)Z^1 + (X^3Y^2 - X^2Y^3 + X^2Y^1 - X^1Y^2)Z^2 - (X^3Y^1 - X^1Y^3)Z^3, \quad (6.11)$$

$$'\bar{T}(Y,X,Z) = (Y^3X^1 - Y^1X^3)Z^1 + (Y^3X^2 - Y^2X^3 + Y^2X^1 - Y^1X^2)Z^2 + (Y^3X^1 - Y^1X^3)Z^3, \quad (6.12)$$

$$'\bar{T}(Y,Z,X) = (Y^3Z^1 - Y^1Z^3)X^1 + (Y^3Z^2 - Y^2Z^3 + Y^2Z^1 - Y^1Z^2)X^2 + (Y^3Z^1 - Y^1Z^3)X^3, \quad (6.13)$$

$$'\bar{T}(Z,X,Y) = (Z^3X^1 - Z^1X^3)Y^1 + (Z^3X^2 - Z^2X^3 + Z^2X^1 - Z^1X^2)Y^2 + (Z^3X^1 - Z^1X^3)Y^3. \quad (6.14)$$

Hence, from the equation (6.11), (6.12), (6.13) and (6.14), we have

$$'\bar{T}(X,Y,Z) + {}'\bar{T}(Y,X,Z) = 0,$$

and

$$'\bar{T}(X,Y,Z) + {}'\bar{T}(Y,Z,X) + {}'\bar{T}(Z,X,Y) = 0.$$

Therefore, Theorem 2.2 is verified.

Also, the curvature tensor $\bar{R}$ with respect to $\bar{D}$ is given by

$$\bar{R}(X,Y) = \bar{D}_X\bar{D}_Y Z - \bar{D}_Y\bar{D}_X Z - \bar{D}_{[X,Y]}Z,$$

holds for all vector fields X, Y, Z and also,

$$\bar{R}(e_1,e_2)e_1 = -5e_2 - 2e_3 \neq 0,$$

$$\bar{R}(e_1,e_2)e_2 = -e_3 - 5e_1 \neq 0,$$

$$\bar{R}(e_i,e_j)e_k \neq 0, \text{ for all } i,j,k = 1,2,3. \quad (6.15)$$

Hence, the Riemannian equipped with a semi-symmetric metric connection $\bar{D}$ is not flat.

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