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On Poisson structures of differentiable manifold and its non-commutative properties

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Abstract

Our main aim is to analyse geometric application for non-commutative Poission structure on Manifold. There exists a relationship between Poisson geometry and deformation theory which was initiated by Ezra Getzler. We derive properties of deformation quantization of a noncommutative Poisson structure with objective to compute infinitesimal deformation on Poisson manifold. J. Block and Ping Xu introduced the notion of non-commutative Poisson structure on an associative algebra. The main focus here is to reformulate its impact on symplectic reflection algebra. It is closely related to Horchs-child algebra based on computations of Gerstenhaber bracket. M. Gertenheber established its connection with the deformation theory of an associative algebra A as well as the Hochschild cohomology. A. Gerstenhaber studied deformation theory on a topological algebra where Gerstenhaber bracket is used in defining non-commutative Poisson structures compatible with differential forms on Hochs child cohomology X. Tang introduced the notion of Lie bracket on the Hochs child cohomology. We show the set of noncommutative Poisson structures on an algebra A has one to one correspondence with the set of infinitesimal deformations of A. We integrate the infinitesimal deformation associated to a noncommutative Poisson structure to a real one which is closely related to the notion of deformation quantization in mathematical physics.

Keywords- Poisson geometry, Clifford algebra, non-commutative geometry, invarieant simplistic manifold

1. Introduction

Let us consider that A is a algebra of smooth functions on a smooth manifold M. An application of the Hochschild-Kostant-Rosenberg theorem, the second Hochschild cohomology classes in HH^2 (A; A) are in one to one correspondence with Poisson structures on M. (ii) we extend the relationship between Poisson geometry and deformation theory. we discuss here the characteristic properties of non-commutative Poisson structures on orbifolds obtained from global quotients. Let M be a compact smooth manifold, and G be a finite group acting on M. The orbifold is the quotient space X = M/G, where X is a topological space with quotient singularities. The algebra $C^{\infty}(M)^{G}$ of G-invariant smooth functions on M is not regular. The basic notions of non-commutative geometry has been introduced by A. Connes to derive algebraic properties of the crossed product algebra $C^{\infty}(M) \rtimes G$. We develop a geometric descriptions for all noncommutative Poisson structures which are symplectic $C^{\infty}(M) \rtimes G$ when M is a symplectic manifold with a symplectic action. Noncommutative Poisson structures on $C^{\infty}(M)\rtimes G$, computed the Hochschild cohomology of $C^{\infty}(M) \rtimes G$ as a vector space.

1.2 Theorem

A noncommutative Poisson structure on an associative algebra A is an element in the second Hochschild cohomology group $H^2(A,A)$ of A, whose Gerstenhaber bracket with itself vanishes, i.e. $[\Pi,\Pi]G = 0$.

$$HH^{\bullet}(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G) = \Gamma^{\infty}(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TM^{g} \otimes \wedge^{l(g)} N^{g})^{G}. \quad \ldots \quad (1)$$

Proof

Let M^g be the fixed point manifold of g, and N^g be the normal bundle of the embedding of M^g in M, where $l(g)$ is the dimension of N^g. The group G acts on disjoint union $\sqcup M^g$ of M^g for all $g \in G$ by the conjugate action where, M^g has different components with different dimensions. We choose the disjoint union of all the components and $l(g)$ to be a local constant function on M^g. $C^{\infty}(M)\rtimes G$ which is considered as a bornological algebra with the bornology defined by the Frechet topology. Let us take HH• to be the continuous Hochschild cohomology of a bornological algebra. It implies that the Hochschild cohomology of $C^{\infty}(M) \rtimes G$ is equal to the space of "vector fields" on $\tilde{X}$. we derive characteristic properties of noncommutative Poisson structures on $C^{\infty}(M)\rtimes G$ for which we compute the Gerstenhaber bracket on $HH^{\bullet}(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G)$. It possesses quasi-isomorphisms between the Hochschild cochain complexes defined as follows.

$$C^{\bullet}(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G) \quad \ldots(2)$$

and

$$\Gamma^{\infty}(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TM^{g} \otimes \wedge^{l(g)} N^{g})^{G}. \quad \ldots(3)$$

A quasi-isomorphism L is defined as follows.

$$L : C^{\bullet}(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G) \longrightarrow \Gamma^{\infty}(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TM^{g} \otimes \wedge^{l(g)} N^{g})^{G}. \quad \ldots(4)$$

The Gerstenhaber brackets on $HH^{\bullet}(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G)$ is determined from (3) & (4). we show that the Gerstenhaber bracket on orbifolds is a generalization of the classical Schouten-Nijenhuis bracket on manifolds. This bracket is the twisted Schouten-Nijenhuis bracket on $\square^{\infty}(\oplus_g \wedge^{\bullet -l(g)} TM^g \otimes \wedge^{l(g)} N^g)^G$. We solve the equation $[\Pi,\Pi]_G = 0$ on $HH^2(C^{\infty}(M) \rtimes G; C^{\infty}(M) \rtimes G)$ expresses as noncommutative Poisson structures on $C^{\infty}(M) \rtimes G$ when M is a symplectic manifold.

Let us consider a complex symplectic vector space V with a symplectic G action. The cocycles are used on symplectic reflection algebras which correspond to a special class of noncommutative Poisson structures on Poly (V) ⋊ G, where Poly(V) is the algebra of polynomials on V, we prove that all these cocycles can be extended to a formal deformation of the algebra Poly(V) ⋊ G. which implies that noncommutative Poisson structures may be extended to formal deformations, and it generalizes the symplectic reflection algebras.

Properties of Poission structure and its algebraic interpretation

Let Poly(V) be the algebra of polynomial functions on a vector space V, and G be a finite group acting linearly on V. The Hochschild cohomology of the crossed product algebra Poly(V)⋊G. The step construction of a quasi-isomorphism is given by

$$L : C^{\bullet}(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow \Gamma^{\infty}(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TV^{g} \otimes \wedge^{l(g)} N^{g})^{G}.$$

constructed implicitly by N. Neunaler explicitly in the following steps.

Step I: Let us choose map

$$L_1 : C^{\bullet}(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow \left(C^{\bullet}(Poly(V), Poly(V) \rtimes G)\right)^{G},$$

where G acts on C•(Poly(V), Poly(V) ⋊ G) by the relation

$$g\Psi(a_1, \ldots, a_n) = U_{g^{-1}} \cdot \Psi(g(a_1), \ldots, g(a_n)) \cdot U_g. \quad \ldots (5)$$

U_g denotes the element g ∈ Poly(V)⋊G. Given a Hochschild cocycle $\Psi \in C^k$(Poly(V) ⋊ G, Poly(V) ⋊ G), Let us define $L_1(\Psi) \in (C^k(Poly(V), Poly(V) \rtimes G))^G$ as follows

$$L_1(\Psi)(f_1, \ldots, f_k) = \frac{1}{|G|}\sum_{g}(g\Psi)(f_1, \ldots, f_k), \quad \forall f_1, \ldots, f_k \in Poly(V), \quad \ldots (6)$$

where |G| is the order of group G.

Step II: (isomorphism)

$$L_2 : \left(C^{\bullet}(Poly(V), Poly(V) \rtimes G)\right)^{G} \longrightarrow \left(\bigoplus_{g\in G}\Gamma^{\infty}(\wedge^{\bullet} TV), \kappa_g \wedge \ \right)^{G}. \quad \ldots (7)$$

Let A_g be a vector space isomorphic to Poly(V) equipped with the Poly(V)- bimodule structure given by the relation

$$a \cdot \xi \cdot (b) = a\xi g(b), \text{ for } a, b \in Poly(V), \ \xi \in Poly(V)_g, \quad \ldots (8)$$

where the right hand side of is the product of a, ξ, and g(b) are elements in Poly(V). Since Poly(V)-Poly(V) bimodule, Poly(V)⋊G has a natural splitting into a direct sum of submodules $\oplus_{g2G}A_g$, The cochain complex $C^{\bullet}(Poly(V), Poly(V) \rtimes G)$ has a natural splitting into $\oplus_{g2G} C^{\bullet}(Poly(V), A_g)$. Let us define L_2 to be the sum of the maps

$$L_2^g : C^{\bullet}(Poly(V), A_g) \longrightarrow \left(\Gamma^{\infty}(\wedge^{\bullet} TV), \kappa_g\right) \quad \dots \quad (9)$$

over all $g \in G$.

We introduce the vector field $(x) = \sum_i x^i \frac{\partial}{\partial x^i}$, on V where the x^i are coordinate functions on V and the vector field $\kappa_g \in \Gamma^{\infty}(TV)$ by the equation

$$\kappa_g(x) = X(g(x)) - X(x). \quad \dots \quad (10)$$

which shows that for a permutation σ of k elements fixing $x \in V$, the product $(x^{i1}_{\sigma(1)} - x^{i1}) \cdot \cdot \cdot (x^{ik}_{\sigma(k)} - x^{ik})$ is a function on $x_1, \cdot \cdot \cdot, x_k \in V$. Now by taking the product of the values of the coordinate functions, given an element $\Psi \in C^k(Poly(V), A_g)$, $L^g_2(\Psi) \in \Gamma^{\infty}(\wedge^k T\ V)$, the usual projection to anti-symmetric linear operators, by

$$L_2^g(\Psi)(x) = \sum_{i_1,\dots,i_k} \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} \Psi\left((x_{\sigma(1)})^{i_1} - x^{i_1}) \cdots (x^{i_k}_{\sigma(k)} - x^{i_k})\right)(x) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$

where S_k is the permutation group of k-elements. The G action on $(\oplus_{g\in 2} \Gamma^{\infty}(\wedge \bullet T\ V), \kappa g \wedge)$ is defined by the relation

$$h(\sum_g \phi_g)|_{h^{-1}gh} = h_*(\phi_g), \qquad \text{for } \sum_g \phi_g \in \bigoplus_g \Gamma^{\infty}(\wedge^{\bullet} TV), \text{ and } h \in G \quad \dots \quad (11)$$

It is always verifiable that L^g_2 is G-equivariant, and it defines a map given by

$$L_2 : (C^{\bullet}(Poly(V), Poly(V) \rtimes G))^G \longrightarrow (\bigoplus_{g \in G} \Gamma^{\infty}(\wedge^{\bullet} TV), \kappa_g \wedge\ )^G.$$

Step III (Irreducible representations)

$$L_3 : \left(\bigoplus_{g\in G} (\Gamma^{\infty}(\wedge^{\bullet} TV), \kappa_g \wedge\ )\right)^G \longrightarrow \left(\bigoplus_{g \in G} (\Gamma^{\infty}(\wedge^{\bullet - l(g)} TV^g \otimes \wedge^{l(g)} N^g), 0\ )\right)^G$$

Let C_g be the cyclic group generated by g, which has a natural action on V. As C_g is abelian, V is decomposed into a direct sum of C_g irreducible representations. Let V^g be the subspace of all trivial C_g-representations in V, and N^g be the sum of all nontrivial irreducible Cg representations in V. Therefore, V may be expressed as $V\ g \oplus N_g$. Let us define L_3 to be the sum of L^g_3, which is given by the relation

$$L_3^g(X) = pr^g(X|_{V^g}), \quad \dots \quad (12)$$

where $X|_{Vg}$ is the restriction of $X \in \wedge \bullet T\ V$ to $\wedge \bullet T\ V\ |_{Vg}$, and pr^g projects $\wedge \bullet T\ V\ |_{Vg}$ to $\wedge \bullet^{-l(g)} T\ V^g \otimes \wedge^{l(g)} N^g$. The space $\oplus_{g\in G} (\Gamma^{\infty}(\wedge \bullet^{-1}(g) TV^g \otimes \wedge^{l(g)} N^g, 0)$ is closed under G action on $(\oplus_{g\in G} \Gamma^1(\wedge \bullet TV), \kappa_g \wedge)$ and therefore inherits a G action. Similarly, under the computations it is also verifiable that L^g_3 is G-equivariant and L_3 defines a map on the G-invariant components. Hence, we obtain that $L = L_3 \circ L_2 \circ L_1$ is a quasi-isomorphism of cochain complexes

$$L : C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow \Gamma^\infty\big(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TV^g \otimes \wedge^{l(g)} N^g\big)^G. \quad \ldots(13)$$

We now discuss its impact for generalization of computation of Hochschild cohomology of Poly(V) ⋊ G using the quasi-isomorphism L. The main thrust is on obtain the process to generalize this construction to $C^\infty(M) \rtimes G$ by defining L to be the given equation

$$L : C^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G) \longrightarrow \Gamma^\infty\big(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TM^g \otimes \wedge^{l(g)} N^g\big)^G \quad \ldots(14)$$

We observe that the map L_1 is a quasi-isomorphismfrom $C^\bullet(A\rtimes G,A\rtimes G)$ to $(C^\bullet(A,A \rtimes G))^G$ for any algebra A with a finite group action. Therefore, the map L_1 extends to the general case $C^\infty(M) \rtimes G$.

$$L_1 : C^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G) \longrightarrow \Big(C^\bullet(C^\infty(M), C^\infty(M) \rtimes G)\Big)^G \quad \ldots(15)$$

Secondly, L_3 is generalized to the manifold case, as the map is written as follows.

$$L_3 : \Big(\bigoplus_{g\in G} (\Gamma^\infty(\wedge^\bullet TM), \kappa \wedge)\Big)^G \longrightarrow \Big(\bigoplus_{g\in G} (\Gamma^\infty(\wedge^{\bullet - l(g)} TM^g \otimes \wedge^{l(g)} N^g), 0)\Big)^G.$$

which is obtained by composing the projection map $\wedge^\bullet TM|_{M^g} \rightarrow \wedge^{\bullet - l(g)} TM^g \otimes \wedge^{l(g)} N^g$ with the restriction map from $\wedge^\bullet TM$ to $\wedge^\bullet TM|_V$. Thirdly, to generalize L_2, Connes' map is taken from the Koszul resolution of $C^\infty(M)$ to its Bar resolution for a manifold with an affine structure. Here L^g_2 is defined from the Hochschild cochain complex $C^\bullet(C^\infty(M),C^\infty(M)_g)$ to $\wedge^\bullet TM$ in a similar way. The Connes construction only works for affine manifolds. Therefore, for a general manifold M, the Hochschild cochain complex of $C^\infty(M) \rtimes G$ is taken as a (pre)sheaf over the orbifold M/G, and use Cech techniques to compute the sheaf cohomology of this (pre)sheaf. L is a quasi-isomorphism of (pre)sheaves which is locally defined as the L_g.

$$HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G) = \Gamma^\infty\big(\bigoplus_{g\in G} \wedge^{\bullet - l(g)} TM^g \otimes \wedge^{l(g)} N^g\big)^G \quad \ldots(16)$$

But g is in the g-centralizer subgroup of G, the g-fixed point component's contribution to $HH^\bullet(C^\infty(M) \rtimes G,C^\infty(M) \rtimes G)$ has to be from g-invariant sections of $\Box^\infty(\wedge^{\bullet - l(g)} TV^g \otimes \wedge^{l(g)} N^g)$. As g acts on TV^g, a g-invariant section of $\wedge^{\bullet - l(g)} TV^g \otimes \wedge^{l(g)} N^g$ must have g-invariant component in $\wedge^{l(g)} N^g$. we thus find that $\wedge^{l(g)} N^g$ is a line bundle over V^g. To have a nonzero g-invariant section, it is required that the g action on $\wedge^{l(g)} N^g$ must be trivial. This implies that $\det(g|_{N^g}) = 1$. We further observe that $g|_{N^g}$'s action on N^g is of finite order. It can be diagonalized. If N^g is of odd dimension, then by the fact that $\det(g|_{N^g}) = 1$, we conclude that $g|_{N^g}$ has at least one eigenvalue equal to 1. This contradicts to the assumption of N^g. Therefore, if $\dim(N^g)$ is odd, there is no nonzero contribution to $HH^\bullet(C^\infty(M) \rtimes G,C^\infty(M) \rtimes G)$ from this g-fixed point component. Hence, the Hochschild cohomology of $C^{\infty}(M)\rtimes G$ has no contribution from g-fixed point submanifolds with odd l(g). Therefore, we conclude that

$$HH^{\bullet}(C^{\infty}(M)\rtimes G; C^{\infty}(M)\rtimes G) = \Gamma^{\infty}\Big(\bigoplus_{\substack{g\in G,\\ l(g)\text{ is even}}} \wedge^{\bullet - l(g)}TM^{g}\otimes \wedge^{l(g)}N^{g}\Big)^{G} \qquad \ldots(17)$$

The quasi-isomorphism

$$T:\Gamma^{\infty}\big(\bigoplus_{g\in G}\wedge^{\bullet-l(g)}TM^{g}\otimes\wedge^{l(g)}N^{g}\big)^{G}\longrightarrow C^{\bullet}(C^{\infty}(M)\rtimes G; C^{\infty}(M)\rtimes G), \qquad \ldots(18)$$

which is a quasi-inverse to the map L, we construct a twisted cocycle g for each element g associated to the determinant line bundle $\wedge^{l(g)}N^{g}$; (ii), the twisted cocycles is taken to construct the map T. Our main focus is to discuss the local case Poly(V)⋊G, and derive method to generalize the construction to general manifolds.

1.3 Theorem

Let C(g) be the centralizer subgroup of g, which acts on Ng. If C(g) action on N^{g} is diagonalizable[2], there is a natural construction of Ω_g such that

$$h(\Omega_g) = \det(h|_{N^g})\Omega_g, \qquad h\in C(g).$$

Proof

As C (g) action on N^{g} is diagonalizable and g commutes with elements in C(g), g and C(g) action on N^{g} are diagonalized simultaneously. Therefore, a set of coordinates $y^1, \cdot\cdot\cdot, y^{l}(g)$ on N^{g}, may be derived which are eigen functions of g and C(g) action. Let us define Ω_g using the coordinates y^i. In particular, $\tilde{y}i = g^i y^i$, and $h(\tilde{y}^i) = h(g^i y^i) = g^i h^i y^i$, where g^i and h^i are eigenvalues of Ω_g and h action on y^i. we combining here expressions of $h(\Omega_g)$, we obtain the equation

$$h(\Omega_g) = \det(h|_{N^g})\Omega_g, \qquad h\in C(g). \qquad \ldots(19)$$

Let us consider two special cases where the conditions assumed in theorem (1.2) are satisfied,

(i) Group G is abelian;

(ii) The codimension l(g) is 1 or 2. C(g) acts on N^{g} by isometry. When l(g) = 1, 2, isometry group of N^{g} is abelian.

The cohomology $HH^{\bullet}(A, A_e)$ is computed are given by

$$HH^{\bullet}(A, A_e) = \begin{cases} 0 & \bullet\neq 1\\ \mathbb{R} & \bullet = 1\end{cases}$$

Where $HH^1(A, Ae)$ is generated by Ω_e.

1.4 Theorem

Given $\xi \in \Gamma^{\infty}(\bigoplus_g \wedge^{k-l(g)} T Vg^P \otimes \wedge^{l(g)} N^g)^G$, we write $\xi = \Sigma_g X_g \otimes \wedge_g$, where Λ_g *is* defined. The composition map $L_2 \circ T_1$ satisfies the following equation

$$L_2(T_1(\xi)) = \sum_g X_g \otimes \Lambda_g = \xi.$$

Proof

It is given that L_2 is G equivariant and ξ is G invariant. Let us compute $L_2(T_1(\xi))(x)$ as follows: $L_2(T_1(\xi))(x) =$

$$= \sum_{i_1,\ldots,i_k} T_1(\xi)((x_1-x)^{i_1}\cdots(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}$$

$$= \sum_g \sum_{i_1,\ldots,i_k} T_1(X_g\otimes\Lambda_g)((x_1-x)^{i_1}\cdots(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}$$

$$= \sum_g \sum_{i_1,\ldots,i_k} X_g((x_1-x)^{i_1},\ldots,(x_{k-l(g)}-x)^{i_{k-l(g)}})$$

$$\times\, \Omega_g((x_{k-l(g)+1}-x)^{i_{k-l(g)+1}},\ldots,(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}$$

$$= \sum_g \sum_{i_1,\ldots,i_k} X_g((x_1-x)^{i_1},\ldots,(x_{k-l(g)}-x)^{i_{k-l(g)}})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(g)}}}$$

$$\otimes\, \Omega_g((x_{k-l(g)+1}-x)^{i_{k-l(g)+1}},\ldots,(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_{k-l(g)+1}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}$$

$$= \sum_g X_g\otimes\Lambda_g. \quad \ldots(20)$$

It shows $\wedge_g$ and Ω_g have the same values on linear functions. Let us define $T = T_2 \circ T_1$, and derive the following theorem.

1.5 Theorem

The map T is a quasi-isomorphism. In particular, $L \circ T = id$.

Proof

We find that $L_1(T_2) = id$ on $C^\bullet(A;A \rtimes G)^G$, and therefore have

$$L\circ T(\xi) = L_3(L_2(L_1(T_2(T_1(\xi))))) = L_3(L_2(T_1(\xi))),$$

which is equal to ξ by

Given $\xi \in \Gamma^{\infty}(\bigoplus_g \wedge^{k-l(g)} T Vg^P \otimes \wedge^{l(g)} N^g)^G$, we write $\xi = \Sigma_g X_g \otimes \wedge_g$, where Λ_g *is* defined. The composition map $L_2 \circ T_1$ satisfies the following equation

$$L_2(T_1(\xi)) = \sum_g X_g \otimes \Lambda_g = \xi.$$

1.6 Lemma

For α, β ∈ G, the condition $V^\alpha + V^\beta = V$ is equivalent to the equality that l(α) + l(β) = l(αβ). And when $V^\alpha + V^\beta = V$, $V^\alpha \cap V^\beta = V^{\alpha\beta}$.

Proof

As $\dim(V^\alpha) + \dim(V^\beta) = \dim(V^\alpha + V^\beta) + \dim(V^\alpha \cap V^\beta)$,

we have that $l(\alpha) + l(\beta) = \dim(V) - \dim(V^\alpha) + \dim(V) - \dim(V^\beta)$

$= \dim(V) - \dim(V^\alpha + V^\beta) + \dim(V) - \dim(V^\alpha \cap V^\beta)$

$\geq \dim(V) - \dim(V^\alpha + V^\beta) + \dim(V) - \dim(V^{\alpha\beta})$

$= \dim(V) - \dim(V^\alpha + V^\beta) + l(\alpha\beta)$, …(21)

where it has been considered that $V^\alpha \cap V^\beta \subset V^{\alpha\beta}$. Therefore, l(α)+l(β) = l(αβ) implies that $V = V^\alpha + V^\beta$.

On the other hand, let us assume that $V^\alpha + V^\beta = V$ and let ⟨ ,⟩ be a G invariant metric on V . For any v ∈ $V^{\alpha\beta}$, we have αβ(v) = v, and accordingly $\beta(v) = \alpha^{-1}(v)$, and $\beta(v) - v = \alpha^{-1}(v) - v$. Furthermore, as the metric ⟨ ,⟩ is G invariant, we find that β(v)−v is orthogonal to V^β with respect to the metric ⟨ ,⟩ and $\alpha^{-1}(v) - v$ is orthogonal to V^α. Therefore $\beta(v) - v = \alpha^{-1}(v) - v$ is orthogonal to $V^\alpha + V^\beta$, which is equal to V by the assumption of lemma. This implies that v must belong to $V^\alpha \cap V^\beta$, and we have $V^\alpha \cap V^\beta = V^{\alpha\beta}$. It implies that

$L(\alpha) + l(\beta) = \dim(V) – \dim(V^\alpha + V^\beta) + \dim(V) – \dim(V^{\alpha\beta}) = l(\alpha\beta)$

Hence, the theorem is proved.

References

1. Eisenhart, L. P. (1992). Riemannian Geometry, Princeton University Press, London.
2. Boucetta, M. (2003). Riemann Poisson manifolds and Kaehler – Riemann foliations C. R. Acade. Sci. Paris 336 (423-28).
3. Connes. A, (1986). Non-commutative differential geometry Publ. Math IHES, 62 (257 - 360).
4. Le, L. B. (1999). Noncommutative compact manifolds constructed from quivers. AMA Algebra Montp. Announc, Paper 1, 5 pp. (electronic).
5. Boeijink, J. and W. D. van Suijlekom,(2011). The noncommutative geometry of Yang Mills fields, *J. Geom. Phys*. 61, 1122-1134.
6. Gracia-Bondia, J. M., J. C. Varilly and H. Figueroa, (2001). Elements of Noncommutative Geometry, Birkhuauser, Boston.
7. Xu, P.(1994). Noncommutative Poisson algebras, *Am. J. Math*. 116, 101–125.
8. Connes (1994). Non-commutative Geometry, Academic Press.

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