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Generalized quasi conformal curvature tensor on K-contact manifolds

Niraj Kumar Gupta and *Bhagwat Prasad
Department of Mathematics, DDU Government College, Saidpur, Ghazipur
***Department of Mathematics, S. M. M. Town P.G. College, Ballia**
Coesponding author email- bhagwatprasad2010@rediffmail.com

Abstract

The object of this paper is to study K-contact manifolds with generalized quasi-conformal curvature tensor. We characterized K-contact manifolds satisfying certain curvature conditions on generalized quasi-conformal curvature tensor.

Key words and phrases

K-contact manifold generalized quasi-conformal curvature tensor, generalized quasi-conformally flat manifold ξ-generalized quasi-conformally flat manifold and irrotational generalized quasi-conformal curvature tensor

1. Introduction

Let (M^n, g), $(n = 2m+1)$ be a contact Riemannian manifold with contact form η, associated vector field ξ, (1-1)- tensor field ϕ and associated Riemannian metric g. If ξ is a killing vector field, then M^n called a K-contact manifold Blair (1976) and Sasaki (1965). K-contact manifolds have been studied by several authors such as Tanno (1966, 1964), Chaki and Ghosh (1972) De and Ghosh (2009) and many others.

Curvature tensor viz., projective curvature, conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor are useful tools to understand the global differential geometric properties of the manifolds with some special structures. Also, they give us some information about the global curvature properties of the manifolds, as they are associated with some special transformations.Keeping importance of these curvature tensor many curvature tensors defined on Riemannian manifolds such as quasi conformal curvature tensor Yano and Sawaki (1968), Pseudo projective curvature tensor Prasad (2002) and quasi-concirculan. Curvature tensor Prasad and Maurya (2007) etc. Continuing these study, a new type of curvature tensor was defined by Prasad, Doulo and Pandey (2011) with the name of generalized quasi-conformal curvature tensor. According to them a generalized qausi conformal curvature tensor was given by

$$G_{qc}(X,Y)Z = aR(X,Y)Z + b\left[Ric(Y,Z)X - Ric(X,Z)Y\right] + c\left[g(Y,Z)QX - g(X,Z)QY\right] - \frac{r}{n}\left[\frac{a}{n-1} + b + c\right]. [g(Y,Z)X - g(X,Z)Y], \quad (1.1)$$

where a, b and c are constants such that $a \neq 0$, $b\neq0$, $c\neq0$ and R(X,Y)Z, Ric(Y,Z)Q and r are the Riemannian curvature tensor of the type (1,3), Ricci tensor of the type (0,2), the Ricci operator operator defined by Ric(Y,Z)=g(QY,Z) and scalar curvature of the manifold respectively.

If b = c then (1.1) takes the form

$$G_{qc}(X,Y)Z = aR(X,Y)Z + b\left[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right]. [g(Y,Z)X - g(X,Z)Y] = \tilde{C}(X,Y)Z,$$

where $\tilde{C}(X,Y)Z$ is quasi-conformal curvature tensor Yano and Sawaki (1968). Thus the quasi-conformal curvature tensor $\tilde{C}$ is a particular case of the tensor $G_{qc}(X,Y)$ Z. For this reason, G_{qc} is called the generalized quasi-conformal curvature tensor. A manifold (M^n,g) $(n>3)$ is called generalized quasi-conformally flat if the $G_{qc}=0$. It is known Prasad, Doulo and Pandey (2011) that the generalized quasi-conformally flat manifold is a manifold of constant curvature, provided $[a+(n-1)b-c]\neq0$. It can easily verified that

$$'Gqc(X,Y,Z,W) + 'Gqc(Y,X,Z,W) = 0,$$
$$'Gqc(X,Y,Z,W) + 'Gqc(X,Y,W,Z) = 0,$$
$$'Gqc(X,Y,Z,W) - 'Gqc(Z,W,X,Y) \neq 0,$$

and

$$'Gqc(X,Y,Z,W) + 'Gqc(Y,Z,X,W) + 'Gqc(Z,X,Y,W) = 0,$$

where $'Gqc(X,Y,Z,W) = g(Gqc(X,Y)Z,W)$.

In particular, the generalized quasi-conformal curvature G_{qc} is reduced to:

1. Conformal curvature tensor C Mishra (1984) if $a = 1, b = c = -\frac{1}{n-2}$,
2. Projective cuvature tensor P (Mishra (1984) if $a = 1, b = -\frac{1}{n-2}$ and $c = 0$,
3. Concircular curvature tensor Mishra (1984) if $a = 1, b = c = 0$,

4. W_2-curvature tensor Pokharyal and Mishra (1972) if $a = 1, b = 0$ and $c = -\frac{1}{n-2}$,
5. Pseudo projective curvature tensor Prasad (2002) if c = 0,
6. Pseudo W_2-curvature tensor $\widetilde{W}_2$Prasad and Maurya (2004) if b = 0.

After preliminaries in section 3, we prove that a generalized quasi-conformally flat K-contact manifold is an η-Einstein manifold. As a consequence of this we obtain scalar curvature under certain condition. Section 4 deals with the study of a K-contact manifold satisfying div G_{qc}=0. In section 5 we prove that a ξ-generalized quasi-conformally flat K-contact manifold is an η-Einstein manifold. Section 6 and 7 devoted with the study of $Ric_1^1 G_{qc}$=0 and generalized quasi-conformally irrotational curvature tensor in K-contact manifold respectively.

2. Preliminaries

Let M^n be an odd dimensional differentiable manifold on which there are defined a real vector valued linear function ϕ, 1- form η and a vector field ξ satisfying :

$\phi^2 = -X + \eta(X)\xi$ and $\eta(\xi)=1$ (2.1)a

$\phi\xi = 0, \eta(X) = 0,$ rank $\phi = n - 1$ (2.1)b

for arbitrary vectors X, Y is called an almost contact manifold Sasaki, 1967 and the structure (ϕ, ξ, η) is called an almost contact structure Balair (1976). An almost contact manifold M^n on which there is a metric tensor g on M^n satisfying

$g(\phi X, \phi Y = g(X,Y) - \eta(X)\eta(Y),$ (2.2)a

and

$g(X, \xi) = \eta(X),$ (2.2)b

is called an almost contact metric manifold and the structure (ϕ, ξ, η, g) is called an almost contact metric structure or contact metric manifold. A contact metric manifold is Sasakian manifold if and only if

$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$ (2.3)

Every Sasakian manifold is K-contact but the converse need not be true, except in dimension 3 June and Kim (1994).

Besides the above relations in K-contact manifold the following relations holds:

$D_X\xi = -\phi X,$ (2.4)

$g(R(\xi,X)Y,\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y),$ (2.5)

$R(\xi,X)\xi = -X + \eta(X)\xi,$ (2.6)

$Ric(X,\xi) = (n-1)\eta(X),$ (2.7)

$D_X\phi = R(\xi,X)Y,$ (2.8)

for any vector fields X and Y.

Further since ξ is a killing vector field, Ric and r remains invariant under it, i,e.,

$$L_{\xi}Ric=0 \text{ and } L_{\xi}r=0, \tag{2.9}$$

where L denotes the Lie-derivation.

Again a K-contact manifold is called Einstein if the Ricci tensor Ric is of the form Ric(X,Y)=λ g(X,Y) where λis constant and η-Einstein if the Ricci tensor Ric is of the form Ric(X,Y)=a′g(X,Y)+b′η(X)η(Y), where a′ and b′ smooth functions on M^n.

3. Generalized quasi conformally flat K-contact manifolds

In this section we consider generalized quasi-conformally flat K-contact manifold. If a K-contact manifold $(M^n,\phi,\xi,\eta,g)$is generalized quasi-confomally flat, then form (1.1), we get

$$aR(X,Y)Z = b[Ric(X,Z)Y - Ric(Y,Z)X] + c[g(X,Z)QY - g(Y,Z)QX] - \frac{r}{n}\left[\frac{a}{n-1} + b + c\right][g(X,Z)Y - g(Y,Z)X]. \tag{3.1}$$

Equation (3.1) can be put as

$$ag(R(X,Y)Z,W) = b[Ric(X,Z)g(Y,W) - Ric(Y,Z)g(X,W)] + c[g(X,Z)Ric(Y,W) - g(Y,Z)Ric(X,W)] - \frac{r}{n}\left[\frac{a}{n-1} + b + c\right][g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \tag{3.2}$$

Putting ξ for X and Z in (3.2), we get

$$ag(R(\xi,Y)\xi,W) = b[Ric(\xi,\xi)g(Y,W) - Ric(Y,\xi)g(\xi,W)] + c[g(\xi,\xi)Ric(Y,W) - g(Y,\xi)Ric(\xi,W)] - \frac{r}{n}\left[\frac{a}{n-1} + b + c\right][g(\xi,\xi)g(Y,W) - g(Y,\xi)g(\xi,W)]. \tag{3.3}$$

In view of (2.1), (2.2), (2.5) and (2.7) in (3.3), we get

$$Ric(Y,W) = Ag(Y,W) + B\eta(Y)\eta(W), \tag{3.4}$$

where A and B are given by

$$A = -\frac{a}{c} - \frac{b}{c}(n-1) + \frac{r}{n.c}\left(\frac{a}{n-1} + b + c\right)$$

and

$$B = +\frac{a}{c} + \frac{b}{c}(n-1) - \frac{r}{n.c}\left(\frac{a}{n-1} + b + c\right) + (n+1). \tag{3.5}$$

Form (3.5), we get

$$A + B = (n-1)$$

Form (3.4), we can state the following theorem:

Theorem (3.1): A generalized quasi-conformally flat K-contqact manifold is an η-Einstein manifold.

Putting Y=W=E_i, where {E_i} is an orthonormal basis of the tangent space at each point of the manifold in (3.4) and taking summotion over i, 1≤i ≤n, we get

$$r = An + B. \tag{3.6}$$

In view of (3.5) and (3.6), we get

$$r = n(n - 1), \text{ provided } a + b(n - 1) - c \neq 0. \tag{3.7}$$

Hence we have the following theorem:

Theorem (3.2): In the generalized quasi conformally flat K-contact manifold, scalar curvature

$$r = n(n - 1), \text{ provided } a + b(n - 1) - c \neq 0.$$

Form (3.4), (3.5) and (3.7), we obtain

$$Ric(Y, W) = (n - 1)g(Y, W). \tag{3.8}$$

In view of (3.2) and (3.8), we get

$$Ric(X, Y, Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W), a \neq 0 .$$

Hence we have the following theorem:

Theorem (3.3): Let (M^n,g) be a K-contact manifold. Then M^n is generalized quasi-conformally flat if and only if it is locally isornetric with a unit sphere $S^n(1)$.

4. K-Contact manifold satisfying div G_{qc}=0:

This section deals with a K-contact manifold satisfying

$$divG_{qc} = 0, \tag{4.1}$$

where div denotes the divergence of generalized satisfying conformal curvature tensor G_{qc}.

Differeantiating (1.1) covariantly along ∪, we obtain

$$(D_U G_{qc})(X, Y)Z = a(D_U R)(X, Y)Z + b\big[(D_U Ric)(Y, Z)X - (D_U Ric)(X, Z)Y\big] + c\big[g(Y, Z)\ (D_U Q)X - g(X, Z)(D_U Q)(Y)\big] - \frac{D_U r}{n}\Big[\frac{a}{n-1} + b + c\Big]\big[g(Y, Z)\ X - g(X, Z)\ Y\big]. \tag{4.2}$$

Contraction of (4.2), we get

$$(divG_{qc})(X, Y)Z = (a + b)[(D_X Ric)(Y, Z) - (D_Y Ric)(X, Z) - \frac{1}{n}\Big[\frac{a}{n-1} - \Big(\frac{n-2}{2}\Big)c + b\Big][g(Y, Z)\ dr(X) - g(X, Z)dr(Y)]. \tag{4.3}$$

Form (4.1) and (4.3), we get

$$(a + b)[(D_X Ric)(Y, Z) - (D_Y Ric)(X, Z)] - \frac{1}{n}\Big[\frac{a}{n-1} - \Big(\frac{n-2}{2}\Big)c + b\Big].$$

$$[g(Y,Z)dr(X) - g(X,Z)\,dr(Y)] = 0. \tag{4.4}$$

Form (2.9), we get

$$(D_\xi\,Ric)(Y,Z) = Ric(D_Y\xi,Z) - Ric(D_Z\xi,Y). \tag{4.5}$$

Putting X=ξ in (4.4) and using (4.5) and dr(ξ)=0, we get

$$(a+b)[Ric(D_Y\,\xi,Z) + Ric(D_Z\,\xi\,,Y) + (D_Y Ric)(\,\xi,Z)] - \frac{1}{n}\left[\frac{a}{n-1} - \left(\frac{n-2}{2}\right)c + b\right]\eta(Z)dr(Y) = 0. \tag{4.6}$$

Form (2.7), we have

$$(D_Y Ric)(\,\xi,Z) = (n-1)g(D_Y\,\xi,Z) - Ric(D_Y\xi,Z). \tag{4.7}$$

Using (4.6) and (4.7), we get

$$(a+b)\big[(n-1)g(D_Y\xi,Z) + Ric(D_Z\xi,Y)\big] = \frac{1}{n}\left[\frac{a}{n-1} + b - \left(\frac{n-2}{2}\right)c\right]\eta\,(Z)\,dr(Y). \tag{4.8}$$

From (2.4) and (4.8), we get

$$(a+b)\big[(n-2)g(\phi Y,Z) + Ric(\phi Z,Y)\big] = \frac{1}{n}\left[\frac{a}{n-1} + b - \left(\frac{n-2}{2}\right)c\right]\eta(Z)dr(Y)\quad . \tag{4.9}$$

Operating ϕ on Z in (4.9), we get

$$(a+b)[Ric(Y,Z) - (n-1)g(Y,Z)] = 0. \tag{4.10}$$

Equation (4.10) implies that

$$Ric(Y,Z) = (n-1)g(Y,Z), a+b \neq 0. \tag{4.11}$$

Hence (4.11), we get

$$QY = (n-1)Y. \tag{4.12}$$

Hence in view of (4.11), (4.12), we get from (1.1)

$$D_{qc}(X,Y)Z = a\big[R(X,Y)Z - g(Y,Z)X - g(X,Z)Y\big].$$

Hence from (4.11), we get the following:

Theorem (4.1): A K-contact manifold with divergence free generalized quasi-conformal curvature tensor is an Einstein manifold, provided a+b≠0.

5. ξ-generalized quasi-conformally flat K-contact manifold

ξ -conformally flat K-contact manifold have been studied (Zhen, Cabrerizo, Fermnandez and Fernanadez, 1997).

Here we study ξ-genralized quasi-conformally flat K-contact manifold.

Definition (5.1): A K-contact manifold is said to be ξ generalized quasi-conformally flat manifold if

$$G_{qc}(X,Y)\xi=0. \tag{5.1}$$

Let as assume that the manifold M^n is ξ-generalized quasi conformally flat. Then using (5.1) in (1.1), we get

$$aR(X,Y)\xi + b(n-1)[\eta(Y)X - \eta(X)Y] + c[\eta(Y)QX - \eta(X)QY]$$
$$-\frac{r}{n}\left[\frac{a}{n-1} + b + c\right][\eta(Y)X - \eta(X)Y] = 0. \quad (5.2)$$

Putting ξ for X in (5.2) and using (2.7) and η(ξ)=1, we get

$$a[-Y + \eta(Y)\xi] + b(n-1)[-Y + \eta(Y)\xi + c(n-1)\eta(Y)\xi - QY]$$
$$-\frac{r}{n}\left[\frac{a}{n-1} + c + b\right][\eta(Y)\,\xi - Y] = 0. \quad (5.3)$$

Simplification of (5.3), we get

$$Ric(Y,Z) = Ag(Y,W) + B\eta(Y)\,\eta(W), \quad (5.4)$$

where A and B are given by

$$A = -\frac{a}{c} - \frac{b}{c}(n-1) + \frac{r}{n}\left[\frac{a}{n-1} + b + c\right], \quad (5.5)$$

$$B = \frac{a}{c} + \frac{b}{c}(n-1) - \frac{r}{n}\ [\frac{a}{n-1} + b + c) + (n-1)]. \quad (5.6)$$

In view of (5.4), we get the following theorem:

Theorem (5.1): A ξ-generalized quasi-conformally flat K-contact manifold is an η–Einstein manifold.

6. K-contact Riemannian manifold satisfying ($Ric_1^1 G_{qc}$)(Y,Z)=0 :

Let us consider that in K-contact Riemannian manifold satisfying

$$(Ric_1^1 G_{qc})(Y,Z) = 0. \quad (6.1)$$

Ric_1^1 denotes the contraction of G_{qc} with respect to X. Thus we have from (1.1)

$$\left(Ric_1^1 G_{qc}\right)(Y,Z) = aRic(Y,Z) + b(n-1)Ric(Y,Z) +$$
$$c\left[rg(Y,Z) - Ric(Y,Z)\right] - \frac{r}{n}\left[\frac{a}{n-1} + b + c\right].$$
$$(n-1)g(Y,Z). \quad (6.2)$$

From (6.1) and (6.2), we get

$$\left[a + b(n-1) - c\right]\left[Ric(Y,Z) - \frac{r}{n}\,g(Y,Z)\right] = 0, \quad (6.3)$$

which gives Ric(Y,Z)=$\frac{r}{n}$g(Y,Z), provided to a+b(n-1)–c≠0.This shows that (M^n,g) be an Einstein manifold.

Putting ξ for Z in (6.3), we get

$$r = n(n-1), \text{ provided } a + (n-1)b - c \neq 0.$$

Theorem (6.1) : If in K-contact manifold the relation ($Ric_1^1 G_{qc}$)(Y,Z)=0 holds, then M^n is an Einstein manifold with the scalar curvature r= n(n-1), provided a+b(n-1)–c≠0

7. Generalized quasi conformaly irrotational curvature tensor in K-contact manifold

The rotation of generalized quasi-conformaly curvature on a Riemannian manifold is given by

$$R_o + G_{qc} =$$
$$\left(D_U G_{qc}\right)(X,Y)\,Z + \left(D_X G_{qc}\right)(U,Y,Z) +$$
$$\left(D_Y G_{qc}\right)(U,X,Z) - \left(D_Z G_{qc}\right)(X,Y,U). \quad (7.1)$$

By virtue of second Bianch identy, we get

$$\left(D_U G_{qc}\right)(X,Y)Z + \left(D_X G_{qc}\right)(U,Y)Z + \left(D_Y G_{qc}\right)(U,X)Z = 0. \quad (7.2)$$

Hence from (7.1) and (7.2), we get

$$\text{Curl } G_{qc} = -\left(D_Z G_{qc}\right)(X,Y)U. \quad (7.3)$$

If the generalized quasi-conformal curvature tensor is irrotational, then

$$\text{Curl } G_{qc} = 0. \quad (7.4)$$

By (4.4), we get

$$\left(D_Z G_{qc}\right)(X,Y)U = G_{qc}(D_Z X, Y)U + G_{qc}(X\, D_Z Y)U + G_{qc}(X,Y)D_Z U. \quad (7.5)$$

Putting U=ξ in (7.5), we get

$$\left(D_Z G_{qc}\right)(X,Y)\xi = G_{qc}(D_Z X, Y)\xi + G_{qc}(X\, D_Z Y)\xi + G_{qc}(X,Y)D_Z \xi. \quad (7.6)$$

Putting Z=ξin (7.5) and using (2.10, (2.3) and (2.7), we get

$$G_{qc}(X,Y)\xi = \left[a + (n-1) - \frac{r}{n}\left(\frac{a}{n-1} + b + c\right)\right]\left[\eta(Y)X - \eta(X)Y\right]$$
$$+ c\left[\eta(Y)QX - \eta(X)QY\right]. \quad (7.7)$$

Using (7.6) and (2.4) in (7.6), we get

$$G_{qc}(X,Y)\phi Z = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + b + c\right)\right]$$
$$\left[g(Y,\phi Z)X - g(X,\phi Z)Y\right] + c[g(Y,\phi Z)\phi X$$
$$- g(X,\phi Z)QY - \eta(Y)(D_Z Q)X + \eta(X)(D_Z Q)$$
$$- \frac{D_Z r}{n}\left(\frac{a}{n-1} + b + c\right)\{\eta(Y)X - \eta(X)Y\}]. \quad (7.7a)$$

Operating ϕ and Z in (7.7a) and using (2.1), we get

$$G_{qc}(X,Y)Z = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + b + c\right)\right]\left[g(Y,Z)X - g(X,Z)Y\right]$$
$$+ c[g(Y,Z)\,QX - g(X,Z)\,QY - \eta(Y)(D_Z Q)(X) + \eta(X)(D\phi_Z Q)$$
$$- \left(\frac{a}{n-1} + b + c\right)\{\eta(Y)X - \eta(X)Y\}]. \quad (7.8)$$

In view of (7.8), we get the following theorem:

Theorem (7.1): If the generalized quasi conformal curvature tensor in a K-contact manifold is irrotational then the generalized quasi conformal curvature $G_{qc}(X, Y)\,Z$ is given by the expression

$$G_{qc}(X,Y)Z = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + b + c\right)\right]\left[g(Y,Z)X - g(X,Z)Y\right]$$
$$+ c[g(Y,Z)\,QX - g(X,Z)\,QY - \eta(Y)(D_Z Q)(X) + \eta(X)(D\phi_Z Q)$$

$$-\left(\frac{a}{n-1}+b+c\right)\{\eta(Y)X-\eta(X)Y\}].$$

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