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Semi-invariant submanifold of Lorentzian Sasakian manifolds

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Abstract

In this paper, we introduce the notion of semi-invariant submanifold of Lorentzian almost contact manifold. The integrability condition for the distributions $\mathrm{D}, \mathrm{D}^{\perp}$ and some properties of semi-invariant submanifolds of Lorentzian Sasakian manifold are found. According to these cases semi-invariant submanifold of Lorentzian Sasakian manifold is categorized and its used to demonstrate that the method presented in this is effective.

1. Introduction

Semi-invariant submanifolds have been studied by many authors (1), (2), (3), (4), (5), (6), (7), (8) and also studied the semi-invariant sub manifolds of Lorentzian Sasakian manifolds by Pablo alegre (10).

An odd dimensional Riemannian manifold (M^{2n+1}, g) is called a Lorentzian almost contact manifold if it is endowed with structure(ϕ, ξ, η, g), where ϕ is a (1.1) tensor, ξ and η a vector field and a 1-form on $\widetilde{M}$ respectively, and g is Lorentzian metric, satisfying

$$\phi^2 X = -X + \eta(X)\xi \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.2}$$

$$\eta(\xi) = 1 \tag{1.3}$$

$$\eta(X) = -g(X, \xi), \tag{1.4}$$

for any vector field X, Y in M. Let ϕ denote the 2 form in $\widetilde{M}$ given by $\phi(X,Y) = g(X, \phi Y)$ if $d\eta = \phi$, $\widetilde{M}$ is called Lorentzian manifold. A normal contact Lorentzian manifold is called Lorentzian Sasakian (8), this is a contact Lorentzian one verifying

$$\left(\widetilde{\nabla}_x \phi\right)Y = -g(X,Y)\xi - \eta(Y)X \qquad (1.5)$$

In this a case, we have

$$\widetilde{\nabla_x}\, \xi = \phi X \qquad (1.6)$$

Let us consider a submanifold M of a Lorentzian almost contact metric manifold $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$, tangent to the structure vector field ξ

Put $\quad \phi X = TX + FX, \qquad (1.7)$

for any vector field X, where TX (resp. FX) denotes the tangential (resp. normal) component of ϕX. Similarly

$$\phi V = tV + fV\ , \qquad (1.8)$$

for any normal vector field V with tV tangent and fV normal to M.

Given a submanifold of a Lorentzian almost contact manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, we also use g for the induced metric on M.

We denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ and by ∇ the induced Levi-civita connection on M. Thus, the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla_x}\, Y = \nabla_x Y + h(X,Y) \qquad (1.9)$$

$$\widetilde{\nabla_x}\, V = -A_V X + \nabla_X^{\perp} V, \qquad (1.10)$$

for vector field X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, $\nabla^{\perp}$ the normal connection and A_V the shape operator in the direction of V. The second fundamental form and shape operator are related by

$$g(h(X,Y), V) = g(A_V X, Y) \qquad (1.11)$$

2.Semi-invariant submanifold

Let Mbe a submanifold of manifold $\widetilde{M}$ tangent to ξ. Then M is called a semi-invariant submanifold of $\widetilde{M}$ if there exists a differentiable distribution

$D : x \longrightarrow D_x \subset T_x M$ on M satisfying the following condition

$$\phi\, D_x \subset D_x \text{ for each } x \in M \qquad (2.1)$$

and complementary orthogonal distribution

$D^{\perp}: x \longrightarrow D_x^{\perp} \subset T_x^{\perp}M$ is totally real i.e.

$$\phi D_x^{\perp} \subset T_x^{\perp}M \text{ for each } x \in M \tag{2.2}$$

We call $D(resp. D^{\perp})$ horizontal (resp. vertical) distribution.

Let $\tilde{R}$ (resp. R) be the curvature tensor of $\widetilde{M}$ (resp. M). Then the equation of Gauss and Codazzi are respectively given by

$$g\big(\tilde{R}(X,Y)Z,W\big) = g(R(X,Y)Z,W) - g\big(B(X,W),B(Y,Z)\big) + g\big(B(Y,W),B(X,Z)\big) \tag{2.3}$$

$$\big(\tilde{R}(X,Y),Z\big)^{\perp} = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \tag{2.4}$$

Note: Throughout this paper we have taken the structure tensor ξ as tangent to the submanifold.

Lemma 2.1 Let M be a submanifold of a Lorentzian Sasakian manifold $\big(\widetilde{M},\phi,\xi,\eta,g\big)$. Then we have the following identities.

$$T^2X = -X + \eta(X)\xi - tFX \tag{2.5}$$

$$FTX + fFX = 0 \tag{2.6}$$

$$TtV + tfV = 0 \tag{2.7}$$

$$FtV + f^2V = -V \tag{2.8}$$

Proof Applying ϕ to (1.7) and (1.8) and separating the tangent and normal parts we get the above identities.

Lemma 2.2 Let M be a submanifols of a Lorentzian Sasakian manifold $\big(\widetilde{M},\phi,\xi,\eta,g\big)$. Then

$$(\nabla_X T)Y = A_{FY}X + th(X,Y) - g(X,Y)\xi - \eta(Y)X \tag{2.9}$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,TY), \tag{2.10}$$

where we have defined $(\nabla_X T)Y$ and $(\nabla_X F)Y$ respectively by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$$

and

$$(\nabla_X F)Y = \nabla_X FY - F\nabla_X Y,$$

for $X, Y \in TM$

Proof Differentiating (1.7) covariantly along X and using (1.1), (1.5), (1.7), (1.8) and then separating the tangential and normal parts we get (2.9) and (2.10) respectively.

Lemma (2.3) Let M be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M}, \phi, \xi, \eta, g)$. Then

$$(\nabla_X t)V = A_{fV}X - TA_V X \tag{2.11}$$

$$(\nabla_X f)V = -FA_V X - h(X, tV), \tag{2.12}$$

where we have defined $(\nabla_X t)V$ and $(\nabla_X^{\perp} f)V$ respectively by

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X V$$

$$(\nabla_X^{\perp} f)V = \nabla_X^{\perp} fV - f\nabla_X^{\perp} V,$$

for any $X \in TM$ and $V \in T^{\perp}M$

Proof Differentiating (1.8) covariantly along X and using (1.5), (1.9) and (1.10) we get (2.11) and (2.12)

Lemma (2.4) Let M be a semi-invariant submanifold of a Lorentzian Sasakian manifold. If D is ξ – horizontal and $W, Z \in D^{\perp}$, then

$$A_{FZ}W = A_{FW}Z \tag{2.13}$$

Proof For any tangent vector field X.

$$g\big((\nabla_X T)(Z), W\big) = g(\nabla_X TZ, W) - g(T\nabla_X Z, W) = 0,$$

from which we obtain on a account of (2.9);

$$A_{FZ}W - A_{FW}Z - \eta(Z)W - \eta(W)Z \tag{2.14}$$

Since D is ξ-horizontal, we get (2.13)

Proposition (2.1): Let M be a semi-invariant submanifold of a Lorentzian Sasakian manifold. Then the vertical distribution $D^{\perp}$ is integrable if and only if

$$A_{FZ}W = \eta(Z)W \tag{2.15}$$

Proof For $Z, W \in D^{\perp}$, we have

$$\phi[Z,W] = T[Z,W] + F[Z,W] = (\nabla_W T)Z - (\nabla_Z T)W + F[Z,W],$$

which with the help of (2.14) yields

$$\phi[Z,W] = F[Z,W] + 2A_{FZ}W - 2\eta(Z)W$$

Hence $D^{\perp}$ is integrable if and only if (2.15) is satisfied.

Proposition (2.2) Let M be a semi-invariant submanifold of a Lorentzian Sasakian manifold. Then the distribution D is completely integrable if and only if

$$h(X,TY) = h(Y,TX), for\ X,Y \in D \tag{2.16}$$

Proof For $X,Y \in D$, we have

$$\phi[X,Y] = T[X,Y] + F[X,Y]$$

$$= T[X,Y] + F\nabla_X Y - F\nabla_Y X$$

$$= T[X,Y] + (\nabla_Y F)X - (\nabla_X F)Y,$$

which with the help of (2.10), gives

$$\phi[X,Y] = T[X,Y] + h(X,TY) - Y(Y,TX)$$

Hence D is integrable if and only if (2.16) holds.

Proposition (2.3) If D is ξ-vertical, then the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M if and only if

$$g(h(W,X),\phi Z) = 0 \text{ for } W,Z \in D^{\perp} \text{ and } Y \in D \tag{2.17}$$

Proof From (2.9), we have

$$\nabla_X TY - T\nabla_X Y = (\nabla_X T)Y = A_{FY}X + th(X,Y) - g(X,Y)\xi - \eta(Y)X$$

Putting $X = W \in D^{\perp}$ and $Y = Z \in D^{\perp}$ and using fact that D is ξ-vertical, we get

$$-T\nabla_W Z = A_{FZ}W + th(W,Z) - g(W,Z) - \eta(Z)W$$

For any $Y \in D$, we get

$$-g(T\nabla_W Z, Y) = g(A_{FZ}W, Y) + g(th(W,Z), Y) -$$

$$g(W,Z)g(\xi,Y) - g(W,Y)\eta(Z).$$

From which we get

$$g(\nabla_W Z, TY) = g(h(W,Y), FZ) = g(h(W,Y), \phi Z)$$

Hence we have our assertion

Definition: A Semi-invariant submanifold of a Lorentzian Sasakian manifold is called semi-invariant product if it is locally a Riemannian product of an invariant submanifold M^T and an anti-invariant submanifold $M^\perp$ of M [9].

Theorem 2.1 Let M be a semi-invariant submanifold of a Lorentzian Sasakian manifold and Let D be ξ – Vertical. Then M is a semi-invariant product if

$$A_{FZ}TX = \eta(Z)TX, \; Z \in D^\perp \; and \; X \in D \tag{2.18}$$

Proof Since M is a semi-invariant product, for any tangent vector S

$$\nabla_S X \in D \; for \; X \in D \; and \; \nabla_S Z \in D^\perp \; for \; Z \in D^\perp$$

Now from (1.1) and (2.9) and using the fact that D is ξ – Vertical, we get

$$g(\nabla_S Z, Y) = g(TA_{FZ}S, Y) = g(Tth(S,Z), Y) + g(S,TY)\eta(Z) + g(S,Z)g(\xi,TY)$$
$$+g(tF\nabla_S Z, Y) - g(\xi, Y)\eta(\nabla_S Z)$$

Hence

$$\nabla_S Z = TA_{FZ}S - \eta(Z)TS + Tth(S,Z) - \eta(\nabla_S Z) \tag{2.19}$$

Again $g(Z, \nabla_S X) = -g(X, \nabla_S Z) = 0$

This, by means of (2.19) and fact that D is ξ vertical, gives

$$g(A_{FZ}TX - \eta(Z)TX, S) = 0$$

i.e. $A_{FZ}TX = \eta(Z)TX$

implying that (2.18) hold good.

Theorem 2.2 Let M be a semi-invariant product of a Sasakian manifold and $D - \xi$ vertical, then

$$g(\tilde{R}(X, \phi X)Z, \phi Z) = 0, \tag{2.20}$$

for unit vector $X \in D$ and $Z \in D^\perp$, where $\tilde{R}$ is the curvature tensor of $\widetilde{M}$.

Proof Since M is a semi-invariant product, for any tangent vector $\nabla_S X \in D$ for $X \in D$ and $\nabla_S Z \in D^\perp$ for $Z \in D^\perp$. This property together with the equation (2.17) gives,

$$g(B(\nabla_X \phi X, Z), \phi Z) = g\big(B\big(\nabla_{\phi X} X, Z\big), \phi Z\big) = g(B(\phi X, \nabla_X Z), \phi Z)$$

$$= g\big(B\big(X, \nabla_{\phi X} Z\big)\ \phi Z\big) = 0$$

The Codazzi equations affords us

$$g\big(\tilde{R}(X, \phi X)Z, \phi Z\big) = g\left((\nabla_X B)(\phi X, Z) - \big(\nabla_{\phi X} B\big)(X, Z), \phi Z\right)$$

$$= g(D_X B(\phi X, Z), \phi Z) - g\big(D_{\phi X} B(X, Z), \phi Z\big)$$

$$= -g(B(\phi X, Z), D_X \phi Z) - g\big(B(X, Z), D_{\phi X} \phi Z\big)$$

$$= -g\big(B(\phi X, Z), \widetilde{\nabla_X} \phi Z\big) + g\big(B(X, Z), \tilde{\nabla}_{\phi X} \phi Z\big)$$

$$= -g\big(B(\phi X, Z), \big(\tilde{\nabla}_X \phi\big) Z\big) - g\big(B(\phi X, Z), \phi \tilde{\nabla}_X Z\big)$$

$$+ g\big(B(X, Z), \big(\tilde{\nabla}_{\phi X} \phi\big) Z\big) + g\big(B(X, Z), \phi \tilde{\nabla}_{\phi X} Z\big),$$

which on using (1.5) and the fact that D is ξ-vertical gives (2.20).

3. Discussion and Analysis of Result

In this paper I have given some results on semi-invariant submanifold of Lorentzian Sasakian manifold and the integrability condition for the distribution $D\ and\ D^{\perp}$ have also been discussed.

4. Conclusion

In this paper I have categorized semi-invariant submanifolds of Lorentzian Sasakian manifold satisfying the conditions,

$\big(\tilde{\nabla}_X \phi\big) Y = -g(X, Y)\xi - \eta(Y) X$ and $\tilde{\nabla}_X \xi = \phi X$. This derivative operators are very important. It provides information about the structure on the manifold.

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