ISSN:0976-4933
Journal of Progressive Science
A Peer-reviewed Research Journal
Vol.15, No.01&02, pp 33-45 (2024)

Hyperbolic Cosymplectic manifold with a type of (α, β) semi-symmetric non-metric connection

Pavan Kumar Maurya and Abhiram Shukla

Faculty of Mathematics and Statistical Sciences

Email: homestudy90@gmail.com and abhiramshukla.maths@srmu.ac.in

Abstract

In the present paper, we introduce a new type of semi-symmetric non-metric connection on a hyperbolic cosymplectic manifold called a type of (α, β) semi-symmetric non-metric connection. Some geometrical properties have been obtained in their paper. Finally, we construct an example of the existence of a type of semi-symmetric non-metric connection and verify our results.

Keywords and phrases: Hyperbolic cosymplectic manifold, quasi-concircualr curvature tensor, nearly Ricci recurrent, nearly recurrent, semi-symmetric non-metric connection, Curvature tensor, symmetric and skew-symmetric.

1. Introduction

The role of almost hyperbolic contact structure in differential geometry is very important. The almost hyperbolic contact structure has been studied by Upadhyay and Dubey in 1973. A large number of authors have been studied about almost hyperbolic contact metric manifold such as Sinha and Yadav (1980), Kalpana and Srivastava (1987), Chinca and Gonzalec (1990), Doğan and Karadog (2014) etc.

Let us consider on odd-dimensional complete real differential manifold M^n of dimension n with a fundamental tensor field ϕ of type (1,1), 1-form η and a vector field ξ, satisfying (Upadhyay and Dubey, 1973)

$$\phi^2 X = X + \eta(X)\xi, \;\; \eta(\phi X) = 0, \;\; \phi\xi = 0, \;\; g(X,\xi) = \eta(X) \tag{1.1}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y), \;\; \eta(\xi) = -1, rank\, \phi = n, \tag{1.2}$$

for all $X, Y \in TM$. Then M^n is called a hyperbolic contact metric manifold and the structure $\{\phi, \xi, \eta, g\}$ are called hyperbolic contact metric structure (Upadhyay and Dubey, 1973).

The 2^{nd} fundamental form F of the structure defined as

$$F(X,Y) = g(\phi X, Y), \tag{1.3}$$

also

$$g(\phi X, Y) + g(X, \phi Y) = 0, \tag{1.4}$$

i.e. $F(X,Y) + F(Y,X) = 0.$

A hyperbolic metric manifold M^n is said to be hyperbolic cosymplectic manifold (Upadhyay and Dubey, 1973; Sinha and Yadav, 1980) if

$$D_X F = 0, \quad D_X \phi = 0, \quad (D_X \eta)(Y) = 0, \quad D_X \xi = 0. \tag{1.5}$$

On the other hand, the curvature tensor have been important role in manifold M^n. With this in mind, Prasad and Maurya (2007) defined a quasi-concircular curvature tensor as follows:

$$\tilde{V}(X,Y)Z = a.R(X,Y)Z + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.6}$$

$\forall X,Y,Z \in$ TM, where r is the scalar curvature tensor and if $a = 1$, $b = -\frac{1}{n-1}$, then eq^n (1.6) i.e. quasi-concircular curvature tensor reduces to

$$\tilde{V}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.7}$$

$$= V(X,Y)Z,$$

which is known as concircular curvature tensor (Yano and Kon, 1984). The quasi-concircualr curvature tensor was studied by many authors such as Prasad and Yadav (2017), Ahmad et al. (2019), Prasad and Yadav (2023), Prasad and Yadav (2024) and many others.

Recently, Prasad and Yadav (2021) introduced a new type of non-flat Ricci recurrent Riemannian manifold whose Ricci tensor S satisfies the condition:

$$(D_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + B(X)g(Y,Z), \tag{1.8}$$

$\forall X,Y,Z \in$ TM, where D denotes the operator of covariant differentiation with respect to metric tensor g and two non-zero 1-form defined as

$$A(X) = g(X,\rho_1) \text{ and } B(X) = g(X,\rho_2), \tag{1.9}$$

and denoted by $N\{R(R_n)\}$ them. The name nearly Ricci recurrent Riemannian manifold was chosen because if $B = 0$ in (1.8) then the manifold reduces to a Ricci recurrent manifold as follows:

$$(D_X S)(Y,Z) = A(X)S(Y,Z), \tag{1.10}$$

which is very close to Ricci recurrent space. This justified the name "Nearly Ricci recurrent manifold" for the manifold defined by (1.8) and the use of the symbol $N\{R(R_n)\}$ for it. Further, only one author, Yadav and Prasad (2023) and others.

Continuing this study, Prasad and Yadav (2023) introduced a nearly recurrent Riemannian manifold which is defined as

$$(D_U R)(X,Y)Z = [A(U) + B(U)]R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.11}$$

$\forall X,Y,Z \in$ TM, where A and B are two non-zero 1-forms and denoted by $(NR)_n$. The name of nearly recurrent was chosen if $B = 0$ in (1.11), then the manifold reduces to recurrent (Ruse, 1946)

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z,$$

which is very closed to recurrent space. This is justified the name "Nearly recurrent manifold" defined by (1.11) and use of the symbol $(NR)_n$. The nearly recurrent manifold have been studied by Yadav and Prasad (2023) and Prasad and Yadav (2024).

Let $\bar{D}$ be a linear connection in a Riemannian manifold M^n. The torsion tensor $\bar{T}$ is given by

$$\bar{T}(X,Y) = \bar{D}_X Y - \bar{D}_Y X - [X,Y].$$

The connection $\bar{D}$ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric. The connection $\bar{D}$ is a metric connection if there is a Riemannian metric g in M^n such that $\bar{D}_X g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In 1924, Friedmann and Schouten introduced the idea of a semi-symmetric linear connection. A linear connection $\bar{D}$ is said to be a semi-symmetric connection if its torsion tensor $\bar{T}$ is of the form

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

$\forall\, X, Y \in$ TM.

Golab introduced the idea of a quarter symmetric linear connection in a differentiable manifold (1925). A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\bar{T}$ is of the form

$$\bar{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is (1,1) tensor field. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter symmetric metric connection reduces to the semi-symmetric metric connection (Friedmann and Schouten, 1924).

In 1970, Yano studied semi-symmetric metric connection on a Riemannian manifold as

$$\bar{D}_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi,$$

whose torsion tensor and metric are

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

and $$(\bar{D}_X g)(Y,Z) = 0.$$

Recently, in 2011, Prasad, Dubey and Yadav studied semi-symmetric non-metric connection on a Riemannian manifold as

$$\bar{D}_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi + u(X)Y,$$

where η and u are two 1-forms whose torsion tensor and metric are

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y + u(X)Y - u(Y)X,$$

and $$(\bar{D}_X g)(Y,Z) = -2u(X)g(Y,Z).$$

The semi-symmetric non-metric connection have been studied by many authors such as Melhotra (2012), Barman (2015), Wu and Wang (2021) and others.

In recent paper De and Sengupta (2000) studied a quarter symmetric metric connection on a Sasakian manifold as

$$\bar{D}_X Y = D_X Y + \eta(X)\phi Y.$$

The quarter symmetric metric connection have been developed by several authors such as Srivastava, Sharma and Prasad (2008), Kumar, Bagewadi and Venkatesha (2011), Haseeb (2015) etc. On the other hand quarter symmetric non-metric connection have been studied by several authors such as Dwivedi (2011), Mondal (2012), Patra and Bhattacharyya (2013), Yadav and Prasad (2023) etc.

The motivation of above ideas, we define a new type of semi-symmetric non-metric connection which are both combination of semi-symmetric non-metric connection and quarter symmetric non-metric connection on hyper cosymplectic manifold as follows:

$$\bar{D}_X Y = D_X Y + \alpha[\eta(X)Y - g(X,Y)\xi] + \beta[\eta(X)\phi Y - g(\phi X, Y)\xi],$$

$\forall\, X, Y \in$ TM. Such a linear connection is called "A type of (α,β) semi-symmetric non-metric connection" whose torsion tensor and metric are

$$\bar{T}(X,Y) = \alpha[\eta(X)Y - \eta(Y)X] + \beta[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi],$$

and

$$(\bar{D}_X g)(Y,Z) = \alpha[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) - 2\eta(X)g(Y,Z)] + \beta[\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)].$$

2. A type of $(\boldsymbol{\alpha},\boldsymbol{\beta})$ semi symmetric non-metric connection

Let M^n be a hyperbolic cosymplectic manifold with Levi-Civita connection D, we define a linear connection $\bar{D}$ on M^n by

$$\bar{D}_X Y = D_X Y + \alpha[\eta(X)Y - g(X,Y)\xi] + \beta[\eta(X)\phi Y - g(\phi X, Y)\xi], \tag{2.1}$$

$\forall\, X, Y \in$ TM,

$$g(X,\xi) = \eta(X). \tag{2.2}$$

Using (2.1), the torsion tensor $\bar{T}$ on M^n w.r.t. the connection $\bar{D}$ is given by

$$\bar{T}(X,Y) = \bar{D}_X Y - \bar{D}_Y X - [X,Y]$$
$$= \alpha[\eta(X)Y - \eta(Y)X] + \beta[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{2.3}$$

A linear connection satisfying (2.3) is called semi-symmetric connection.

Again, using (1.1), (1.2), (1.4), (1.5) and (2.1), we get

$$(\bar{D}_X g)(Y,Z) = Xg(Y,Z) - g(\bar{D}_X Y, Z) - g(Y, \bar{D}_X Z),$$

which gives

$$(\bar{D}_X g)(Y,Z) = \alpha[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) - 2\eta(X)g(Y,Z)] + \beta[\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)]. \tag{2.4}$$

A linear connection $\bar{D}$ defined by (2.1) satisfying (2.3) and (2.4) is called a type of (α,β) semi-symmetric non-metric connection. Conversely, we will show that a linear connection $\bar{D}$ defined by on M^n satisfying (2.3) and (2.4) is given by (2.1).

Let $\bar{D}$ is a linear connection M^n given by

$$\bar{D}_X Y = D_X Y + H(X,Y). \tag{2.5}$$

Now, we shall determine the tensor H s.t. $\bar{D}$ satisfies (2.3) and (2.4). In view of (2.5), we get

$$\bar{T}(X,Y) = H(X,Y) - H(Y,X). \tag{2.6}$$

We have

$$(\bar{D}_X g)(Y,Z) = Xg(Y,Z) - g(\bar{D}_X Y,Z) - g(Y,\bar{D}_X Z). \tag{2.7}$$

In view of (2.4), (2.5) and (2.7), we get

$$g(H(X,Y),Z) + g(H(Z,X),Y) = -\alpha[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) - 2\eta(X)g(Y,Z)] \\ -\beta[\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)]. \tag{2.8}$$

From (2.5), (2.6) and (2.8), we get

$$g(\bar{T}(X,Y),Z) + g(\bar{T}(Z,X),Y) + g(\bar{T}(Z,Y),X) = 2g(H(X,Y),Z) - [gH(X,Y),Z) + \\ g(H(X,Z),Y)] - [g(H(Y,X),Z) + \\ g(H(Y,Z),X)] + [g(H(Z,Y),X) + \\ g(H(Z,X),Y)]. \tag{2.9}$$

In view of (2.9), we get

$$H(X,Y) = \frac{1}{2}[\,\bar{T}(X,Y) + '\bar{T}(X,Y) + '\bar{T}(Y,X)] + \alpha[\eta(X)Y + \eta(Y)X - 2g(X,Y)\xi] \\ +\beta[-\eta(Y)\phi X - \eta(X)\phi Y], \tag{2.10}$$

where $'\bar{T}$ be the tensor of type (1, 2) defined by

$$g('\bar{T}(X,Y),Z) = g(\bar{T}(Z,X),Y) = \alpha[\eta(Z)g(X,Y) - \eta(X)g(Z,Y)] + \beta[\eta(Z)g(\phi X,Y) \\ -\eta(X)g(\phi Z,Y) - 2\eta(Y)g(\phi Z,X). \tag{2.11}$$

In view of (2.10) and (2.11), we get

$$H(X,Y) = \alpha[\eta(X)Y - g(X,Y)\xi] + \beta[\eta(X)\phi Y - g(\phi X,Y)\xi].$$

This implies that

$$\bar{D}_X Y = D_X Y + \alpha[\eta(X)Y - g(X,Y)\xi] + \beta[\eta(X)\phi Y - g(\phi X,Y)\xi].$$

Hence, we have the following theorem:

Theorem 2.1: Let M^n be a hyperbolic cosymplectic manifold with hyperbolic cosymplectic structure $\{\phi,\xi,\eta,g\}$ a type of (α,β) semi-symmetric non-metric connection which satisfies (2.3) and (2.4). Then a type of (α,β) semi-symmetric non-metric connection is given by

$$\bar{D}_X Y = D_X Y + \alpha[\eta(X)Y - g(X,Y)\xi] + \beta[\eta(X)\phi Y - g(\phi X,Y)\xi].$$

3. Curvature tensor of hyperbolic cosymplectic manifold with respect to a type of $(\boldsymbol{\alpha},\boldsymbol{\beta})$ semi-symmetric non-metric connection $\bar{\boldsymbol{D}}$

Let R and $\bar{R}$ be the curvature tensor of the connection D and $\bar{D}$ respectively, then

$$\bar{R}(X,Y)Z = \bar{D}_X\bar{D}_Y Z - \bar{D}_Y\bar{D}_X Z - \bar{D}_{[X,Y]}Z. \tag{3.1}$$

In view of (2.1) and (3.1), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha[(D_X\eta)(Y)Z - (D_Y\eta)(X)Z - g(Y,Z)D_X\xi +$$

$$g(X,Z)D_Y\xi] + \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi + \beta[(D_X\eta)(Y)\phi Z - (D_Y\eta)(X)\phi Z + \eta(Y)(D_X\phi)(Z) - \eta(X)(D_Y\phi)(Z) - g\big((D_X\phi)Y,Z\big)\xi + g\big((D_Y\phi)X,Z\big)\xi - g(\phi Y,Z)D_X\xi + g(\phi X,Z)D_Y\xi] + \beta^2[\{g(\phi Y,Z)g(\phi X,\xi) - g(\phi X,Z)g(\phi Y,\xi)\}\xi - \{g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)\}\phi\xi + \{g(\phi Y,\phi Z)\eta(X) - g(\phi X,\phi Z)\eta(Y)\}\xi] + \alpha\beta[\{g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)\}\xi + \{g(Y,Z)g(\phi X,\xi) - g(X,Z)g(\phi Y,\xi)\}\xi + \{g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)\}\xi - \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi\xi]. \tag{3.2}$$

In view of (1.1)-(1.5) and (3.2), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z + (\alpha^2 - \beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi. \tag{3.3}$$

Hence, we have the following theorem:

Theorem 3.1: The curvature tensor $\bar{R}(X,Y)Z$ hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\bar{D}$ is given by (3.3).

Now, let $\bar{R}(X,Y)Z = 0$, then eq^n (3.3) will be

$$R(X,Y)Z = (\alpha^2 - \beta^2)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi. \tag{3.4}$$

Hence, we have the following theorem:

Theorem 3.2: If the curvature tensor of $\bar{R}(X,Y)Z$ hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\bar{D}$ vanishes, then the curvature tensor of $R(X,Y)Z$ of Levi-Civita connection D is constant curvature.

In view of (3.3), we get

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,X,Z,W) = 0, \tag{3.5}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$.

Also,

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(X,Y,W,Z) = (\alpha^2 - \beta^2)[\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W) + \{\eta(Y)g(X,W) - \eta(X)g(Y,W)\}\eta(Z)], \tag{3.6}$$

$$'\bar{R}(X,Y,Z,W) - '\bar{R}(Z,W,X,Y) = (\alpha^2 - \beta^2)[\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W) - \{\eta(Z)g(W,X) - \eta(W)g(Z,X)\}\eta(Y)], \tag{3.7}$$

and

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,Z,X,W) + '\bar{R}(Z,X,Y,W) = 0, \tag{3.8}$$

$\forall\, X,Y \in$ TM.

Hence, we have the following theorem:

Theorem 3.3: The curvature tensor $\bar{R}(X,Y)Z$ hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ satisfies the following:

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,X,Z,W) = 0,$$

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(X,Y,W,Z) = (\alpha^2-\beta^2)[\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}\eta(W) + \{\eta(Y)g(X,W)-\eta(X)g(Y,W)\}\eta(Z)],$$

$$'\bar{R}(X,Y,Z,W) - '\bar{R}(Z,W,X,Y) = (\alpha^2-\beta^2)[\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}\eta(W) - \{\eta(Z)g(W,X)-\eta(W)g(Z,X)\}\eta(Y)],$$

and

$$'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,Z,X,W) + '\bar{R}(Z,X,Y,W) = 0.$$

On contraction of (3.3), we get

$$\bar{S}(Y,Z) = S(Y,Z) - (\alpha^2-\beta^2)[g(Y,Z)-\eta(Y)\eta(Z)], \tag{3.9}$$

and

$$\bar{r} = r - (\alpha^2-\beta^2)(n-1). \tag{3.10}$$

In view of (3.8), we get

$$\bar{S}(Y,Z) - \bar{S}(Z,Y) = 0.$$

Hence, we have the following theorem:

Theorem 3.4: The Ricci tensor $\bar{S}$ hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ is symmetric.

4. ξ −Quasi-concircularly flat

The quasi-concircular curvature tensor of the manifold of the type (1,3) with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ is given by

$$\bar{\tilde{V}}(X,Y)Z = a.\bar{R}(X,Y)Z + \frac{\bar{r}}{n}\left(\frac{a}{n-1}+2b\right)[g(Y,Z)X - g(X,Z)Y]. \tag{4.1}$$

Putting $Z=\xi$ in (4.1), we get

$$\bar{\tilde{V}}(X,Y)\xi = a.\bar{R}(X,Y)\xi + \frac{\bar{r}}{n}\left(\frac{a}{n-1}+2b\right)[g(Y,\xi)X - g(X,\xi)Y]. \tag{4.2}$$

In view of (3.3), (3.9), (3.10), (3.10) and (4.2), we get

$$\bar{\tilde{V}}(X,Y)\xi = \tilde{V}(X,Y)\xi - \left(\frac{n-1}{n}\right)(\alpha^2-\beta^2)\left(\frac{a}{n-1}+2b\right)[\eta(Y)X - \eta(X)Y].$$

Hence, we have the following theorem:

Theorem 4.1: A n-dimensional hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ is $\xi-$ quasi-concircularly flat if and only if

$$\tilde{V}(X,Y)\xi = \left(\frac{n-1}{n}\right)(\alpha^2-\beta^2)\left(\frac{a}{n-1}+2b\right)[\eta(Y)X - \eta(X)Y].$$

5. Nearly recurrent hyperbolic cosymplectic manifold

Definition 5.1: A hyperbolic cosymplectic manifold is called nearly recurrent hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\bar{D}$, if its curvature $\bar{R}$ of $\bar{D}$ satisfies the condition:

$$(\bar{D}_X\bar{R})(U,Y)Z = [A(X) + B(X)]\bar{R}(U,Y)Z + B(X)[g(Y,Z)U - g(U,Z)Y]. \tag{5.1}$$

On contraction of (5.1), we have

$$(\bar{D}_X\bar{S})(Y,Z) = [A(X) + B(X)]\bar{S}(Y,Z) + (n-1)B(X)g(Y,Z). \tag{5.2}$$

In view of (3.9) and (5.2), we get

$$(\bar{D}_X\bar{S})(Y,Z) = [A(X) + B(X)][S(Y,Z) - (\alpha^2 - \beta^2)\{g(Y,Z) - \eta(Y)\eta(Z)\}] + (n-1)B(X)g(Y,Z). \tag{5.3}$$

We have

$$(\bar{D}_X\bar{S})(Y,Z) = X\bar{S}(Y,Z) - \bar{S}(\bar{D}_XY,Z) - \bar{S}(Y,\bar{D}_XZ). \tag{5.4}$$

In view of (1.5), (2.1), (3.8) and (5.4), we get

$$\begin{aligned}(\bar{D}_X\bar{S})(Y,Z) = {}& (D_XS)(Y,Z) - \alpha[2S(Y,Z)\eta(X) - g(X,Y)S(\xi,Z)] + \\ & \beta[\{S(\phi Y,Z) + S(Y,\phi Z)\}\eta(X) - g(\phi X,Y)S(\xi,Z) - \\ & g(X,Z)\eta(Y)] + (\alpha^2-\beta^2)[\alpha\{2g(Y,Z)\eta(X) - g(X,Y)\eta(Z) - \\ & g(X,Z)\eta(Y)\} - \beta\{g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)\}].\end{aligned} \tag{5.5}$$

From (5.3) and (5.5), we get

$$\begin{aligned}(D_XS)(Y,Z) = {}& [\{A(X) + B(X)\}S(Y,Z) + (n-1)B(X)g(Y,Z)] - \\ & (\alpha^2-\beta^2)\{A(X) + B(X)\}[\{g(Y,Z) - \eta(Y)\eta(Z)\} + \\ & \alpha\{2g(Y,Z)\eta(X) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\} - \\ & \beta\{g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)\}] + \\ & \alpha[2S(Y,Z)\eta(X) - g(X,Y)S(\xi,Z)] - \\ & \beta[\{S(\phi Y,Z) + S(Y,\phi Z)\}\eta(X) - \\ & g(\phi X,Y)S(\xi,Z) - g(X,Z)\eta(Y)].\end{aligned} \tag{5.6}$$

Hence, we have the following theorem:

Theorem 5.1: A nearly recurrent hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\bar{D}$ is nearly Ricci recurrent if and only if

$$\begin{aligned}& (\alpha^2-\beta^2)\{A(X) + B(X)\}[\{g(Y,Z) - \eta(Y)\eta(Z)\} + \\ & \alpha\{2g(Y,Z)\eta(X) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\} - \\ & \beta\{g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)\}] - \alpha[2S(Y,Z)\eta(X) - g(X,Y)S(\xi,Z)] + \\ & \beta[\{S(\phi Y,Z) + S(Y,\phi Z)\}\eta(X) - g(\phi X,Y)S(\xi,Z) - g(X,Z)\eta(Y)] = 0.\end{aligned}$$

In view of (3.3) and (5.1), we get

$$(\bar{D}_U\bar{R})(X,Y)Z = [A(U)+B(U)][R(X,Y)Z+(\alpha^2-\beta^2)\{\eta(X)g(Y,Z)- \eta(Y)g(X,Z)\}\xi] + B(U)[g(Y,Z)X-g(X,Z)Y]. \quad (5.7)$$

We have

$$(\bar{D}_U\bar{R})(X,Y)Z = \bar{D}_U\bar{R}(X,Y)Z - \bar{R}(\bar{D}_UX,Y)Z - \bar{R}(X,\bar{D}_UY)Z - \bar{R}(X,Y)\bar{D}_UZ. \quad (5.8)$$

In view of (1.1)-(1.5) , (2.1), (3.3) and (5.8), we get

$$\begin{aligned}(\bar{D}_U\bar{R})(X,Y)Z = {} & (D_UR)(X,Y)Z + \alpha[g(U,X)R(\xi,Y)Z + \\ & g(U,Y)R(\xi,X)Z + g(U,Z)R(X,Y)\xi - 2\eta(U)R(X,Y)Z] + \\ & \beta[\{g(\phi U,X)R(\xi,Y)Z + g(\phi U,Y)R(\xi,X)Z + \\ & g(\phi U,Z)R(X,Y)\xi\} - \{R(\phi X,Y)Z + R(X,\phi Y)Z\}\eta(U)] + \\ & \alpha(\alpha^2-\beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\eta(U)\xi - \\ & \{\eta(U)\eta(X) + g(U,X)\}g(Y,Z)\xi - \{2\eta(U)g(Y,Z) - \\ & g(U,Y)\eta(Z) - g(U,Z)\eta(Y)\}\eta(X)\xi + \{g(X,Z)\eta(U) - \\ & g(U,Z)\eta(X)\}\eta(Y)\xi + \{\eta(U)\eta(Y) + g(U,Y)\}g(X,Z)\xi + \\ & \beta(\alpha^2-\beta^2)[\{g(\phi U,X)\eta(Z) + g(\phi U,Z)\eta(X)\}\eta(Y)\xi + \\ & \{g(\phi U,Z)\eta(Y) + g(\phi U,Y)\eta(Z)\}\eta(X)\xi + g(\phi U,Y)g(X,Z)\xi]. \quad (5.9)\end{aligned}$$

From (5.7) and (5.9), we get

$$\begin{aligned}& (D_UR)(X,Y)Z + \alpha[g(U,X)R(\xi,Y)Z + g(U,Y)R(\xi,X)Z + g(U,Z)R(X,Y)\xi \\ & -2\eta(U)R(X,Y)Z] + \beta[\{g(\phi U,X)R(\xi,Y)Z + g(\phi U,Y)R(\xi,X)Z + \\ & g(\phi U,Z)R(X,Y)\xi\} - \{R(\phi X,Y)Z + R(X,\phi Y)Z\}\eta(U)] + \\ & \alpha(\alpha^2-\beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\eta(U)\xi - \\ & \{\eta(U)\eta(X) + g(U,X)\}g(Y,Z)\xi - \{2\eta(U)g(Y,Z) - \\ & g(U,Y)\eta(Z) - g(U,Z)\eta(Y)\}\eta(X)\xi + \{g(X,Z)\eta(U) - \\ & g(U,Z)\eta(X)\}\eta(Y)\xi + \{\eta(U)\eta(Y) + g(U,Y)\}g(X,Z)\xi + \\ & \beta(\alpha^2-\beta^2)[\{g(\phi U,X)\eta(Z) + g(\phi U,Z)\eta(X)\}\eta(Y)\xi + \\ & \{g(\phi U,Z)\eta(Y) + g(\phi U,Y)\eta(Z)\}\eta(X)\xi + g(\phi U,Y)g(X,Z)\xi] \\ & = [A(U)+B(U)][R(X,Y)Z + (\alpha^2-\beta^2)\{\eta(X)g(Y,Z) - \\ & \eta(Y)g(X,Z)\}\xi] + B(U)[g(Y,Z)X - g(X,Z)Y]. \quad (5.10)\end{aligned}$$

If U, X, Y and Z are orthogonal to ξ, then (5.10) will be

$$\begin{aligned}(D_UR)(X,Y)Z = {} & [A(U)+B(U)]R(X,Y)Z + \\ & B(U)[g(Y,Z)X - g(X,Z)Y] - \alpha[g(U,X)R(\xi,Y)Z +\end{aligned}$$

$$g(U,Y)R(\xi,X)Z + g(U,Z)R(X,Y)\xi - 2\eta(U)R(X,Y)Z] -$$
$$\beta[\{g(\phi U,X)R(\xi,Y)Z + g(\phi U,Y)R(\xi,X)Z +$$
$$g(\phi U,Z)R(X,Y)\xi\}] - \alpha(\alpha^2-\beta^2)[g(U,X)g(Y,Z) +$$
$$g(U,Y)g(X,Z)]\xi - \beta(\alpha^2-\beta^2)g(\phi U,Y)g(X,Z)\xi. \quad (5.11)$$

Hence, we have the following theorem:

Theorem 5.2: A nearly recurrent hyperbolic cosmplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ is locally nearly recurrent if and only if

$$\alpha[g(U,X)R(\xi,Y)Z + g(U,Y)R(\xi,X)Z + g(U,Z)R(X,Y)\xi - 2\eta(U)R(X,Y)Z] +$$
$$\beta[\{g(\phi U,X)R(\xi,Y)Z + g(\phi U,Y)R(\xi,X)Z + g(\phi U,Z)R(X,Y)\xi\}] +$$
$$\alpha(\alpha^2-\beta^2)[g(U,X)g(Y,Z) + g(U,Y)g(X,Z)]\xi + \beta(\alpha^2-\beta^2)g(\phi U,Y)g(X,Z)\xi = 0.$$

6. Nearly Ricci recurrent hyperbolic cosymplectic manifold

Definition 5.1: A hyperbolic cosymplectic manifold is called nearly Ricci recurrent hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$, if its Ricci tensor $\overline{S}$ of $\overline{D}$ satisfies the condition:

$$(\overline{D}_X\bar{S})(Y,Z) = [A(X) + B(X)]\bar{S}(Y,Z) + B(X)g(Y,Z). \quad (6.1)$$

In view of (3.9) and (6.1), we get

$$(\overline{D}_X\bar{S})(Y,Z) = [A(X) + B(X)]S(Y,Z) + B(X)g(Y,Z) -$$
$$(\alpha^2-\beta^2)\{g(Y,Z) + \eta(Y)\eta(Z)\}]. \quad (6.2)$$

From (5.5) and (6.2), we get

$$(D_XS)(Y,Z) = [A(X) + B(X)]S(Y,Z) + B(X)g(Y,Z) -$$
$$(\alpha^2-\beta^2)\{g(Y,Z) + \eta(Y)\eta(Z)\}] +$$
$$\alpha[2S(Y,Z)\eta(X) - g(X,Y)S(\xi,Z)] -$$
$$\beta[\{S(\phi Y,Z) + S(Y,\phi Z)\}\eta(X) - g(\phi X,Y)S(\xi,Z) -$$
$$g(X,Z)\eta(Y)] - (\alpha^2-\beta^2)[\alpha\{2g(Y,Z)\eta(X) - g(X,Y)\eta(Z) -$$
$$g(X,Z)\eta(Y)\} + \beta\{g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)\}]. \quad (6.3)$$

Theorem 6.1: A nearly recurrent hyperbolic cosymplectic manifold with respect to a type of (α,β) semi-symmetric non-metric connection $\overline{D}$ is locally nearly recurrent if and only if

$$(\alpha^2-\beta^2)\{g(Y,Z) + \eta(Y)\eta(Z)\}] + \alpha[2S(Y,Z)\eta(X) - g(X,Y)S(\xi,Z)] -$$
$$\beta[\{S(\phi Y,Z) + S(Y,\phi Z)\}\eta(X) - g(\phi X,Y)S(\xi,Z) - g(X,Z)\eta(Y)]$$
$$-(\alpha^2-\beta^2)[\alpha\{2g(Y,Z)\eta(X) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\} +$$
$$\beta\{g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)\}] = 0.$$

7. Example

Let us consider a 3-dimensional manifold $M^3 = \{(x,y,z)\epsilon R^3, z \neq 0\}$,where $\{x,y,z\}$ are standard co-ordinate in R^3.

We choose the vector fields

$$e_1 = e^{kz}\frac{\partial}{\partial x}, e_2 = e^{z}\frac{\partial}{\partial y} \text{ and } e_3 = \frac{\partial}{\partial z}, \tag{7.1}$$

which are linearly independent at each point of M^3, where k is any real numbers.

Let g be the Riemannian metric defined by

$$g(e_1,e_1) = 1, \;\; g(e_2,e_1) = 1, \;\; g(e_3,e_3) = -1. \tag{7.2}$$

Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for $U \in \chi(M^3)$. Let ϕ be the tensor field of type (1,1) defined by

$$\phi e_1 = e_2, \;\; \phi e_2 = -e_1, \;\; \phi e_3 = 0, \tag{7.3}$$

$$\eta(e_3) = -1, \;\; \phi^2 U = U + \eta(U)\xi, \;\; g(\phi U, \phi W) = -g(U,W) - \eta(U)\eta(W), \tag{7.4}$$

for $U, W \in \chi(M^3)$. Thus, $e_3 = \xi$, (ϕ, ξ, η, g) defined hyperbolic contact metric structure on M.

Let D be the Levi-Civita connection with respect to metric g. Then from (7.1), we have

$$[e_1,e_2] = 0, \;\; [e_1,e_3] = -e_2, [e_2,e_3] = -ke_1. \tag{7.5}$$

The Riemannian connection D of the metric g is given by

$$2g(D_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \tag{7.6}$$

which is known as Kaszul's formula. Using (7.2) and (7.5) in (7.6), we get

$$\left.\begin{array}{lll} D_{e_1}e_1 = -ke_3, & D_{e_1}e_2 = 0, & D_{e_1}e_3 = -ke_1, \\ D_{e_2}e_1 = 0, & D_{e_2}e_2 = -e_3, & D_{e_2}e_3 = e_2, \\ D_{e_3}e_1 = 0, & D_{e_3}e_2 = 0, & D_{e_3}e_3 = 0. \end{array}\right\} \tag{7.7}$$

We know $D_{e_3}e_3 = -\phi e_3 = 0$.

Also,

$$\left.\begin{array}{lll} \bar{D}_{e_1}e_1 = -(k+\alpha)e_3, & \bar{D}_{e_1}e_2 = 0, & \bar{D}_{e_1}e_3 = -ke_1, \\ \bar{D}_{e_2}e_1 = 0, & \bar{D}_{e_2}e_2 = -(1+\alpha)e_3, & \bar{D}_{e_2}e_3 = e_2, \\ \bar{D}_{e_3}e_1 = -(\alpha e_1 + \beta e_3), & \bar{D}_{e_3}e_2 = (-\alpha e_2 + \beta e_2), & \bar{D}_{e_3}e_3 = 0. \end{array}\right\} \tag{7.8}$$

From (7.8), we have

$$\bar{T}(e_1,e_3) = \alpha e_1 + \beta e_2 \neq 0 \;\; \& \; \alpha[\eta(e_1)e_3 - \eta(e_3)e_1] + \beta[\eta(e_1)\phi e_3 - \eta(e_3)\phi e_1] = \alpha e_1 + \beta e_2.$$

This show that the linear connection $\bar{D}$ defined as (2.1) is a semi-symmetric connection on (M^3, g). Also

$$(\bar{D}_{e_1}g)(e_1,e_3) = -\alpha \neq 0, (\bar{D}_{e_2}g)(e_1,e_3) = \beta \neq 0 \text{ and } (\bar{D}_{e_3}g)(e_1,e_3) = -\alpha + \beta \neq 0.$$

Hence, the above show that the semi-symmetric connection $\bar{D}$ is non-metric connection. This verifies Theorem (2.1).

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Received on 12.04.2024 and accepted on 26.06.2024