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hv-recurrent Finsler Connections with Deflection and Torsion

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Abstract

The purpose of the present paper is to determine a Finsler connection which is nither h –metrical nor v-matrical. The presented paper we determine only those Finsler connection whose torsion tensor field S vanishes.

1.Introduction

In 1934 E. Cartan (Motsumoto, 1986) published his monograph 'Les especes de Finsler' and fixed his method to define a notion of connection in the geometry of Finsler spaces. In 1966 his method was reconsidered by M. Matsumoto (Hashiguchi, 1969) and determined uniquely the Cartan's connection by assuming four elegent axioms

(1) The connection is metrical,
(2) The deflection tensor field vanishes,
(3) The torsion tensor field T vanishes,
(4) The torsion tensor field S vanishes

Hashiguchi (1969) replaced the condition (Carton, 1934) by some weaker condition and determined a Finsler connection with the given deflection tensor field. In 1975 (Hashiguchi, 1969) he also determined uniquely a Finsler connection by replacing the condition (Hashiguchi, 1969). In almost all these works it has been assumed that the connection is metrical so that covariant differentiation commutes with the raising and lowering of the indices. Prasad, *et al*., (1990) determine a Finsler connection which is h-recurrent but v-metrical that is the hv-covariant derivative of the metric tensor is recurrent and v□covariant derivative of metric tensor vanishes. The purpose of the present paper is to determine a Finsler connection which is neither h□metrical nor vh metrical. A Finsler connection will be called hv-recurrent Finsler connection if the h and v covariant derivatives of the metric tensor is recurrent with respect to two different covariant vector fields. In this paper we determine only those Finsler connection whose torsion tensor field S vanishes.

2.Fundamental Formulae

A Finsler manifold $F^n = (M^n, L)$ of dimension n is a manifold M^n associated with a fundamental function L(x, y) where x (= x^i) denote positional variable of F^n and y (= y^i) denote the components of a tangent vector with respect to x^i. The metric tensor of F^n is given by $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$

A Finsler connection of F^n is a triad (F^i_{jk}, N^i_k, C^i_{jk}) of a v-connection F^i_{jk}, a non- linear connection N^i_k and v-connection C^i_{jk} (Motsumoto,1970). If a Finsler connection is given, the h and v covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j|k} = d_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk} \quad (2.1)$$

$$V^i_j|_k = \dot{\partial}_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk} \quad (2.2)$$

where $d_k = \partial_k - N^m_k \dot{\partial}_m$, $\partial_k = \frac{\partial}{\partial x^k}$

For any Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) we have five torsion tensor and three curvature tensors which are given by

(h) h-torsion: $T^i_{jk} = F^i_{jk} - F^i_{kj}$ (2.3)

(v) v-torsion: $S^i_{jk} = C^i_{jk} - C^i_{kj}$ (2.4)

(h) hv-torsion: C^i_{jk} = as the connection C^i_{jk} (2.5)

(v) h-torsion: $R^i_{jk} = d_k N^i_j - d_j N^i_k$ (2.6)

(v) hv-torsion: $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$ (2.7)

h-curvature: $R^i_{hjk} = d_k F^i_{hj} - d_j F^i_{hk} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj} + C^i_{hm} R^m_{jk}$ (2.8)

hv- curvature: $P^i_{hjk} = \dot{\partial}_k F^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk}$ (2.9)

v-curvature: $S^i_{hjk} = C^i_{hj} C^i_{mk} - C^m_{hk} C^i_{mj} + \dot{\partial}_k C^i_{hj} - \dot{\partial}_j C^i_{hk}$. (2.10)

The deflection tensor field D^i_k of a Finsler connection is given by

$$D^i_k = y^j F^i_{jk} - N^i_k. \quad (2.11)$$

When a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection. We shall use Cartan's connection which will be denoted by (Γ^{*i}_{jk}, G^i_k, g^i_{jk}) this connection is uniquely determined from the metric function L by the following five axioms:

(C_1) The connection is h-metrical i.e., $g_{ij|k} = 0$,

(C_2) The connection is u metrical i.e., $g_{ij}|_k = 0$,

(C_3) The Deflection tensor field D^i_k vanishes,

(C_4) The torsion tensor field T^i_{jk} vanishes,

(C_5) The torsion tensor field S^i_{jk} vanishes,

and are given by (Motsumoto,1966)

$$\Gamma^{*i}_{jk} = \frac{1}{2} g^{ih} (d_k g_{jh} + d_j g_{kh} - d_h g_{jk}) \tag{2.12}$$

$$\text{(a)} \quad G^i_k = \dot{\partial}_k G^i \qquad \text{(b)} \quad G^i = \frac{1}{2} Y^i_{00} \tag{2.13}$$

$$g^i_{jk} = g^{ih} g_{jhk}, \quad g_{jhk} = \frac{1}{2} \dot{\partial}_h g_{jk} = \frac{1}{4} \dot{\partial}_h \dot{\partial}_j \dot{\partial}_k L^2 \tag{2.14}$$

where

$$Y^i_{jk} = \frac{1}{2} g^{ih} (\partial_k g_{jh} + \partial_j g_{hk} - \partial_h g_{jk}) \tag{2.15}$$

is the Christoffel symbol of F^n and '0' denote contraction with y^i.

In this paper we replace the four conditions (C_1), (C_2), (C_3) and (C_5) and investigate general Finsler connections with given deflection tensor field and torsion tensor field which are hv-recurrent as well as vh-recurrent that is

$$\text{(a)} \quad g_{ij|k} = a_k g_{ij} \qquad \text{(b)} \quad g_{ij}|_k = b_k g_{ij} \tag{2.16}$$

where a_k and b_k are components of a given covariant vector fields in F^n.

3.hv-recurrent connections with deflection and torsion

Since our Finsler connection is h-recurrent as well as vh-recurrent we have to notice that some formulae have the style different from the ones familiar to us. For example the h□recurrent, $g_{ij|k} = a_k g_{ij}$, of the metric tensor g_{ij} gives the formula

$$g_{ijk|l} = g^h_{ik|l} g_{hj} + a_l g_{ijk} \tag{3.1}$$

and v-recurrent, $g_{ij}|_k = b_k g_{ij}$, of the metric tensor g_{ij} gives the formula

$$g_{ijk}|_l = g^h_{ik}|_l g_{hj} + b_l g_{ijk} . \tag{3.2}$$

Theorem (3.1). The vertical connection of the Finsler space F^n is uniquely determined from the metric function L (x, y) by the following axioms:

$g_{ij}|_k = b_k g_{ij}$, with given covariant vector field b_k, (1)

$S^i_{jk} = 0$. (2)

Proof: From the axiom (1) it follows that

$$\dot{\partial}_k g_{ij} - g_{hj} C^h_{ik} - g_{ih} C^h_{jk} = b_k g_{ij} .$$

Applying Christoffel process and using axiom (2), we obtain

$$C^i_{jk} = g^i_{jk} - \frac{1}{2} (b_j \delta^i_k + b_k \delta^i_j - b^i g_{jk}) \tag{3.3}$$

where $b^i = g^{ij} y_j$. Hence we have the theorem.

Since for the Carton's vertical connection g^i_{jk} satisfies C-conditions i.e., $y^j = 0$, we get

$$C^i_{jk}\, y^j = \frac{1}{2}(b^i y_k - b_0 \delta^i_k - b_k y^i) \quad (3.4)$$

where $y_k = g_{jk}\, y^j$.

We use the following results which can be proved by the equations (2.9), (3.4) and axiom (C_5).

$$y^i_{|j} = D^i_j, \quad y^i|_j = (1-\frac{1}{2}b_0)\delta^i_j + \frac{1}{2}(b^i\, y_j - b_j\, y^i) \quad (3.5)$$

$$P^i_{0jk} = P^i_{jk} + D^i_h C^h_{jk} + D^i_j|_k + y^h C^i_{km} P^m_{jk} - (y^h C^i_{hk})_{|j} \quad (3.6)$$

Now we shall prove the following:

Theorem (3.2). If a Finsler connection is h v□recurrent with respect to given recurrence vectors a_k and b_k and the connection coefficient are symmetric, then it holds for the components $P_{ijkl} = g_{jh}\, P^h_{ikl}$ of the h v□recurrent tensor field.

$$P_{ijkl} + P_{jikl} + a_h\, g_{ij}\, g^h_{kl} + a_k|_l\; g_{ij} + b_h P^h_{kl}\, g_{ij} - b_{l|k}\, g_{ij} - \frac{1}{2}(b_k a_l + b_l a_k - a_h b^h g_{kl})g_{ij} = 0 \quad (3.7)$$

$$P_{ijkl} = G_{(ij)}\{g_{jkl|i} - g_{jkm} P^m_{il} - a_i\, g_{jkl}\} + \lambda_{ijkl} + \mu_{ijkl} \quad (3.8)$$

where

$$\lambda_{ijkl} = \frac{1}{2}[(T_{ijk} + T_{kij} + T_{ikj} + a_j\, g_{ik} - a_k\, g_{ij} - a_i\, g_{jk})|_l + (T_{ijm} - T_{jim} + T_{jmi} - a_m\, g_{ij})g^m_{kl} \quad (3.9)$$
$$+ G_{(ij)}\{(T_{kim} + T_{ikm} + T_{kmi} + a_m\, g_{ki})g^m_{jl}\}]$$

$$\mu_{ijkl} = \frac{1}{2}[(P^m_{kl}(b_j g_{im} + b_m g_{ij} - b_i g_{jm}) + (b_{i|k} g_{jl} + b_{l|k} g_{ij} - b_{j|k} g_{il}) \quad (3.10)$$
$$+ \frac{1}{2}(b_l a_i + b_i a_l - a_m b^m g_{il})g_{jk} + \frac{1}{2}(b_l a_k + b_k a_l - a_m b^m g_{lk})g_{ij}$$
$$- \frac{1}{2}(b_j a_l + b_l a_j - a_m b^m g_{jl})g_{ik} + \frac{1}{2}b^m G_{(ij)}\{T_{kim} g_{jl} + T_{ikm} g_{jl} + T_{ijm} g_{lk}\}$$
$$+ \frac{1}{2}b_m(T^m_{kj} g_{il} + T^m_{ik} g_{jl} - T^m_{ij} g_{kl}) + \frac{1}{2}b_l(T_{jki} + T_{kij} + T_{kji}) + \frac{1}{2}b_i(T_{jkl} + T_{jlk} + T_{kjl})$$
$$+ \frac{1}{2}b_k(T_{jli} + T_{jil} + T_{lji}) + \frac{1}{2}b_j(T_{lki} + T_{lik} + T_{kli}) + \frac{1}{2}b_l(a_i g_{jk} + a_k g_{ij} - a_j g_{ik})]$$

where $T_{jhk} = g_{ih}\; T^i_{jk}$ and $G_{(ij)}\{\}$ denotes the interchange of the indices i and j and subtraction.

Proof : Applying the Ricci identity (Motsumoto,1966) for the metric tensor g_{ij} we get

$$g_{ij}|_{l|k} - g_{ij|k}|_l = g_{ij|h} C^h_{kl} + g_{ij}|_h\; P^h_{kl} + g_{hj} P^h_{ikl} + g_{ih} P^h_{jkl}$$

which in view of $g_{ij|k} = a_k\, g_{ij}$ and $g_{ij}|_k = b_k\, g_{ij}$ gives (3.7).

Again contracting one of the Bianchi identity [6]

$$T^h_{ij}|_k - C^h_{mk} T^m_{ij} + G_{(ij)}\{T^h_{im} C^m_{jk} + C^h_{jk|i} + C^h_{im} P^m_{jk} - P^h_{ijk}\} = 0$$

with g_{hl} and applying Christoffel process with respect to i, l and j, we get (3.8).

Theorem (3.3). Given a non–linear connection N^i_k, a skew-symmetric Finsler (1, 2) tensor field T^i_{jk} and covariant vector fields a_k and b_k in a Finsler space there exists a unique Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) satistying the axiom (C_5) and (C'_1)The connection is h-recurrent that is $g_{ij|k} = a_k\, g_{ij}$, (C'_2) The connection v- recurrent that is $g_{ij}|_k = b_k\, g_{ij}$,
(C'_3) The non-linear connection is the given , (C'_4) The (h)-h torsion tensor field is the given.

Proof: From the axioms (C_5) and (C'_2), it follows that the vertical connection C^i_{jk} is given by (3.3).

From the axiom (C'_1), we have

$$\partial_k g_{ij} - N^m_k \dot{\partial}_m g_{ij} - g_{mj} F^m_{ik} - g_{im} F^m_{jk} = a_k g_{ij}.$$

Applying Christoffel process to the above equation and using axiom (C'_4) and expression (2.3) for (h)-h torsion, we get

$$F^i_{jk} = Y^i_{jk} - (g^i_{km} N^m_j + g^i_{jm} N^m_k - g^{hi} g_{jkm} N^m_h) - \frac{1}{2}(a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk}) + A^i_{jk} \tag{3.11}$$

where

$$A^i_{jk} = \frac{1}{2}(T_{kjh} g^{hi} + T_{jkh} g^{hi} + T^i_{jk}). \tag{3.12}$$

In view of (3.3) and (3.11) it is clear that the Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) is uniquely determined from the metric function L and from given vector fields a_k, b_k, T^i_{jk} and N^i_k. For the above connection the deflection tensor field D^i_k defined in (2.11) is obtained by contraction of (3.11) with y^j, to get

$$D^i_k = G^i_k + 2g^i_{km} G^m - g^i_{km} N^m_0 - N^i_k - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k) + A^i_{0k}. \tag{3.13}$$

Contracting (3.13) with y^k, we get

$$N^i_0 = 2G^i - D^i_0 - a_0 y^i + \frac{1}{2} a^i L^2 + A^i_{00}. \tag{3.14}$$

Substituting the value N^i_0 in (3.13) and using c-conditions, we get

$$N^i_k = G^i_k - g^i_{km}(A^m_{00} - D^m_0 + \frac{1}{2} a^m L^2) + (A^i_{0k} - D^i_k) - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k).$$

Hence we have the following:

Theorem (3.4). Given a Finsler (1, 1) tensor field D^i_k, a covariant vector fields a_k,b_k and a skew-symmetric Finsler (1, 2) tensor field T^i_{jk} in a Finsler space there exixts a unique Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) satisfying the axioms (C'_1), (C'_2), (C'_4) and (C_5) and (C'_3) the deflection tensor field is the given D^i_k.

The v-connection F^i_{jk} is given by (3.11) in which the non-linear connection is given by

$$N^i_k = G^i_k - g^i_{km} B^m_0 + B^i_k \quad (3.15)$$

where

$$B^i_k = A^i_{0k} - D^i_k - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \quad (3.16)$$

The vertical connection is given by (3.3).

As a special case of the above theorem if we impose the axiom (C_3) instead of (C'_3), the B^i_k in (3.16) become

$$B^i_k = A^i_{0k} - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k) \quad (3.17)$$

and we have the following:

Theorem (3.5). Given a skew-symmetric Finsler (1, 2) tensor field T^i_{jk} and covariant vector fields a_k, b_k in a Finsler space there exists a unique Finsler connection ($F^i_{jk}, N^i_k, C^i_{jk}$) satisfying the axioms (C'_1), (C'_2),(C'_3), (C'_4) and (C_5). These coefficients are given by (3.11), (3.3) and

$$N^i_k = G^i_k - g^i_{km}(A^m_{00} - \frac{1}{2} a^m L^2) + A^i_{0k} - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \quad (3.18)$$

If we assume that $B^i_k = 0$, equation (3.15) reduces to $N^i_k = G^i_k$, and we have the following which gives the Finsler connection with deflection and torsion.

Theorem (3.6). Given a skew-symmetric Finsler (1, 2) tensor field T^i_{jk} and covariant vector fields a_k, b_k in a Finsler space there exists a unique Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) satisfying the axioms (C'_1), (C'_2), (C'_4), (C_5) and (C'''_3): The non linear connection N^i_k is the one given by E. Cartan. The coefficient are given by

$$F^i_{jk} = Y^i_{jk} - (g^i_{km} G^m_j + g^i_{jm} G^m_k - g^{hi} g_{jkm} G^m_h) - \frac{1}{2}(a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk}) + A^i_{jk}. \quad (3.19)$$

The deflection tensor field D^i_k is expressed as

$$D^i_k = A^i_{0k} - \frac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \quad (3.20)$$

As a special case of Theorem (3.5) if we impose the axiom (C_4) instead of (C'_4), we have the following:

Theorem (3.7). Given covariant vector fields a_k, b_k there exists a unique Finsler connection (F^i_{jk}, N^i_k, C^i_{jk}) satisfying the axioms (C'_1), (C'_2), (C_3), (C_4) and (C_5). These coefficients are given by

$$F^i_{jk} = \Gamma^{*i}_{jk} + Q^i_{jk} \quad (3.21)$$

$$N^i_k = G^i_k + T^i_k \quad (3.22)$$

$$C^i_{jk} = g^i_{jk} - \frac{1}{2}(b_j \delta^i_k + b_k \delta^i_j - b^i g_{jk}) \quad (3.23)$$

where Γ^{*i}_{jk} and G^i_k are given by (2.12) and (2.13) and

$$Q^i_{jk} = \frac{1}{2}\{a_0 g^i_{jk} + L^2(g^i_{jm}g^m_{kh} + g^i_{km}g^m_{jh} - g^m_{jk}g^i_{mh})\,a^h \qquad (3.24)$$

$$-(g^i_{jh}y_k + g^i_{kh}y_j - g_{hjk}y^i)\,a^h - (a_k\delta^i_j + a_j\delta^i_k - a^i g_{jk})\}$$

$$T^i_k = \frac{1}{2}(a^i y_k - a_k y^i - a_0\delta^i_k - L^2 g^i_{jk}a^j)\,. \qquad (3.25)$$

For simplicity we shall use the following terminology

(i) A Finsler connection will be called an h v□recurrent Finsler connection if it is h and v□recurrent and its (h) – h torsion tensor field vanishes.

(ii) A Finsler connection will be called an h v-recurrent Finsler connection with torsion if it is h and v-recurrent and its (h)□ h torsion tensor field does not vanish.

4.hv-recurrent generalized Berwald spaces

A Berwald space is a Finsler space in which coefficients Γ^{*i}_{jk} of Cartan's connection depend on the position alone (Motsumo1966). Hashiguchi (1975) generalized the concept of Berwald space and defined a generalized Berwald space as a Finsler space in which the metrical Finsler connection F^i_{jk} with torsion depends on the position alone. We shall generalize these two concepts and give the following:

Definition 1. A Finsler space is called an h v-recurrent Berwald space if there is possible to introduce an h and v-recurrent Finsler connection in such a way that the connection coefficients F^i_{jk} depends on the position alone.

Definition 2. A Finsler space is called an h v-recurrent generalized Berwald space if there is possible to introduce an h and v-recurrent Finsler connection with torsion in such away that the connection coefficient F^i_{jk} depends the position alone.

Now we shall find the condition under which a Finsler space become an hv-recurrent generalized Berwald space.

Theorem (4.1). The h v-recurrent Finsler connection F^i_{jk} with torsion satisfies the condition $\dot{\partial}_l F^i_{jk} = 0$, then

$$g_{ijl}D^i_k = 0 \qquad (4.1)$$

$$(2b_m P^m_{kl} + b_l a_k + \frac{1}{2}b_k a_l - \frac{1}{2}a_m b^m g_{lk})\,g_{ij} + \frac{1}{2}b_l(T_{kij} + T_{kji}) = 0 \qquad (4.2)$$

Proof: From (2.9), it follows that the condition

$$\dot{\partial}_l F^i_{jk} = 0 \qquad (4.3)$$

is equivalent to

$$P^i_{jkl} = -C^i_{jl|k} + C^i_{jm}P^m_{kl} \qquad (4.4)$$

which in view of (3.1) and (3.3), we get

$$P^{i}_{jkl} = a_k g_{ijl} - g_{ijl|k} + g_{ijm} P^{i}_{kl} + \frac{1}{2}(b_{i|k} g_{jl} + b_{l|k} g_{ij} - b_{j|k} g_{il}) \tag{4.5}$$

$$-\frac{1}{2}(b_i\, g_{jm} + b_m\, g_{ij} - b_j\, g_{im}) P^{m}_{kl}$$

Substituting this value in (3.7), we get

$$(a_m\, g^{m}_{kl} + a_k|_l) g_{ij} + 2(a_k g_{ijl} - g_{ijl|k} + g_{ijm} P^{m}_{kl}) - \frac{1}{2}(b_k a_l + b_l a_k - a_n b^h g_{kl}) g_{ij} = 0 \qquad .(4.6)$$

Contracting this equation with y^i and using (3.5) and c-conditions, we get

$$2 g_{ijl}\, D^{i}_{k} + (a_m\, g^{m}_{kl} + a_k|_l) y_j - \frac{1}{2}(b_k a_l + b_l a_k - a_n b^h g_{kl})\, y_j = 0\,. \tag{4.7}$$

Again contraction with y^j and using c-conditions, we get

$$a_m\, g^{m}_{kl} + a_k|_l = \frac{1}{2}(b_k a_l + b_l a_k - a_n b^h g_{kl}) \tag{4.8}$$

which when substituted in (4.7), gives (4.1). Substitution of (4.8) in (3.7) gives

$$P_{ijkl} + P_{jikl} = (b_{l|k} - b_h P^{h}_{kl})\, g_{ij}\,. \tag{4.9}$$

Then equation (3.8) and (4.9), gives (4.2).

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