ISSN:0976-4933
Journal of Progressive Science
Vol.11, No.01 & 02, pp 29-34 (2020)

Quarter-symmetric non-metric connection in P-cosymplectic manifold

Niraj Kumar Gupta and *Bhagwat Prasad
Department of Mathematics, DDU Govt. Degree College, Saidpur, Ghazipur
***Department of Mathematics, S. M. M. Town P.G. College, Ballia-277001**
Email - niraj.gdc@gmail.com

Abstract

The object of the present paper is to study P-cosymlectic manifolds with respect to the quarter-symmetric metric connection.

1. Introduction

Let (M, g) be an n-dimensional connected semi-Riemannian manifold of class C^∞ and D be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R, the projective curvature tensor J, the concircular curvature tensor V, the conharmonic curvature tensor H and the conformal curvature tensor C of (M, g) are defined by Yano and Kon (1984)

$$R(X, Y) Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z, \quad (1.1)$$

$$J(X, Y) Z = R(X, Y) Z - \frac{1}{n-1} (Ric (Y, Z) X - Ric(X, Z) Y], \quad (1.2)$$

$$V(X, Y) Z = R(X, Y) Z - \frac{r}{n(n-1)} [g(Y, Z) X - g(X, Z) Y], \quad (1.3)$$

$$H(X, Y,) Z = R(X, Y) Z - \frac{1}{n-1} [Ric (Y, Z) X - Ric (X, Z) Y + g(Y, Z) \phi X - (X, Z)\phi Y, \quad (1.4)$$

$$C(X, Y) Z = R(X, Y) Z - \frac{1}{n-1} [Ric(Y, Z) X - Ric(X, Z) Y + g(Y, Z) \phi X - g(X, Z) \phi Y] + \frac{r}{(n-1)(n-2)} [g(Y, Z) X - g(X, Z) Y], \quad (1.5)$$

respectively, where r is the scalar curvature, Ric is the Ricci tensor and Q is the Ricci operator such that Ric(X, Y) = g(QX, Y).

Let M and C^∞ semi-Riemannian manifold. A (1.3) tensor field R on M is said to curvature-like tensor, if it has all the formal properties of curvature operator. That is, it satisfies the following properties Maltz (1972):

$$'R(X, Y, Z, W) + 'R(X, Y, W, Z) = 0,$$
$$'R(X, Y, Z, W) + 'R(Y, X, W, Z) = 0,$$
$$'R(X, Y, Z, W) + 'R(Z, W, Y, X) = 0, \quad (1.6)$$
$$'R(X, Y, Z, W) + 'R(Y, Z, X, W) + 'R(Z, X, Y, W) = 0,$$

where

$$g(R(X, Y) Z, W) = 'R(X, Y, Z, W).$$

A linear connection $\overline{D}$ defined on (M, g) is said to be a quarter-symmetric connection Golab (1975) of its torsion tensor $\overline{T}$ of $\overline{D}$

$$\overline{T}(X, Y) = \overline{D}_X Y - \overline{D}_Y X - [X, Y],$$

satisfies

$$\bar{T}(X, Y) = \eta(Y)\,\phi X - \eta(X)\,\phi Y, \qquad (1.7)$$

where η is a 1-form and ϕ is a (1,1)-tensor field. Moreover, if a quarter-symmetric connection $\bar{D}$ satisfies the condition

$$(\bar{D}_X g)(Y, Z) = 0, \qquad (1.8)$$

where X, Y, Z $\in\chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M, then $\bar{D}$ is said to be a quarter symmetric metric connection. If we change ϕX by X, then the connection is known as semi-symmetric metric connection Friedmann and Schoutern (1924). Thus the notion of quarter-symmetric generalizes the nation of semi-symmetric connection. A quarter-symmetric metric connection have studied by many geometers in several ways to a different extent such as Ahamd, Jun and Haseeb (2009), Berman (2015), De, Mandol and Mandal (2016), Haseeb (2015), Prasad and Haseeb (2016) and many others.

A relation between the quarter-symmetric metric connection $\bar{D}$ and the Lievi-Civita connection D in a P-cosymplectic manifold is given by Haseeb and Prasad (2021)

$$\bar{D}_X Y = D_X Y + \eta(Y)\,\phi X - g(\phi X, \xi). \qquad (1.9)$$

The paper is organized as follows: After introduction in section 2, we give a brief account of P-cosymplectic manifolds.In section 3, we establish the relation between curvature tensors and Ricci tensor of the Riemannian connection D and the quarter-symmetric metric connection $\bar{D}$ in a P-cosymplectic manifold. In section 4, 5, 6 and 7 we studied ξ–projectively flat, ξ-concircularly flat, ξ-conharmonically flat and ξ-conformaly flat P-cosymplectic manifold with respect to quarter-symmetric metric connection $\bar{D}$ respectively.

2. Preliminaries

Let M be an η–dimensional differentiable manifold on which there are defined a tensor fieldϕ of a type (1.1), a 1- form η and a vector field ξ satisfying

$$\phi^2 X = X - \eta(X)\,\xi \text{ and } \eta(\xi) = 1, \qquad (2.1)$$

Also $\quad \phi(\xi) = 0 \quad$ and $\quad \eta(\phi X) = 0. \qquad (2.2)$

Than (M, n) is called an almost para-contact manifold Sato (1976).

Let g be the Riemannian metric satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \qquad (2.3)$$

and

$$g(X, \xi) = \eta(X), \qquad (2.4)$$

then the structure (ϕ, ξ, η, g) satisfying (2.1), (2.2), (2.3) and (2.4) is called an almost contact para-contact metric manifold Sato (1976). If we define $F(X, Y) = g(\phi X, Y)$, then the following relations exist :

$$F(X, Y) = F(Y, X), \qquad (2.5)$$

and

$$F(\phi X, \phi Y) = g(\phi^2 X, \phi Y) = g(X, \phi Y) = g(\phi X, Y) = F(X, Y). \qquad (2.6)$$

An almost contact para-contact manifold (P-contact manifold) is called P-cosymlectic manifold if

$$D_X\phi = 0 \Leftrightarrow (D_X F)(Y, Z) = 0. \qquad (2.7)$$

On this manifold, we have

$(D_X\eta)(Y) = 0$ and $(D_X\xi) = 0.$ (2.8)

Definition (2.1): A P-cosymplectic manifold M is said to be an η-Einstein manifold if its Ricci tensor Ric is of the form Blair (1976)

$$Ric(X, Y) = a\, g(X, Y) + b\eta(X)\eta(Y),$$

where a and b are scalar functions on M.

A P-cosymplectic manifold M is said to be a generalized η–Einstein manifold if its Ricci tensor Ric is of the form Haseeb and Prasad (2018)

$$Ric(X, Y) = a\, g(X, Y) + b\, \eta(X)\, \eta(Y) + c\, F(X, Y),$$

where a, b, c are scalar functions on M and $F(X, Y) = g(\phi X, Y)$. If c = 0 then the manifold reduces to be an η–Einstein manifold.

3. Curvature tensor of P-cosymplectic manifold with respect to the quarter-symmetric metric connection $\overline{D}$

The curvature tensor $\overline{R}$ of a P-cosymplectic manifolds with respect to the quarter-symmetric metric connection $\overline{D}$ is given by.

$$\overline{R}(X, Y) Z = \overline{D}_X\overline{D}_Y Z - \overline{D}_Y\overline{D}_X Z - \overline{D}_{[X, Y]} Z. \quad (3.1)$$

From equations (1.1), (1.9) and (3.1), we get

$$\overline{D}(X, Y) Z = R(X, Y) Z + [(D_X\eta) (Z) \phi Y - (D_Y\eta) (Z) \phi X + \eta(Z) \{(D_X\phi) (Y) - (D_Y\phi) (X)\}] - [(D_XF) (Y, Z)\xi - (D_YF) (X, Z)\xi + F(Y, Z) (D_X\xi) - F(X, Z) (D_Y\xi)] - [F(Y, Z)\eta(\xi)\phi X - F(X, Z)\, \eta(\xi)\, \phi Y + \eta(Z)F(X, \phi Y)\xi - \eta(Z)\, F(Y, \phi X)\xi]. \quad (3.2)$$

Form (2.1), (2.8) and (3.2), we get

$$\overline{R}(X, Y) Z = R(X, Y) + g(\phi X, Z)\, \phi Y - g(\phi Y, Z)\, \phi X. \quad (3.3)$$

Taking the inner product of (3.3), we get

$$\overline{R}(X, Y, Z, W) = {}'\overline{R}(X, Y, Z, W) + g(\phi X, Z)\, g(\phi Y, W) - g(\phi Y, Z)\, g(\phi X, W), \quad (3.4)$$

where $'\overline{R}(X, Y, Z, W) = g(\overline{R}(X, Y) Z, W)$ and $'R(X, Y, Z, W) = g(R(X, Y) Z, W)$.

contracting (3.3) over X and W, we get

$$\overline{R}ic (Y, Z) = Ric(Y, Z) + g(Y, Z) - \eta(Y)\, \eta(Z) - g(\phi Y, Z)\psi. \quad (3.5)$$

where Ric and $\overline{R}ic$ are the Ricci tensors with respect to the connection D and $\overline{D}$ respectively on M and ψ= trace φ.

From (3.5), we get

$$\phi Y = QY + Y - \eta(Y)\xi - \phi Y.\ \psi. \quad (3.6)$$

where Q and $\overline{Q}$are the Ricci operators with the respect to the connection D and $\overline{D}$, respectively on M, contracting (3.5), we get

$$\overline{r} = r + (n - 1) - \psi^2. \quad (3.7)$$

where r and $\overline{r}$ are the scalar curvatures with respect to the connection D and $\overline{D}$. respectively on M.

in view of (1.6) and (3.6), we get

$$\left.\begin{aligned} &'\overline{R}(X, Y, Z, W) + {}'\overline{R}(Y, X, Z, W) = 0,\\ &'\overline{R}(X, Y, Z, W) + {}'\overline{R}(X, Y, W, Z) = 0,\\ &'\overline{R}(X, Y, Z, W) - {}'\overline{R}(Z, W, X, Y) = 0,\\ &\text{and}\\ &'\overline{R}(X, Y, Z, W) + {}'\overline{R}(Y, Z, X, W) + {}'\overline{R}(Z, X, Y, W) = 0. \end{aligned}\right\} \quad (3.8)$$

Thus in view of equation (3.8), we can state the following theorem:

Theorem (3.1): The curvature tensor of type (0, 4) of a P-cosymplectic manifold with respect to quarter-symmetric metric connection is curvature like tensor.

Form (3.5) we can state the following theorem:

Theorem (3.2) : (i) If the Ricci tensor of P-cosymplectic manifold with respect to quarter-symmetric metric connection vanishes then the Ricci tensor of the manifold with respect to Levi-Civita connection reduces in generalized η–Einstein manifold,

(ii) Ricci tensor $\overline{\text{Ric}}$ (Y, Z) is symmetric,

(iii) Ricci tensor $\overline{\text{Ric}}$ (Y, Z) is skew-symmetric if and only if Ricci tensor Ric(Y, Z) of Levi-Civita connection given by the expression

$$\text{Ric}(Y, Z) = \eta(Y)\,\eta(Z) + g(Y, Z) - g(\phi Y, Z)\,\psi.$$

4. ξ-projectively flat P-cosymplectic manifolds with respect quarter-symmetric metric connection

Definition (4.1): A P-cosymplectic manifold M with respect to the quarter-symmetric metric connection is said to be –protectively flat if

$$\bar{J}(X, Y)\xi = 0, \tag{4.1}$$

for all vector fields X, Y $\in\chi(M)$, $\chi(M)$ is the set of all differentiable vector fields an M.

In view of (1.2), (3.3), (3.5) we get

$$\bar{J}(X, Y) Z = J(X, Y) Z - g(\phi Y, Z)\,\phi X + g(\phi X, Z)\,\phi Y - \frac{\psi}{n-1}[g(\phi X, Z)\,Y - g(\phi Y, Z)\,X] - \frac{1}{n-1}[g(Y, Z)\,X - \eta(Y)\,\eta(Z)\,X - g(X, Z)\,Y + \eta(X)\,\eta(Z)\,Y]. \tag{4.2}$$

Putting ξ for Z in (4.2) and using (2.1) and (2.2), we get

$$\bar{J}(X, Y)\xi = J(X, Y)\xi. \tag{4.3}$$

In view of (4.3), we have the following theorem:

Theorem (4.1): An n–dimensional P-cosymplectic manifold with respect to the quarter-symmetric metric connection is ξ–protectively flat if and only if the manifold with respect to the Levi-Civita connection is also ξ–protectively flat.

5. ξ–concircularly flat P-cosymplectic manifolds with respect to the quarter-symmetric metric connection

Definition (5.1): A P-cosymplectic manifold M with respect to the quarter-symmetric metric connection is said to be ξ–concircularly flat if

$$\bar{V}(X, Y)\xi = 0, \tag{5.1}$$

for all vector fileds X, Y $\in\chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In view of (1.3), (3.3), (3.5) and (3.7), we get

$$\bar{V}(X, Y) Z = V(X, Y) Z - g(\phi Y, Z)\,\phi X + g(\phi X, Z)\,\phi Y - \frac{(n-1)-\psi^2}{n(n-1)}[g(Y, Z)\,X - g(X, Z)\,Y]. \tag{5.2}$$

Putting ξ for Z in (5.2) and using (2.1) and (2.2), we get

$$\bar{V}(X, Y)\xi = V(X, Y)\,\xi - \frac{(n-1)-\psi^2}{n(n-1)}[\eta(Y)\,X - \eta(X)\,Y]. \tag{5.3}$$

In view of (5.3), we van state the following theorem:

Theorem (5.1) : A n-dimensional P-cosymplectic manifold with respect to the quarter-symmetric metric connection is ξ–concircularly flat if and only if the manifold with respect to the Levi-Civita connection is also ξ–concirularly flat, provided $\psi^2 = (n - 1)$.

6. ξ–conharmanically flat P-cosymplectic manifold with respect to quarter-symmetric metric connection

Definition (6.1) A P-cosymplectic manifold M with respect to the quarter-symmetric metric connection is said to be ξ–conharmonically flat if

$$\bar{H}(X, Y)\xi = 0 \tag{6.1}$$

for all vector fields X, Y $\in \chi(M)$ is the set of all differentiable vector fields on M.

In view of (1.4), (3.3), (3.5) and (3.6), we get

$$\begin{aligned}\bar{H}(X, Y) Z = {} & H(X, Y) Z - g(\phi Y, Z)\, \phi X + g(\phi X, Z)\, \phi Y - \\ & \frac{\psi}{n-2} [g(\phi X, Z)\, \phi Y - g(\phi Y, Z)\, X + g(Y, Z)\, \phi X \\ & - g(X, Z)\, \phi Y] - \frac{1}{n-2} [2g(Y, Z)\, X - 2g(X, Z)\, Y \\ & - \eta(Y)\, \eta(Z)\, X + \eta(X)\, \eta(Z)\, Y - \eta(X)\, g(Y, Z)\xi \\ & + g(X, Z)\, \eta(Y)\xi].\end{aligned} \tag{6.2}$$

Puttingξ for Z in (6.2) and using (2.1) and (2.2), we get

$$\bar{H}(X, Y)\xi = H(X, Y)\, \xi - \frac{\psi}{n-2}[\eta(Y)\, \phi X - \eta(X)\, \phi Y] - \frac{1}{n-2}[\eta(Y)\, X - \eta(X)\, Y]. \tag{6.3}$$

In view of (6.3), we can write the following theorem:

Theorem (6.1) : A n–dimensional P-cosymplectic manifold with respect to quarter-symmetric metric connection is ξ-conharmonically flat if and only if the manifold with respect to the Levi-Civita connection is also ξ-conharmonically flat, provided X and Y orthogonal to ξ.

7. ξ-Conformally flat P-cosymplectic manifold with respect to quarter-symmetric metric connection:

Definition (7.1): A P-cosymplectic manifold M with respect to the quarter0symmetric metric connection is said to be ξ–conformally flat of

$$\bar{C}(X, Y)\xi = 0, \tag{7.1}$$

for all vector fields X and Y $\in \chi(M)$ is the set of all differentiable vector fields on manifold.

In view of (1.5),(3.3), (3.5), (3.6) and (3.7), we get

$$\begin{aligned}\bar{C}(X, Y) Z = {} & C(X, Y) Z - g(\phi Y, Z)\, \phi X + g(\phi X, Z)\, \phi Y - \\ & \frac{\psi}{n-2}[g(\phi X, Z)\, Y - g(\phi Y, Z)\, X + g(Y, Z)\, \phi X \\ & - g(X, Z)\, \phi Y] - \frac{1}{n-2}[2g(Y, Z)\, X - 2g(X, Z)\, Y \\ & - \eta(Y)\, \eta(Z)\, X + \eta(X)\, \eta(Z)\, Y - \eta(X)\, g(Y, Z)\xi \\ & + g(X, Z)\, \eta(Y)\xi] + \frac{(n-1)-\psi^2}{(n-1)(n-2)}[g(Y, Z) - g(X, Z)\, Y].\end{aligned} \tag{7.2}$$

Putting ξ for Z in (7.2) and using (2.1) and (2.2), we get

$$\begin{aligned}\bar{C}(X, Y)\xi = {} & C(X, Y)\xi - \frac{\psi}{n-2}[\eta(Y)\, \phi X - \eta(X)\, \phi Y] - \\ & \frac{\psi^2}{(n\ 0\ 1)(n-2)} [\eta(Y)\, X - \eta(X)\, Y].\end{aligned} \tag{7.3}$$

In view of (7.3), we can state the following theorem:

Theorem (7.1) : A n-dimensional P-cosymplectic manifold with respect to quarter-symmetric metric connection is ξ–conformally flat if and only if the manifold with respect to the Levi-Civita connection is also ξ-conformally flat, provided ψ= 0.

References

1. Ahmad, M., Jun, J. B. and Haseeb, A. (2009). Hypersurfaces of an almost r-Para contact Riemannian manifold endowed with a quarter-symmetric metric connection, Bull. Korean Math. Soc., 46:477-487.
2. Berman, A. (2015). Weakly symmetric and Weakly Ricci-symmetric is Sasakian manifolds admitting a quarter-symmetric metric connection Novi. Sod. J. Math. 45:143-153.
3. De, U. C., Mandal, D. and Mandal, K. (2016). Some characterizations of Kenmotsu manifolds admitting a quarter-symmetric metric connection, Bull. Trans. Unir. Brasov, 9(1) : 39-52.
4. Friendmann, A. and Schoutern J.A. (1924). Uber die Geometric der halbsymmetrischen Ubertrogung, Math. Z., 21:211-223.
5. Golab, S. (1975). On semi-symmetric and quarter-symmetric linear connections, Tensor N.S., 29:249-254.
6. Haseeb, A. (2015). Some new results on Para-Sasakian manifolds with a quarter-symmetric metric connection, Facta Unversitatis (NIS), Ser. Math. Inform 30:765-776.
7. Haseeb, A. ans Prasad, R. (2018). On concircular curvature tensor in Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection, Acta Et. Comm. Univ. Tart. De. Mathematics, 22(2) : 279-292.
8. Haseeb, A. and Prasad, R. (2021). Certain results on Lorentzian Para-Kenmotsu manifolds, Bol. Soc. Paran. Mst., 39(3) : 201-220.
9. MoltZ, R. (1972). The nullity spaces of curvature-like tensors, Diff. Geo. 7:579-523.
10. Prasad, R. and Haseeb, A. (2016). On a LP-Sasakian manifolds with respect to the quarter-symmetric metric connection, Novi Sad J. Math., 46:103-116.
11. Sato, I. (1976). On a structure similar to almost contact structure I Tensor, N.S., 30(2) : 219-224.
12. Yano, K. and Kon, M. (1984). Structures on Manifolds, Seriesin Pure Mata., World Scientific,Vol. 3.

Received on 14.8.2020 and accepted on 29.10.2020