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Comparative structure of Riemannian manifold and spin manifold

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Abstract

We establish here connection between curvature of a generalized cylinder with geometric data on M with spinor metric structure by comparing the Dirac operators for two different metrics based on identification and existence of semi-Riemannian metric. Specific objective is to investigate properties of spinors on a manifold foliated by semi-Riemannian hypersurfaces using commutator expansion and its normal derivative. We derive algebraic properties of semi-Riemannian manifold initiated by H. Baunn by taking non-degenerate symmetric bilinear form. The two semi-Riemannian metrics on a manifold cannot always be joined by a continuous path of metrics even if they have the same signature. we show here that for a Codazzi tensor, the manifold can be embedded as a hypersurface into a Ricci flat manifold equipped with a parallel spinor which generalizes the case of Killing spinors. The classification of manifolds admitting Killing spinors that the cone over such a manifold possesses a parallel spinor.

Keywords- Spinor manifold, Semi-Riemannian, Ricci curvature, Hyper surfaces, and, energy, momentum tensor

1.1 Introduction

The modified version of spin structure has the advantage of being independent of the choice of any semi-Riemannian metric on X. An oriented manifold together with a spin structure is called a spin manifold. Let M be a manifold and let g_t be a smooth 1-parameter family of semi-Riemannian metrics on M, such that $t \in I \subset R$. The manifold $Z = I \times M$ with the metric $dt^2 + g_t$ is called a generalized cylinder over M. In a semi-Riemannian hypersurface with spacelike normal bundle it is always possible that every semi-Riemannian manifold. The spacelike normal bundle is equivalent to the case of a time like normal bundle under restricted conditions which is closely related to the geometries of M and Z. which characterized by the equation $\nabla^{\Sigma}{}_{M}{}^{X}\psi = 1/2A(X)\cdot\psi$ where A is a given symmetric endomorphism field. Let us identify spinors for 1-parameter families of semi-Riemannian metrics such that by taking a 1-parameter family of metrics the corresponding generalized cylinder and parallel transport on this cylinder coincide. The identification is the same as the one for Riemannian metrics. We apply variation formula to compute the energy-momentum tensor for spinors.

$$\langle v, w\rangle := \sum_{i=1}^{r} v^i w^i - \sum_{i=r+1}^{n} v^i w^i \quad (1)$$

on R^n. The corresponding orthogonal group is defined as follows.

$O(r, s) := \{A \in GL(n,R) \mid \langle Av, Aw\rangle = \langle v, w\rangle$ *for all* $v, w \in R^n\}$… (2)

where as the special orthogonal group is defined by the relation

$SO(r, s) := \{A \in O(r, s) \mid det(A) = 1\}$. … (3)

If r = 0 or s = 0, then SO(r, s) is connected, otherwise it has two connected components. let $Cl_{r,s}$ be the Clifford algebra corresponding to the symmetric bilinear form $\langle\cdot, \cdot\rangle$. Which is called the unital algebra generated by R^n satisfying the relations

$v \cdot w + w \cdot v + 2 \langle v,w\rangle \cdot 1 = 0$ … (4)

for all $v,w \in R^n$. Let us consider a decomposition into even and odd elements given by

$$\mathrm{Cl}_{r,s} = \mathrm{Cl}^0_{r,s} \oplus \mathrm{Cl}^1_{r,s} \quad …(5)$$

such that R injects naturally into $Cl^0_{r,s}$ and R^n into $Cl^1_{r,s}$. The spin group is defined by the relation

$$\mathrm{Spin}(r,s) := \{v_1 \cdots v_k \in \mathrm{Cl}^0_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j\rangle = \pm 1 \text{ and } k \text{ is even}\} \quad …(6)$$

with multiplication inherited from $C^l_{r,s}$. Given $v \in R^n$ such that $\langle v, v\rangle \neq 0$ and arbitrary $w \in R^n$ we obtain that $v^{-1} = - v / \langle v,v\rangle$ and the equation

$$\mathrm{Ad}_v(w) := v^{-1} \cdot w \cdot v = -w + 2\frac{\langle v, w\rangle}{\langle v, v\rangle} v. \quad …(7)$$

Hence $-\mathrm{Ad}_v$ is the reflection across the hyperplane $v^\perp$ In particular, leaves $R^n \subset Cl_{r,s}$ invariant. Thus conjugation gives an action of Spin(r, s) on R^n by an even number of reflections across hyperplanes. This yields the exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{1, -1\} \longrightarrow \mathrm{Spin}(r,s) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(r,s) \longrightarrow 1.$$

Case – i) If n = r + s is even the Clifford algebra possesses an irreducible complex module $\Sigma_{r,s}$ of complex dimension dimension $2^{n/2}$, the complex *spinor module*. In case of $Cl^0_{r,s}$ the spinor module decomposes into

$$\Sigma_{r,s} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}, \quad …(8)$$

the submodules of spinors of positive resp. negative chirality. In particular, the spin group Spin(r, s) $\subset Cl^0_{r,s}$ acts on $\Sigma^+_{r,s}$ and on $\Sigma^-_{r,s}$. This action is given by

$$\rho = \rho^+ \oplus \rho^- : \mathrm{Spin}(r,s) \to \mathrm{Aut}(\Sigma^+_{r,s}) \times \mathrm{Aut}(\Sigma^-_{r,s}) \subset \mathrm{Aut}(\Sigma_{r,s}) \quad …(9)$$

Which is called the spinor representation of Spin(r, s). Given an orientation on R^n the $Cl^0_{r,s}$-submodules $\Sigma^+_{r,s}$ and $\Sigma^-_{r,s}$ may be characterized by the action of the volume element vol := $e_1 \cdot\cdot\cdot e_n \in Cl^0_{r,s}$ which acts on $\Sigma^+_{r,s}$ as $+i^{s+n(n+1)/2}$id and on $\Sigma^-_{r,s}$ as $-i^{s+n(n+1)/2}$id where $e_1, . . . , e_n$ is a positively oriented orthonormal basis of R^n.

Case-ii) If n is odd, then $Cl_{r,s}$ has two inequivalent irreducible modules $\Sigma^0_{r,s}$ and $\Sigma^1_{r,s}$, both of complex dimension $2^{(n-1)/2}$. These two modules are again distinguished by the action of the volume element vol = $e_1 \cdot\cdot\cdot e_n \in Cl^1_{r,s}$, namely vol acts as $+i^{s+n(n+1)/2}$id on $\Sigma^0_{r,s}$ and as $-i^{s+n(n+1)/2}$id on $\Sigma^1_{r,s}$. When restricted to $Cl^0_{r,s}$ the two modules become equivalent and let us write $\Sigma_{r,s} := \Sigma^0_{r,s}$. Now the spinor representation

$$\rho : \mathrm{Spin}(r,s) \to \mathrm{Aut}(\Sigma_{r,s}) \quad …(10)$$

is irreducible. All spinor modules carry nondegenerate symmetric sesquilinear forms $\langle\cdot, \cdot\rangle$ which are invariant under the action of Spin(r, s). The action of a vector $v \in R^n \subset Cl_{r,s}$ on $\Sigma_{r,s}$ is skewsymmetric with respect to $\langle\cdot, \cdot\rangle$, i.e. $\langle v\cdot\sigma_1, \sigma_2\rangle = -\langle\sigma_1, v \cdot \sigma_2\rangle$.

1.2 Differentiable manifold and its comparison with spin manifold

Let us choose X to denote an oriented n-dimensional differentiable manifold. The bundle $P_{GL+}(X)$ of positively oriented tangent frames forms a $GL^+(n,R)$-principal bundle over X. $GL^+(n,R)$ denotes the group of real n×n-matrices with positive determinante and A : $\widetilde{GL}^+(n,R) \rightarrow GL^+(n,R)$ its connected twofold covering group. A *spin structure* of X is a $\widetilde{GL}^+$ (n,R)-principal bundle $P_{\widetilde{GL}}$ (X) over X together with a twofold covering map $\Sigma : P_{\widetilde{GL}}{}^+(X) \rightarrow P_{GL}{}^+(X)$.

1.3 Spin Manifold and its algebraic representation

Let X has a semi-Riemannian metric of signature (r, s), r + s = n. The bundle $P_{SO}(X) \subset P_{GL}+(X)$ of positively oriented *orthonormal* tangent frames forms an SO(r, s)-principal bundle over X. we restrict the mapping A : $P_{\widetilde{GL}}{}^+$ (n,R) → $GL^+(n,R)$ to the preimage of SO(r, s) ⊂ $GL^+(n,R)$ ⟹Ad : Spin(r, s) → SO(r, s). Putting $P_{Spin}(X) := \Theta^{-1}(P_{SO}(X))$. Semi-Riemannian manifold $P_{Spin}(X)$ is called a spin structure of X and together with $P_{Spin}(X)$ is called a semi-Riemannian spin manifold. we define the spinor bundle of X as the complex vector bundle associated to the spinor representation, i. e.

$$\Sigma X := P_{\mathrm{Spin}}(X) \times_\rho \Sigma_{r,s}. \quad \ldots(11)$$

Hence, for p ∈ X the fiber of $\Sigma_p X$ of ΣX over p consists of equivalence classes of pairs [b, σ] where b ∈ $P_{Spin}(X)p$ and $\sigma \in \Sigma_{r,s}$ subject to condition that

$$[b, \sigma] = [bg^{-1}, g\sigma] \quad \ldots(12)$$

for all g ∈ Spin(r, s). But, the spinor bundle cannot be defined independently of the metric using $P_{\widetilde{GL}}{}^+(X)$ instead of $P_{Spin}(X)$ because the spinor representation ρ of Spin(r, s) on $\Sigma_{r,s}$ does not extend to a representation of $\widetilde{GL}^+$ (n,R) on $\Sigma_{r,s}$.. The tangent bundle is written as, $TX = P_{SO}(X)\times_\tau R^n$ where τ is the standard representation of SO(r, s) on R^n. The Clifford multiplication $T_pX \otimes \Sigma_pX \rightarrow \Sigma_pX$ is defined by the relation

$$[\Theta(b), v] \cdot [b, \sigma] := [b, v \cdot \sigma] \quad \ldots(13)$$

where b ∈ $P_{Spin}(X)_p$, $v \in R^n$, and $\sigma \in \Sigma_{r,s}$. For g ∈ Spin(r, s) we obtain the relation

$$\begin{aligned}[\Theta(bg), v] \cdot [bg, \sigma] &= [\Theta(b)\mathrm{Ad}_g, v] \cdot [bg, \sigma] = [\Theta(b), \mathrm{Ad}_g v] \cdot [b, g\sigma] \\ &= [b, gvg^{-1}g\sigma] = [b, gv\sigma] = [bg, v\sigma]\end{aligned}$$

which does not hold for non-oriented manifolds and pin structures.

1.4 Impact of Clifford algebra on spinor manifold and its metric structure

The spinor bundle of even dimension splits into the positive and the negative half-spinor bundles,

$$\Sigma X = \Sigma^+ X \oplus \Sigma^- X$$

where $\Sigma_\pm X = P_{Spin}(X)\times_{\rho\pm}\}\Sigma^\pm{}_{r,s}$. But, Clifford multiplication by a tangent vector interchanges Σ^+X and Σ^-X. The Spin(r, s)-invariant nondegenerate symmetric squilinear forms on $\Sigma_{r,s}$ and $\Sigma^\pm{}_{r,s}$ induce inner products on ΣX and $\Sigma^\pm X$ which is denoted by □·, ·□. The connection 1-formω^X on $P_{SO}(X)$ for the Levi-Civita connection ∇^X are lifted via Θ to $P_{Spin}(X)$, i. e. $\omega^{\Sigma X} := \mathrm{Ad}^{-1}{}_* \circ \Theta^*(\omega^X)$ after composing with $\mathrm{Ad}^{-1}{}_*$. The connection 1-form on $P_{Spin}(X)$ take values in the Lie algebra of Spin(r, s) rather than in that of SO(r, s). $\omega^{\Sigma X}$ induces a covariant derivative $\nabla^{\Sigma X}$ on ΣX covariant derivative. we now define covariant derivative $\nabla^{\Sigma X}$. If b is a local section in $P_{Spin}(X)$, then $\Theta(b) = (e_1, \ldots, e_n)$ is a local oriented orthonormal tangent frame, $□e_i, e_j□ \equiv \varepsilon_i\delta_{ij}$ where $\varepsilon_i = \pm 1$. The Christoffel symbols of ∇^X with respect to this frame are given by

$$\nabla^X_{e_i} e_j = \sum_{k=1}^{n} \Gamma^k_{ij}\, e_k. \quad (14)$$

But the covariant derivative of a locally defined spinor field φ = [b, σ], σ a function with values in $\Sigma_{r,s}$, is given by

$$\nabla^{\Sigma X}_{e_i}\varphi = \left[b, d_{e_i}\sigma + \frac{1}{2}\sum_{j<k}\Gamma^k_{ij}\,\varepsilon_j\, e_j\cdot e_k\cdot\sigma\right]$$

Here $\nabla^{\Sigma X}$ is a metric connection which gives the splitting in even dimensions invariant.

$$\nabla^{\Sigma X}_Z(Y\cdot\varphi) = (\nabla^X_Z Y)\cdot\varphi + Y\cdot\nabla^{\Sigma X}_Z\varphi$$

for all vector fields Z and Y and all spinor fields φ. The curvature tensor $R^{\Sigma X}$ of $\nabla^{\Sigma X}$ is evaluated in terms of the curvature tensor R^X of the Levi-Civita connection,

$$R^{\Sigma X}(Y,Z)\varphi = \frac{1}{2}\sum_{i<j}\varepsilon_i\varepsilon_j\left\langle R^X(Y,Z)e_i, e_j\right\rangle e_i\cdot e_j\cdot\varphi.$$

By an appropriate application of first Bianchi identity, relation is derived

$$\sum_{i=1}^{n}\varepsilon_i\, e_i\cdot R^{\Sigma X}(e_i,Y)\varphi = \frac{1}{2}\mathrm{Ric}^X(Y)\cdot\varphi. \qquad \ldots(15)$$

where Ric^X denotes the Ricci curvature known as an endomorphism field on TM. The Ricci curvature is a symmetric bilinear form expressed by ricX(Y,Z) = □Ric^X(Y),Z□.

1.5 Spin Manifold and its hyper-surfaces

Let us choose Z to be an oriented (n + 1)-dimensional semi-Riemannian spin manifold. Let ⊖ : $P_{Spin}(Z) \to P_{SO}(Z)$ be a spin structure on Z. Let M ⊂ Z be a semi- Riemannian hypersurfacewith trivial spacelike normal bundle. Hence, there is a vector field ν on Z along M satisfying □ν, ν□ = +1 and □ν, TM□ = 0. If the signature of M is (r, s), then the signature of Z is (r + 1, s). M inherits a spin structure. The bundle of oriented orthonormal frames of M, $P_{SO}(M)$, can be embedded into the bundle of oriented orthonormal frames of Z restricted to M, $P_{SO}(Z)|_M$, by the map ι : $(e_1, \ldots, e_n) \to (\nu, e_1, \ldots, e_n)$. Then $P_{Spin}(M) :=$ $⊖^{-1}(\iota(P_{SO}(M)))$ defines a spin structure on M. Let us assume that this spin structure be taken on M. The algebraic structure of spin manifold shown that if n is even, then

$$\Sigma\mathcal{Z}|_M = \Sigma M$$

where the Clifford multiplication with respect to M is given by X ⊗ϕ → ν ·X · ϕ "·". If n is odd, then

$$\Sigma^+\mathcal{Z}|_M = \Sigma M$$

and again Clifford multiplication with respect to M is given by X ⊗ ϕ → ν · X · ϕ where

$$\Sigma^-\mathcal{Z}|_M = \Sigma M \qquad \ldots(16)$$

with Clifford multiplication with respect to M given by X ⊗ ϕ → −ν · X · ϕ. The minus sign comes in odd dimensions $\Sigma_{r,s} = \Sigma^0_{r,s}$ while $\Sigma^1_{r,s}$ leads to the opposite sign for the Clifford multiplication. The identifications preserve the natural inner products □·, ·□. Let W denote the Weingarten map with respect to ν, i. e.

$$\nabla^{\mathcal{Z}}_X Y = \nabla^M_X Y + \langle W(X), Y\rangle\nu \qquad \ldots(17)$$

for all vector fields X and Y on M. The Weingarten map is symmetric with respect to the semi-Riemannian metric, □W(X), Y□ = □X,W(Y)□ and is given by $W(X) = -\nabla^Z_X\nu$. The Christoffel symbols of M with respect to a local orthogonal tangent frame $(e_1, \ldots, e_n)$ is denoted by $\Gamma^{M,k}_{ij}$ and the Christoffel symbols of Z with respect to $(e_0, e_1, \ldots, e_n)$, $e_0 = \nu$, by $\Gamma^{Z,k}_{ij}$, which implies that for $1 \le i, j, k \le n$ the following relations are satisfied.

$$\begin{aligned}\Gamma_{ij}^{\mathcal{Z},k} &= \Gamma_{ij}^{M,k}, \\ \Gamma_{ij}^{\mathcal{Z},0} &= \langle W(e_i), e_j\rangle, \\ \Gamma_{i0}^{\mathcal{Z},k} &= -\varepsilon_0\varepsilon_k\Gamma_{ik}^{\mathcal{Z},0} = -\varepsilon_k\langle W(e_i), e_k\rangle\end{aligned}$$

…(18)

Combining above equations (18) we obtain the relations on a section $\phi = [b, \sigma]$ of $\Sigma Z|_M$ $(1 \le i \le n)$

$$\begin{aligned}\nabla_{e_i}^{\Sigma\mathcal{Z}}\varphi &= \left[b, d_{e_i}\sigma + \frac{1}{2}\left(-\sum_{k=1}^{n}\varepsilon_k\langle W(e_i), e_k\rangle\varepsilon_0 e_0\cdot e_k + \sum_{1\le j<k\le n}\Gamma_{ij}^{M,k}\varepsilon_j e_j\cdot e_k\right)\cdot\sigma\right] \\ &= \left[b, d_{e_i}\sigma + \frac{1}{2}\left(-e_0\cdot W(e_i) + \sum_{1\le j<k\le n}\Gamma_{ij}^{M,k}\varepsilon_j e_0\cdot e_j\cdot e_0\cdot e_k\right)\cdot\sigma\right] \\ &= \nabla_{e_i}^{\Sigma M}\varphi - \frac{1}{2}\nu\cdot W(e_i)\cdot\varphi.\end{aligned}$$

…(19)

Hence, for each $X \in TM$ and each section ϕ of $\Sigma Z|_M$,

$$\nabla_X^{\Sigma\mathcal{Z}}\varphi = \nabla_X^{\Sigma M}\varphi - \frac{1}{2}\nu\cdot W(X)\cdot\varphi.$$

Let ϕ be a section of ΣZ defined in a neighborhood of M, then

$$D^{\mathcal{Z}}\varphi = \sum_{i=1}^{n}\varepsilon_i e_i\cdot\nabla_{e_i}^{\Sigma\mathcal{Z}}\varphi + \nu\cdot\nabla_{\nu}^{\Sigma\mathcal{Z}}\varphi.$$

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i e_i\cdot\nabla_{e_i}^{\Sigma\mathcal{Z}}\varphi &= \sum_{i=1}^{n}\varepsilon_i e_i\cdot\nabla_{e_i}^{\Sigma M}\varphi - \frac{1}{2}\sum_{i=1}^{n}\varepsilon_i e_i\cdot\nu\cdot W(e_i)\cdot\varphi \\ &= -\nu\cdot\sum_{i=1}^{n}\varepsilon_i\nu\cdot e_i\cdot\nabla_{e_i}^{\Sigma M}\varphi + \frac{1}{2}\sum_{i=1}^{n}\varepsilon_i\nu\cdot e_i\cdot W(e_i)\cdot\varphi \\ &= -\nu\cdot\tilde{D}^M - \frac{1}{2}\operatorname{tr}(W)\nu\cdot\varphi\end{aligned}$$

…(20)

where $\tilde{D}^M = D^M$ if n is even and $\tilde{D}^M = \begin{pmatrix} D^M & 0 \\ 0 & -D^M\end{pmatrix}$ if n is odd. Thus the Dirac operators on M and on Z are related by

$$\nu\cdot D^{\mathcal{Z}} = \tilde{D}^M + \frac{n}{2}H - \nabla_{\nu}^{\Sigma\mathcal{Z}}$$

…(21)

where $H = 1/n\ \mathrm{tr}(W)$ denotes the mean curvature.

1.6 Theorem

Let Z be an (n +1)-dimensional semi-Riemannian spin manifold. Let Z carry a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, such that ⟨ν, ν⟩ = 1 and $\nabla^Z_\nu\nu = 0$.

Proof

Let W denote the Weingarten map of the leaves with respect to ν and let $H = 1/n\ \mathrm{tr}(W)$ be the mean curvature. Let us choose a local oriented orthonormal tangent frame $(e_1, \ldots, e_n)$ for the leaves and assume that $\nabla^Z_\nu e_i = 0$. The following relation is satisfied and known as the Riccati equation.

$$\begin{aligned}[\nabla_{\nu}^{\Sigma\mathcal{Z}}, \tilde{D}^M]\varphi &= \sum_{i=1}^{n}\varepsilon_i\left(\nabla_{\nu}^{\Sigma\mathcal{Z}}(\nu\cdot e_i\cdot\nabla_{e_i}^{\Sigma M}\varphi) - \nu\cdot e_i\cdot\nabla_{e_i}^{\Sigma M}\nabla_{\nu}^{\Sigma\mathcal{Z}}\varphi\right) \\ &= \sum_{i=1}^{n}\varepsilon_i\nu\cdot e_i\cdot\left(\nabla_{\nu}^{\Sigma\mathcal{Z}}\nabla_{e_i}^{\Sigma M}\varphi - \nabla_{e_i}^{\Sigma M}\nabla_{\nu}^{\Sigma\mathcal{Z}}\varphi\right)\end{aligned}$$

$$\sum_{i=1}^{n} \varepsilon_i \nu \cdot e_i \cdot \Big(\nabla_{\nu}^{\Sigma Z} (\nabla_{e_i}^{\Sigma Z} + \frac{1}{2}\nu \cdot W(e_i)) \quad -(\nabla_{e_i}^{\Sigma Z} + \frac{1}{2}\nu \cdot W(e_i)) \nabla_{\nu}^{\Sigma Z} \Big) \varphi$$

$$= \quad \sum_{i=1}^{n} \varepsilon_i \nu \cdot e_i \cdot \Big(R^{\Sigma Z}(\nu, e_i) + \nabla_{[\nu, e_i]}^{\Sigma Z} + \frac{1}{2}\nu \cdot (\nabla_{\nu}^{Z} W)(e_i) \Big) \varphi$$

$$-\frac{1}{2}\nu \cdot \mathrm{Ric}^{Z}(\nu) \cdot \varphi + \sum_{i=1}^{n} \varepsilon_i \nu \cdot e_i \cdot \Big(\nabla_{W(e_i)}^{\Sigma Z} + \frac{1}{2}\nu \cdot (\nabla_{\nu}^{Z} W)(e_i) \Big) \varphi$$

$$-\frac{1}{2}\nu \cdot \mathrm{Ric}^{Z}(\nu) \cdot \varphi$$

$$+\sum_{i=1}^{n} \varepsilon_i \nu \cdot e_i \cdot \Big(\nabla_{W(e_i)}^{\Sigma M} - \frac{1}{2}\nu \cdot W^2(e_i) + \frac{1}{2}\nu \cdot (\nabla_{\nu}^{Z} W)(e_i) \Big) \varphi$$

$$= \quad -\frac{1}{2}\nu \cdot \mathrm{Ric}^{Z}(\nu) \cdot \varphi + \mathfrak{D}^{W} \varphi \quad + \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i e_i \cdot \Big(- W^2(e_i) + (\nabla_{\nu}^{Z} W)(e_i) \Big) \varphi. \qquad \ldots(22)$$

The Riccati equation for the Weingarten map $(\nabla^{Z}_{\nu}W)(X) = R^{Z}(X, \nu)\nu + W^{2}(X)$ implies that the following relation holds.

$$[\nabla_{\nu}^{\Sigma Z}, \tilde{D}^{M}] \varphi \quad = \quad -\frac{1}{2}\nu \cdot \mathrm{Ric}^{Z}(\nu) \cdot \varphi + \mathfrak{D}^{W} \varphi + \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i e_i \cdot (R^{Z}(e_i, \nu)\nu) \cdot \varphi$$

$$= \quad -\frac{1}{2}\nu \cdot \mathrm{Ric}^{Z}(\nu) \cdot \varphi + \mathfrak{D}^{W} \varphi + \frac{1}{2}\mathrm{ric}^{Z}(\nu, \nu) \varphi$$

$$= \quad \mathfrak{D}^{W} \varphi - \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \, \mathrm{ric}^{Z}(\nu, e_i) \, \nu \cdot e_i \cdot \varphi. \qquad \ldots(23)$$

The Codazzi-Mainardi equation as given by B.O' Neill for $X, Y, V \in T_pM$ is expressed as

$$\langle R^{Z}(X, Y)V, \nu \rangle = \langle (\nabla_{X}^{M} W)(Y), V \rangle - \langle (\nabla_{Y}^{M} W)(X), V \rangle .$$

Thus, we obtain the following equation

$$\mathrm{ric}^{Z}(\nu, X) \quad = \quad \sum_{i=1}^{n} \varepsilon_i \langle R^{Z}(X, e_i)e_i, \nu \rangle$$

$$= \quad \sum_{i=1}^{n} \varepsilon_i \left(\langle (\nabla_{X}^{M} W)(e_i), e_i \rangle - \langle (\nabla_{e_i}^{M} W)(X), e_i \rangle \right)$$

$$= \quad \mathrm{tr}(\nabla_{X}^{M} W) - \Big\langle \mathrm{div}^{M}(W), X \Big\rangle \qquad \ldots(24)$$

Combining these two equations, we get

$$[\nabla_{\nu}^{\Sigma Z}, \tilde{D}^{M}] \varphi \quad = \quad \mathfrak{D}^{W} \varphi - \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \, \Big(\mathrm{tr}(\nabla_{e_i}^{M} W) - \Big\langle \mathrm{div}^{M}(W), e_i \Big\rangle \Big) \, \nu \cdot e_i \cdot \varphi$$

$$= \quad \mathfrak{D}^{W} \varphi - \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \, d_{e_i} \, \mathrm{tr}(W) \nu \cdot e_i \cdot \varphi + \frac{1}{2}\nu \cdot \mathrm{div}^{M}(W) \cdot \varphi$$

$$= \quad \mathfrak{D}^{W} \varphi - \frac{n}{2}\nu \cdot \mathrm{grad}^{M}(H) \cdot \varphi + \frac{1}{2}\nu \cdot \mathrm{div}^{M}(W) \cdot \varphi.$$

Hence, the theorem is proved.

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