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Generalized concircular curvature tensor and Spacetimes of general relativity

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Abstract

In This paper, we introduced a new tensor named generalized concircular curvature tensor on a Riemannian manifold which generalized the concircular curvature tensor. First, we deduced some basic geometric properties of generalized concircular curvature tensor. Further, a symmetric investigation of generalized concircular curvature tensor has been made on the four-dimensional spacetime of general relativity. The spacetime fulfilling Einstein field equations with the vanishing of generalized concircular curvature tensor is being considered and the existence of killing and Conformal killing vectors on such spacetime have been established. At last, we extend the similar case for the investigation of cosmological models with dust and perfect fluid spacetime.

Keywords and phrases: Spacetime, quadratic killing tensor, quadratic Conformal Killing tensor, concircular curvature tensor, quasi-concircular curvature tensor, Q-curvature tensor, generalized concircular curvature tensor.

1. Introduction

The study spacetime of general and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed, by basing its study on Lorentzian manifold the general theorem of relativity opens the way to the study of global questions about it Beem and Ehrlicle (1981), Clarke (1986), Geroch (1971), Hawking and Ellis (1973), Joshi (1993), many others. In general relativity, the matter content of spacetime is described by the energy-momentum tens $\mathcal{T}$ which is to be determined from physical considerations dealing with the distribution of matter and energy.

As we known that the symmetric spaces play an important role in differential geometry, the geometrical symmetries of the spacetimes are expressible through vanishing of the Lie derivative of certain tensors with respect to a vector. These symmetries are also known as collineations were first introduced by Katzi

Livine and Devis (1969). Further studies of collineations by Z. Ahsan (1995 and 1996), M. Ali etal (2019), Pundeer (2020) among many others. The spacetime symmetries are used in the study of exact solutions of Einstein's field equations in general relativity. A killing vector field is one of the most important type or symmetries and defined to be a smooth vector field that preserves the metric tensor. We have all the tools needed to workout Einstein's field equation, which explains have to metric responds to energy and momentum. The Einstein's equation Neil (1983), imply that the energy momentum tensor is of vanishing divergence. This requirement satisfied if the energy momentum tensor is covariant constant Chaki and Roy (1996). In 1996, Chaki and Roy proved that a general relativistic spacetime with covariant constant energy-momentum tensor is Ricci symmetric, that is, $D.Ric = 0$, where Ric is the Ricci tensor of the spacetime. Several authors studied spacetmes in different way such as spacetimes with semi-symmetric energy momentum tensor De and Velimirovic (2015), m-projectively semi-symmetric spacetimes by Zengin (2012). M-projectively semi-symmetric Lorentzian $\alpha -$Sasakian manifold by Prakasha etal, pseudo Z-symmetric spacetimes by Montica and Suh (2104), quasi-conformally, pseudo cohormonically symmetric spacetimes by Zengin, Tasci (2018), pseudo projectivelly spacetimes by Mallick, Suh and De (2016), pseudo-quasi-conformal curvature tensor and spacetimes of general relativity by Suh, Chavan, and Pundeer (2021), spacetime admitting generalized conhormonic curvature tensor (2022) and many others. In the general theorem of relativity, the matter content of the spacetime is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion. In a perfect fluid space-time, the energy momentum tensor $\mathcal{T}$ of the type (0,2) is of the form Neill (1989)

$$\mathcal{T}(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y)\ , \tag{1.1}$$

where p is the isotropic pressure, σ is the energy density and A is a non-zero one form such that $g(X,\mu) = A(X), \forall\ X \in TM,$ where μ is the velocity field such that $g(\mu,\mu) = -1$ and A is mathematically equivalent to a unit space-like vector field. The field is called perfect because of the absence of heat condition tensors and stress terms corresponding to viscos's perfect-fluid space-times in a language of differential geometry are called quasi-Einstein spaces De and Shenaury (2019). If the isotropic pressure p vanishes in perfect fluid then it is said to be a dust fluid. In a dust fluid space-time, the energy momentum tensor $\mathcal{T}$ of the type (0,2) is of the form Neill (1983)

$$\mathcal{T}(X,Y) = \sigma A(X)A(Y). \tag{1.2}$$

The Einstein's field equation with cosmology to constant is given by Neill (1981)

$$Ric(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k\mathcal{T}(X,Y)\ , \tag{1.3}$$

where Ric and r denotes the Ricci tensor and scalar curvature respectively, λ is the cosmological constant and $k \neq 0$.

The Einstein's field equation without cosmological constant is given by Neill (1993)

$$Ric(X,Y) - \frac{r}{2}g(X,Y) = k\mathcal{T}(X,Y). \tag{1.4}$$

The Einstein's field equations (1.3) and (1.4) imply that the energy-momentum tensor is conservative. This requirement is satisfied if the energy-momentum tensor is covariant Chaki and Roy (1996).

The geometrical symmetries of spacetime are expressed through the equation

$$\mathcal{L}_\xi A' - 2\Omega' A' = 0, \tag{1.4a}$$

where A' represents a geometrical/physical quantity, $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ and Ω' is a scalar. One of the most simple and widely used example is the metric inheritance symmetric for which $A' = g$ in (1.4a); and for this case, ξ is killing vector field if Ω' is zero.

Definition 1: Let (M^n, g) be a spacetime manifold with Levi-Civita connection D. A quadratic killing tensor is a generalization of a Killing vector and is defined as a second order symmetric tensor $\mathcal{T}$ satisfies the condition:

$$(D_X\mathcal{T})(Y,Z) + (D_Y\mathcal{T})(Z,X) + (D_Z\mathcal{T})(X,Y) = 0. \tag{1.4b}$$

Definition 2: A quadratic Conformal Killing tensor is analogous generalization of a Conformal Killing vector and is defined as a second order symmetric tensor $\mathcal{T}$ satisfying the condition:

$$(D_X\mathcal{T})(Y,Z) + (D_Y\mathcal{T})(Z,X) + (D_Z\mathcal{T})(X,Y) = a(X)(Y,Z) + a(Y)(Z,X) + a(Z)(X,Y), \tag{1.4c}$$

for a smooth 1-form on (M^n, g).

In general relativity the matter content of the space-time is described by the energy-momentum tensor. The matter content is assumed to be field having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion.

A transformation of a Riemannian manifold which transforms every geodesic circle of manifold into a geodesic circle is called a concircular transformation and the geometry which deals with such transformation is called the concircular geometry Yano (1940). A concircular transformation is always a conformal transformation Yano (1940). Hear concircular geodesic circle means a curve in manifold whose first curvature is constant and whose second curvature is identically zero. Yano and Kon (1984) defined the concircular curvature tensors as a (1,3) type tensor $V(X,Y)Z$ that stays invariant under concircular transformation for an n-dimensional Riemannian manifold

$$V(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.5}$$

Equation (1.5) can be written as of type

$$'V(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{1.6}$$

where r is the scalar curvature tensor and

$$'V(X,Y,Z,W) = g(V(X,Y)Z,W) \text{ and } {}'R(X,Y,Z,W) = g(R(X,Y)Z,W). \tag{1.7}$$

In (2012), Prasad and Maurya, defined quasi-concircular curvature tensor by the expression:

$$\tilde{V}(X,Y)Z = a.R(X,Y)Z + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.8}$$

where a and b are constants such that $a, b \neq 0$. Quasi-concircualr curvature tensor has been extended to LP-Sasakian manifold, P-Sasakian manifold and Lorentzian β −Kenmotsu manifold by Narain et al. (2009), Kumar et al. (2009) and Ahmad et al. (2019) respectively.

Subsequently in 2013 Montica and Suh introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ denoted by Q and defined by

$$Q(X,Y)Z = R(X,Y)Z - \frac{\psi}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.9}$$

where ψ is an arbitrary scalar function. Such a tensor Q is known as Q −curvature tensor. The notion of Q −tensor is also suitable to reinterpret some differential structures on a Riemannian manifold.

Motivated by the above studies in the present paper we define generalized concircular curvature tensor L of type (1,3) as follows:

$$L(X,Y)Z = a_1.R(X,Y)Z + a_2.g(Y,Z)X + a_3.g(X,Z)Y, \tag{1.10}$$

where a_1, a_2 and a_3 are constants such that $a_1, a_2, a_3 \neq 0$.

In particular, if

(i) $a_1 = 1\,,\quad a_2 = -a_3 = -\frac{r}{n(n-1)}$, then from (1.10)

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] = V(X,Y)Z = \text{concircular}$$

(ii) $a_1 = a\,,\quad a_2 = -a_3 = -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)$, then from (1.10)

$$L(X,Y)Z = a.R(X,Y)Z + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] = \tilde{V}(X,Y)Z$$ =Quasi-concicular

(iii) $a_1 = 1\,,\quad a_2 = -a_3 = -\frac{\psi}{n(n-1)}$, then from (1.10)

$$L(X,Y)Z = R(X,Y)Z - \frac{\psi}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] = Q.$$

Thus, we see that $V, \tilde{V}$ and Q −tensors are particular case of the tensor L. For this reason curvature tensor L is called generalized concircular curvature tensor. If $a_1 = 1\,,\ a_2 = a_3 = 0\,,$ then generalized concircular curvature tensor and curvature tensor are equivalent. We can express (1.10) as follows:

$$'L(X,Y,Z,W) = a_1.'R(X,Y,Z,W) + a_2.g(Y,Z)g(X,W) + a_3.g(X,Z)g(Y,W), \tag{1.11}$$

where $'L(X,Y,Z)W = g(V(X,Y)Z,W)$ and $'R(X,Y,Z)W = g(R(X,Y)Z,W)$.

A symmetric (0,2) Ricci tensor Ric as a Riemannian manifold (M^n, g) is said to be a Codazzi type tensor if it satisfies the following equation:

$$(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z), \tag{1.12}$$

for arbitrary vector fields X, Y and Z.

The geometrical and topological consequences of the existence of a non-trival Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen (1983).

The present paper is organized as follows:

2. Some properties of generalized concircular curvature tensor L

Let Ric and r denote the Ricci tensor of the type (0,2) and the scalar curvature respectively and Q' denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric, that is,

$$g(Q'X,Y) = Ric(X,Y). \tag{2.1}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$. In a Riemannian manifold the Ricci tensor Ric is defined by

$Ric(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)e_i,Y)$ and $r = \sum_{i=1}^{n} Ric(e_i, e_i)$ is the scalar curvature tensor.

From (1.11), we have the following properties:

$$\left.\begin{array}{l} (i)\ \ 'L(X,Y,Z,W)+'L(Y,X,Z,W) = (a_2+a_3)[g(Y,Z)g(X,W)+g(X,Z)g(Y,W)] \\ (ii)\ \ 'L(X,Y,Z,W)+'L(X,Y,W,Z) = (a_1+a_3)[g(Y,Z)g(X,W)+g(X,Z)g(Y,W)], \\ (iii)\ \ 'L(X,Y,Z,W)-'L(Z,W,X,Y) = 0, \\ (iv)\ \ 'L(X,Y,Z,W)+'L(Y,Z,X,W)+'L(Z,X,Y,W) = (a_2+a_3)\,[g(Y,Z)g(X,W)+ \\ \qquad g(X,Z)g(Y,W)+g(X,Y)g(Z,W)]\,. \end{array}\right\} \tag{2.2}$$

Also, from (1.11), we have

$$\left.\begin{array}{l} (i)\ \ (Ric_1^1L) = 'L(X,Y,e_i,e_i) = (a_2+a_3)g(Y,Z) \\ (ii)\ \ (Ric_2^1L) = 'L(e_i,Y,Z,e_i) = a_1Ric(Y,Z)+(na_2+a_3)g(Y,Z), \\ (iii)\ \ (Ric_3^1L) = 'L(X,e_i,Z,e_i) = -a_1Ric(X,Z)+(na_3+a_2)g(X,Z), \\ (iv)\ \ 'L(X,e_i,e_i,W) = a_1Ric(X,Z)+(a_2+a_3)(na_2+a_3)g(X,W) \end{array}\right\} \tag{2.3}$$

where (Ric_1^1L), (Ric_2^1L) and (Ric_3^1L) on the contraction with respect to X, Y and Z respectively.

For generalized concircularly flat manifold, we get

$$Ric(X,Y) = \lambda' g(X,Y). \tag{2.4}$$

where $\lambda' = -\left(\frac{na_2+a_3}{a_1}\right), a_1 \neq 0.$

Again equation (2.5) gives

$$r = n\lambda'. \tag{2.5}$$

In view of (2.2), (2.3), (2.4) and (2.5), we have

Theorem 2.1: A generalized concircular curvature on (M^n, g) is

(A) skew symmetric in first two slots if $a_2 + a_3 = 0$,

(B) skew symmetric in last two slots if $a_2 + a_3 = 0$,

(C) symmetric in pair of slots,

(D) satisfies Bianchi's first identity if $a_2 + a_3 = 0$.

Theorem 2.2: A generalized concircularly flat manifold are

(A) Einstein manifold,

(B) Ricci symmetric provided $a_1 \neq 0$,

(C) scalar curvature tensor is not zero,

(D) not a constant curvature.

Further, taking covariant derivative of equation (1.10), we get

$$(D_U L)(X,Y)Z = a_1 (D_U R)(X,Y)Z. \tag{2.6}$$

Equation (2.6) can be put as

$$(D_U{}'L)(X,Y,Z,W) = a_1 (D_U{}'R)(X,Y,Z,W). \tag{2.7}$$

Contraction (2.6), we get

$$(divL)(X,Y)Z = a_1 (divR)(X,Y)Z. \tag{2.8}$$

From (2.6), (2.7) and (2.8), we have the following theorem:

Theorem 2.3 (i) A generalized concircular curvature on (M^n, g) satisfies Bianchi's second identity.

(ii) For generalized concircular curvature tensor $(divL)(X,Y)Z = 0$ and $(divV)(X,Y)Z = 0$ are equivalent if scalar curvature is constant provided $a_1 \neq 0$.

(iii) A generalized concircular curvature tensor is divergence free if and only if it is of Codazzi type tensor.

3. Spacetime with vanishing generalized concircular curvature tensor (M^4, g)

Here, we denote generalized concircularly spacetime by the notation $(GCFS)_4$.

Theorem 3.1: For $(GCFS)_4$, the energy-momentum tensor satisfying the Einstein field equations with a cosmological constant is the form

$$\frac{1}{k}(\lambda - \lambda')g(X,Y) = \mathcal{T}(X,Y).$$

Proof: From (1.3), we get

$$Ric(X,Y) + \left(\lambda - \frac{r}{2}\right) g(X,Y) = k\mathcal{T}(X,Y). \tag{3.1}$$

Here, we assume that our manifold $(GCFS)_4$, then from (2.4), (2.5) and (3.1). we get

$$\mathcal{T}(X,Y) = \frac{1}{k}(\lambda - \lambda')g(X,Y). \tag{3.2}$$

Thus, from (3.2), the proof is completed.

Theorem 3.2: For $(GCFS)_4$ satisfying the Einstein field equations with a cosmological constant, ∃ a killing vector field ξ if and only if the Lie derivative of the energy-momentum tensor along that vector field is zero.

Proof: Taking Lie derivative of both sides of (3.2), we get

$$k\left(\mathcal{L}_\xi \mathcal{T}\right)(X,Y) = (\lambda - \lambda')\left(\mathcal{L}_\xi g\right)(X,Y). \tag{3.3}$$

If ξ is killing vector field then, we have

$$\left(\mathcal{L}_\xi g\right)(X,Y) = 0. \tag{3.4}$$

In view of (3.3) and (3.4), we get

$$\left(\mathcal{L}_\xi \mathcal{T}\right)(X,Y) = 0,\ k \neq 0. \tag{3.5}$$

Conversely, if (3.5) holds, then from (3.3), we get

$$\left(\mathcal{L}_\xi g\right)(X,Y) = 0,\ (\lambda - \lambda') \neq 0.$$

Thus, we can state that ξ is killing vector field. The proof is completed.

Theorem 3.3: $(GCFS)_4$ obeying the Einstein field equations with a cosmological term, ∃ a Conformal killing vector field ξ if and only if the energy-momentum tensor has the symmetry in heritance property.

Proof: If ξ satisfies the condition

$$\left(\mathcal{L}_\xi g\right)(X,Y) = 2\Omega' g(X,Y). \tag{3.6}$$

Then, it is called a Conformal killing vector field. Now, we assume that ξ is a Conformal killing vector field of $(GCFS)_4$. Thus from (3.3) and (3.6), we get

$$\left(\mathcal{L}_\xi \mathcal{T}\right)(X,Y) = 2\Omega' g(X,Y). \tag{3.7}$$

In this case, it can be said that the energy-momentum tensor has the symmetry in heritance property. Conversely, if (3.7) holds, then it follows that the equation (3.6) holds, i.e. the vector field ξ is a Conformal killing vector field.

Theorem 3.4: The energy-momentum tensor of $(GCFS)_4$ satisfying the Einstein field equation with a cosmologically term is locally symmetric.

Proof: Let us consider that our space is $(GCFS)_4$. If we take the covariant derivative of (3.2), then we find that

$$(D_X \mathcal{T})(Y,Z) = 0. \tag{3.8}$$

Thus, we see that the energy-momentum tensor satisfies the equation (1.4b) is locally symmetric. In this case, the proof is completed.

Theorem 3.5: $(GCFS)_4$ cannot admit a quadratic Conformal killing energy-momentum tensor satisfying the Einstein field equation with a cosmologically constant.

Proof: If put (3.8) in (1.4c), we get

$$a(X)(Y,Z) + a(Y)(Z,X) + a(Z)(X,Y) = 0. \tag{3.9}$$

Contraction of (3.9) with respect to X and Y leads to

$$a(X) = 0. \tag{3.10}$$

Thus, we can say that the energy-momentum tensor of this manifold cannot be a quadratic Conformal killing tensor. This result completes the proof.

4. Perfect fluid spacetime with vanishing generalized concircular curvature tensor

In this section we consider $(GCFS)_4$ obeying Einstein's field equation with cosmological constant.

Theorem 4.1: In $(GCFS)_4$ satisfying the Einstein field with a cosmological term, the matter contains of the spacetime satisfy the vacuum-like equation of state.

Proof: For $(GCFS)_4$ with the help of (1.1) and (3.2) the Einstein field equations are found as

$$(\lambda - \lambda' - kp)g(X,Y) = k(p+\rho)A(X)A(Y). \tag{4.1}$$

Contraction of (4.1) over X and Y, we get

$$\lambda = \lambda' - \frac{k}{4}(\rho - 3p). \tag{4.2}$$

Again if we put μ for X and Y in (4.1), we have

$$\lambda = \lambda' - k\sigma. \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\sigma + p = 0. \tag{4.4}$$

The proof is completed.

Theorem 4.2: The $(GCFS)_4$ admitting a dust for a perfect fluid is field with radiation.

Proof: If we assume a dust in a perfect fluid, we have

$$\sigma = 3p. \tag{4.5}$$

From (4.4) and (4.5), we get

$$p = 0. \tag{4.6}$$

Thus, this leads the proof.

5. Dust fluid spacetime with $(GCFS)_4$

In a dust or pressure less fluid spacetime, the energy momentum tensor is in the form

$$\mathcal{T}(X,Y) = \sigma A(X)A(Y), \tag{5.1}$$

where σ is the energy density of dust-like matter and A os non-zero 1-form such that $g(X,\mu) = a(X)$ for all X, μ being the velocity vector field of the flow, that is, $(\mu,\ \mu) = -1$.

Theorem 5.1: A relativistic $(GCFS)_4$ satisfying the Einstein field equation with a cosmological terms is vacuum.

Proof: In consequences of (3.2) and (5.1), we get

$$(\lambda - \lambda')g(X,Y) = k\sigma A(X)A(Y). \tag{5.2}$$

Contraction of (5.2) over X and Y gives

$$\lambda = \lambda' - \frac{k\sigma}{4}. \tag{5.3}$$

Again, if we put μ for X and Y in (5.2), we have

$$\lambda = \lambda' - k\sigma. \tag{5.4}$$

Combining (4.3) and (4.4), we get

$$\sigma = 0. \tag{5.5}$$

Hence, from (5.1) and (5.5), we get

$$\mathcal{T}(X,Y) = 0.$$

This means that the spacetime is devoid of the matter. This result completes the proof.

6. Cosmological models with vanishing generalized concircular curvature tensor

In this section, we consider a perfect fluid spacetime with $(GCFS)_4$ satisfying Einstein's field equation without cosmological constant.

Theorem 6.1: A spacetime satisfying Einstein's field equation without cosmological constant and having vanishing generalized concircular curvature tensor represented a dust cosmological model, if the energy density does not vanishing.

Proof: Now, making the use of equation (1.4), (2.4) and (5.1), we get

$$\left(\lambda' - \frac{r}{2}\right) g(X,Y) = k\sigma A(X)A(Y). \tag{6.1}$$

Putting e_i for X and Y in (6.1), where e_i the orthonormal of the basis of the tangent space at each point of the manifold and taking summation over $1 \leq i \leq 4$, we get

$$r = 2\lambda' + \frac{k\sigma}{2}. \tag{6.2}$$

Again if we put μ for X and Y in (6.1), we have

$$r = 2\lambda' + 2k\sigma. \tag{6.3}$$

From (6.2) and (6.3), we get

$$\sigma = 0.$$

This proof the theorem (6.1).

Furthermore, for a spacetime with radiating perfect fluid the resulting universe be isotropic and homogeneous Ellis (1971).

Making the use of (2.4) and (1.3), we get

$$\left(\lambda + \lambda' - \frac{r}{2}\right) g(X,Y) = k\,\mathcal{T}(X,Y). \tag{6.4}$$

In view of (1.1) and (6.4), we get

$$\left(\lambda + \lambda' - \frac{r}{2} - kp\right) g(X,Y) = (p+\rho)A(X)A(Y). \tag{6.5}$$

Now, using the condition of a spacetime with radiative perfect fluid i.e. $\sigma = 3p$ in equation (6.5)

$$\left(\lambda + \lambda' - \frac{r}{2} - \frac{k\sigma}{3}\right) g(X,Y) = \frac{4}{3}\sigma k A(X)A(Y). \tag{6.6}$$

Contraction X and Y in (6.6), we get

$$r = 2(\lambda + \lambda'). \tag{6.7}$$

Again put μ for X and Y in (6.6), we have

$$r = 2(k\sigma + \lambda + \lambda'). \tag{6.8}$$

From (6.7) and (6.8), we get

$\sigma = 0,$

which is not possible by our assumption.

Thus, we may state the following theorem:

Theorem 6.2: A spacetime with vanishing generalized curvature tensor and satisfying Einstein's field equation with cosmological constant is an isotropic and homogeneous spacetime if energy density of the fluid does not vanish.

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