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The four dimensional C^h- symmetric finsler space with constant unified main scalar

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Abstract

F. Ikeda introduced the properties of Finsler spaces satisfying the condition $L^2C^2 = f(x)$ in the year 1984, where L is the fundamental function and C is the length of the torsion vector C_i. In 1991, Ikeda introduced the condition: L^2C^2 = non-zero constant. In 1977, Matsumoto and Miron introduced the theory of intrinsic orthonormal frame field on n-dimensional Finsler space, as a generalization of Berwald' and Moor's ideas on two-dimensional and three-dimensional Finsler space respectively. Ikeda in the year 1991 and Singh and Kumari in the year 2000 have studied the three-dimensional Finsler space with constant unified main scalar. In 2007, Prasad, Chaubey and Patel have discussed the theory of the four-dimensional Finsler space with constant unified main scalar. A Finsler space F^n is called C^h-symmetric Finsler space if $C_{ijklh} = C_{ijhlk}$. In the presentpaper,we have discussed the theoryofthe four-dimensional C^h-symmetric Finsler space with contant unified main scalar .

Keywords- C^h-symmetric Finsler space, Miron frame, unified main scalar, Landsberg space

Introduction

Ikeda (1984) has discussed the properties of Finsler spaces satisfying the condition $L^2C^2 = f(x)$, where L is the fundamental function and C is the length of the torsion vector C_i. In 1991, Ikeda has considerd the condition: L^2C^2 = non-zero constant which is stronger than the corresponding condition considered in 1984. A two-dimensional Berwald space is an example of such a Finsler space with constant function LC. A theory of intrinsic orthonormal frame field on n -dimensional Finsler space, as a generalization of Berwald's and Moor's ideas on two-dimensional and three-dimensional Finsler space respectively, has been studied by Matsumoto and Miron (1977). The three-dimensional Finsler space with constant unified main scalar has been studied by Ikeda (1991) and Singh and Kumari (2000). Recently, Prasad, Chaubey and Patel (2007) has discussed the theory of the four-dimensional Finsler space with constant unified main scalar.They especially found the scalar components of v-scalar curvature S and the conditions under which v-connection vectors vanish with respect to Cartan's connection $C\Gamma$· The purpose of the present paper is to obtain the condition under which h-connection vectors vanish with respect to the Cartan's connection $C\Gamma$ of the four-dimensional C^h-symmetric Finsler space with constant unified main scalar. Also, the h-connection vectors of C-reducible, semi C-reducible and C2-like four-dimensional C^h-symmetric Finsler space with constant unified main scalar has been determined. The orthonormal frame field (l^i, m^i, n^i, p^i) called the Miron frame plays an important role in four-dimensional Finsler space.

Scalar components in Miron frame- Let us consider a four-dimensional Finsler space F^4 with the fundamental function L(x,y). The metric tensor g_{ij} and C-tensor C_{ijk} of F^4 are defined by

$$g_{ij} = (1/2)\dot\partial_i\dot\partial_j L^2, \quad C_{ijk} = (1/4)\dot\partial_i\dot\partial_j\dot\partial_k L^2.$$

Throughout this paper we use the symbols $\dot\partial_i = \partial/\partial y^i$ and $\partial_i = \partial/\partial x^i$. The frame $\{e^i_{(\alpha)}\}$, α = 1, 2, 3, 4 is called Miron frame of F^4, where $e^i_{(1)} = l^i = y^i/L$ is the normalized supporting element, $e^i_{(2)} = m^i = C^i/C$ is the normalized torsion vector, $e^i_{(3)} = n^i$, $e^i_{(4)} = p^i$ are constructed by $g_{ij}e^i_{(\alpha)}e^j_{(\beta)} = \delta_{\alpha\beta}$. Here C is the length of torsion vector $C_i = C_{ijk}g^{jk}$. The Greek letters α, β, γ, δ vary from 1 to 4 throughout the paper. Summation convension is applied for both the Greek and Latin indices.

In the Miron's frame an arbitrary tensor can be expressed by scalar components along the unit vectors l^i, m^i, n^i, p^i. For instance, let $T = T^i_j$ be a tensor field of (1,1) type, then the scalar components $T_{\alpha\beta}$ of T^i_j are defined by

$$T_{\alpha\beta} = T^i_j\, e_{(\alpha)i}e^j_{(\beta)}$$

and the components T^i_j of the tensor T are expressed as

$$T^i_j = T_{\alpha\beta}e^i_{(\alpha)}e_{(\beta)j}\ .$$

From the equations $g_{ij}e^i_{(\alpha)}e^j_{(\beta)} = \delta_{\alpha\beta}$, we have

$$g_{ij} = l_il_j + m_im_j + n_in_j + p_ip_j\ . \tag{2.1}$$

Next the C-tensor $C_{ijk} = (1/2)\partial_k g_{ij}$ satisfies $C_{ijk}l^i = 0$ and is symmetric

in i, j, k, therefore if $C_{\alpha\beta\gamma}$ are scalar components of LC_{ijk}, that is if

$$LC_{ijk} = C_{\alpha\beta\gamma}e_{(\alpha)i}e_{(\beta)j}e_{(\gamma)k}, \tag{2.2}$$

Then we have

$$LC_{ijk} = C_{222}m_im_jm_k + C_{233}\pi_{(ijk)}\{m_in_jn_k\} + C_{244}\pi_{(ijk)}\{m_ip_jp_k\} + C_{322}\pi_{(ijk)}\{m_im_jn_k\} + C_{333}n_in_jn_k + C_{344}\pi_{(ijk)}\{n_ip_jp_k\} + C_{422}\pi_{(ijk)}\{m_im_jp_k\} + C_{433}\pi_{(ijk)}\{n_in_jp_k\} + C_{444}p_ip_jp_k + C_{234}\pi_{(ijk)}\{m_i(n_jp_k + n_kp_j)\}, \tag{2.3}$$

where $\pi_{(ijk)}\{\ldots.\}$ denote the cyclic interchange of i, j,k and summation. For instance

$$\pi_{(ijk)}\{A_iB_jC_k\} = A_iB_jC_k + A_jB_kC_i + A_kB_iC_j\ .$$

Contracting (2.2) with g^{jk}, we get $LC\, m_i = C_{\alpha\beta\beta}e_{(\alpha)i}$. Thus if we put

$$C_{222} = H,\ C_{233} = I,\ C_{244} = K,\ C_{333} = J,$$

$$C_{344} = J',\ C_{444} = H',\ C_{433} = I',\ C_{234} = K' \tag{2.4}$$

then we have (Pandey and Divedi, 1997)

$$H + I + K = LC, \quad C_{322} = -(J + J'), \quad C_{422} = -(H' + I') \tag{2.5}$$

The eight scalars H, I, J, K, H', I', J', K' are called the main scalars of a four-dimen-

sional Finsler space. We shall use Cartan's connection $C\Gamma = (\Gamma^i_{jk}, G^i_j, C^i_{jk})$ in the following section of this paper. The h-covariant derivative of the frame field $e_{(\alpha)i}$ are given by (Matsumoto, 1986)

$$e_{(\alpha)i|j} = H_{(\alpha)\beta\gamma} e_{(\beta)i} e_{(\gamma)j}\ , \tag{2.6}$$

where $H_{(\alpha)\beta\gamma}$, γ being fixed, are given by

$$H_{(\alpha)\beta\gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\gamma & J_\gamma \\ 0 & -h_\gamma & 0 & k_\gamma \\ 0 & -J_\gamma & -k_\gamma & 0 \end{bmatrix} \tag{2.7}$$

$$H_{(2)3\gamma} = -H_{(3)2\gamma} = h_\gamma$$

$$H_{(2)4\gamma} = -H_{(4)2\gamma} = J_\gamma$$

$$H_{(3)4\gamma} = -H_{(4)3\gamma} = k_\gamma$$

Thus, in four-dimensional Finsler space there exist three h-connection vect- ors h_i, J_i, k_i whose scalar components with respect to Miron frame are h_γ, J_γ, k_γ that is

$$h_i = h_\gamma e_{(\gamma)i},\ \ J_i = J_\gamma e_{(\gamma)i},\ \ k_i = k_\gamma e_{(\gamma)i} \tag{2.8}$$

A Finsler space F^n is called C^h-symmetric Finsler space if

$$C_{ijk|h} = C_{ijh|k} \tag{2.9}$$

where | denote h-covariant derivative with respect to Cartan's connection.

With the help of equations (2.7) and (2.8), the equation (2.6) can be explicitly written as

$$l_{i|j} = 0,\ \ m_{i|j} = n_i h_j + p_i J_j\ ,\ \ n_{i|j} = p_i k_j - m_i h_j\ ,\ \ p_{i|j} = -m_i J_j - n_i k_j \tag{2.10}$$

The h-scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor T^i_j of (1,1) type is defined as (Matsumoto, 1986)

$$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta)} e^k{}_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma} , \tag{2.11}$$

where $\delta_k = \partial_k - G^r{}_k \partial_r$.

Thus $T_{\alpha\beta,\gamma}$ is adopted components of $T^i{}_{j|k}$ that is

$$T^i{}_{j|k} = T_{\alpha\beta,\gamma} e^i{}_{(\alpha)} e_{(\beta)j} e_{(\gamma)k} . \tag{2.12}$$

From (2.2) it follows that

$$LC_{hij|k} = C_{\alpha\beta\gamma,\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k} \tag{2.13}$$

The explicit form of $C_{\alpha\beta\gamma,\delta}$ is easily obtained :

(a) $C_{1\beta\gamma,\delta} = 0$, (b) $C_{222,\delta} = H_{,\delta} + 3(J + J')h_\delta + 3(H' + I')J_{\delta},$

(c) $C_{223,\delta} = -(J + J')_{,\delta} + (H - 2I)h_\delta - 2K'J_\delta + (H' + I')k_{\delta},$ (2.14)

(d) $C_{233,\delta} = I_{,\delta} - (3J + 2J')h_\delta - I'J_\delta - 2K'k_{\delta},$

(e) $C_{224,\delta} = -(H' + I')_{,\delta} - 2K'h_\delta + (H - 2K)J_\delta - (J + J')k_{\delta},$

(f) $C_{234,\delta} = K'_{,\delta} - (H' + 2I')h_\delta - (J + 2J')J_\delta + (I - K)k_{\delta},$

(g) $C_{244,\delta} = K_{,\delta} - J'h_\delta - (3H + 2I')J_\delta + 2K'k_{\delta},$

(h) $C_{333,\delta} = J_{,\delta} + 3Ih_\delta - 3I'k_{\delta},$

(i) $C_{334,\delta} = I'_{,\delta} + 2K'h_\delta + IJ_\delta + (J - 2J')k_{\delta},$

(j) $C_{344,\delta} = J'_{,\delta} + Kh_\delta + 2K'J_\delta + (2I' - H')k_{\delta},$

(k) $C_{444,\delta} = H'_{,\delta} + 3KJ_\delta + 3J'k_{\delta},$

where $H_{,\delta}$, for instance, is the h-scalar derivative of the single scalar H, namely, $H_{,\delta} = (\delta_i H)e^i{}_{\delta)}$.

Making use of equation (2.9), equation (2.13) yields

$$C_{\alpha\beta\gamma,\delta} - C_{\alpha\beta\delta,\gamma} = 0. \tag{2.15}$$

This equation is explicitly written as

(a) $-(J + J')_{,2} + (H - 2I)h_2 - 2K'J_2 + (H' + I')k_2 = H_{,3} + 3(J + J')h_3 + 3(H' + I')J_3,$

(b) $I_{,2} - (3J + 2J')h_2 - I'J_2 - 2K'k_2 = -(J + J')_{,3} + (H - 2I)h_3 - 2K'J_3 + (H' + I')k_3 ,$

(c) $K'_{,2} - (H' - 2I')h_2 - (J + 2J')J_2 + (I - K)k_2$

$= -(J + J')_{,4} + (H - 2I)h_4 - 2K'J_4 + (H' + I')k_4$

$= -(H' + I')_{,3} - 2K'h_3 + (H - 2K)J_3 - (J + J')k_3,$

(d) $J_{,2} + 3Ih_2 - 3I'k_2 = I_{,3} - (3J + 2J')h_3 - I'J_3 - 2K'k_3,$ (2.16)

(e) $I'_{,2} + 2K'h_2 + IJ_2 + (J - 2J')k_2 = I_{,4} - (3J + 2J')h_4 - I'J_4 - 2K'k_4$

$= K'_{,3} - (H' + 2I')h_3 - (J + 2J')J_3 + (I - K)k_3,$

(f) $J'_{,2} + Kh_2 + 2K'J_2 + (2I' - H')k_2 = K_{,3} - J'h_3 - (3H' + 2I')J_3 + 2K'k_3$

$= K'_{,4} - (H' + 2I')h_4 - (J + J')J_4 + (I - K)k_4,$

(g) $-(H' + I')_{,2} - 2K'h_2 + (H - 2K)J2 - (J + J')k_2 = H_{,4} + 3(J + J')h_4 + 3(H' + I')J_4,$

(h) $K_{,2} - J'h_2 - (3H' + 2I')J_2 + 2K'k_2 = -(H' + I')_{,4} - 2K'h_4 + (H - 2K)_{J4} - (J + J')k_4,$

(i) $H'_{,2} + 3KJ_2 + 3J'k_2 = K_{,4} - J'h_4 - (3H' + 2I')J_4 + 2K'k_4,$

(j) $I'_{,3} + 2K'h_3 + IJ_3 + (J - 2J')k_3 = J_{,4} + 3Ih_4 - 3I'k_4,$

(k) $J'_{,3} + Kh_3 + 2K'J3 + (2I' - H')k_3 = I'_{,4} + 2K'h_4 + IJ_4 + (J - 2J')k_4,$

(l) $H'_{,3} + 3KJ_3 + 3J'k_3 = J'_{,4} + Kh_4 + 2K'J_4 + (2I' - H')k_4,$

(m) $(3J + 2J')h_1 + I'J_1 + 2K'k_1 = 0.$

Since $C_{ijh}y^h = 0$ and y^h_{lk}=0. Hence from (2.9) it follows that $C_{ijklh}y^h$=0

that is P_{ijk}=0. Therefore we have the following :

THEOREM 2.1 A C^h-symmetric Finsler space is a Landsberg space. The converse of theorem (2.1) is not necessary true so the C^h-symmetric Finsler space is more general than the Landsberg space.

3. The constant unified main scalar

In a four-dimensional Finsler space, H + I + K = LC is called unified main scalar. Now, we consider four-dimensional C^h-symmetric Finsler space with non-zero constant unified main scalar. Therefore, we have

$$(H + I + K)_{,\alpha} = (LC)_{,\alpha} = 0 \text{ for } \alpha = 1, 2, 3, 4. \quad (3.1)$$

Adding equations (2.16)(a), (2,16)(d), the first part of (2.16)(f) and applying (3.1), we get $h_2 = 0$. Similarly, adding equations (2.16)(g), (2,16)(i) and first part of (2.16)(e) and applying equation (3.1), we get $J_2 = 0$. Again adding (2.16) (j), (2.16)(l), the last equation of (2.16)(c) and applying equation (3.1), we get get

$J_3 = h_4$. Hence we have the following:

THEOREM 3.1 In a four-dimensional C^h-symmetric Finsler space with non-zero

constant unified main scalar, the scalar components of h-connection vectors h_i and J_i are given by

$h_i = h_1l_i + h_3n_i + h_4p_i$, $J_i = J_1l_i + J_3n_i + J_4p_i$, where $J_3 = h_4$.

In view of theorem (3.1), the independent equation in (2.16) can be rewritten as

$$
\begin{aligned}
&(a)\ I_{,2} - 2K'k_2 = -(J + J')_{,3} + (H - 2I)h_3 - 2K'J_3 + (H' + I')k_3,\\
&(b)\ J_{,2} - 3I'k_2 + 2K'k_3 = I_{,3} - (3J + 2J')h_3 - I'h_4,\\
&(c)\ I'_{,2} + (J - 2J')k_2 - (I - K)k_3 = K'_{,3} - (H' + 2I')h_3 - (J + 2J')h_4,\\
&(d)\ J'_{,2} + (2I' - H')k_2 - 2K'k_3 = K_{,3} - J'h_3 - (3H' + 2I')J_3,\\
&(e)\ K'_{,2} + (I - K)k_2 = -(J + J')_{,4} + (H - 2I)J_3 - 2K'J_4 + (H' + I')k_4, \qquad (3.2)\\
&(f)\ K_{,2} + 2K'k_2 + (J + J')k_4 = -(H' + I')_{,4} - 2K'J_3 + (H - 2K)J_4,\\
&(g)\ I'_{,2} + (J - 2J')k_2 + 2K'k_4 = I_{,4} - (3J + 2J')J_3 - I'J_4,\\
&(h)\ J'_{,2} + (2I' - H')k_2 - (I - K)k_4 = K'_4 - (H' + 2I')J_3 - (J + 2J')J_4,\\
&(i)\ H'_{,2} + 3J'k_2 - 2K'k_4 = K_{,4} - J'J_3 - (3H' + 2I')J_4,\\
&(j)\ J_{,4} + 2IJ_3 = I'_{,3} + 2K'h_3 + (J - 2J')k_3 + 3I'k_4,\\
&(k)\ I'_{,4} + IJ_4 = J'_{,3} + Kh_3 + (2I' - H')k_3 - (J - 2J')k_4,\\
&(l)\ J'_{,4} - 2KJ_3 + 2K'J_4 = H'_{,3} + 3J'k_3 - (2I' - H')k_4.
\end{aligned}
$$

If we suppose that the non-vanishing main scalars are H, I, K then we have

$J = J' = H' = I' = K' = 0$. Then equations (2.16) reduce to

$$
\begin{aligned}
&(a)\ H_{,3} = (H - 2I)h_2, \qquad (b)\ I_{,2} = (H - 2I)h_3,\\
&(c)\ (H - 2K)J_3 = (H - 2I)h_4 = (I - K)k_2,\\
&(d)\ I_{,3} = 3Ih_2, \qquad (e)\ I_{,4} = IJ_2 = (I - K)k_3, \qquad (3.3)\\
&(f)\ K_{,3} = Kh_2 = (I - K)k_4, \qquad (g)\ H_{,4} = (H - 2K)J_2,\\
&(h)\ K_{,2} = (H - 2K)J_4, \qquad (i)\ K_{,4} = 3KJ_2,\\
&(j)\ J_3 = 3h_4, \qquad (k)\ Kh_3 = IJ_4,\\
&(l)\ 3J_3 = h_4\ .
\end{aligned}
$$

From (3.3) (c), (j) and (e) it can be seen that $J_3 = h_4 = 0$ and either $k_2 = 0$ or $I = K$. To solve the remaining aforesaid equations, we consider the following cases:

(I) $H \neq 2I \neq 2K$, (II) $H = 2I \neq 2K$, (III) $H \neq 2I = 2K$,

(IV) $H = 2K \neq 2I$, (V) $H = 2I \neq 2K$,

Case(1). Solving equations in (3.3) for scalar component of h-connection vectors with the help of $H + I + K = LC$, we have

(a) h_1 is arbitrary, $h_2 = \{(LC)_{,3}\}/(LC)$, $h_3 = I_{,2}/(H-2I)$, $h_4 = 0$,

(b) J_1 is arbitrary, $J_2 = \{(LC)_{,4}\}/(LC)$, $J_3 = 0$, (3.4)

$J_4 = (KI_{,4})/\{I(H-2I)\} = (K_{,2})/(H-2K)$,

(c) k_1 is arbitrary, $k_2 = 0$, $k_3 = (I_{,4})/(I-K) = \{I(LC)_{,4}\}/\{(I-K)LC\}$,

$k_4 = \{K(LC)_{,3}\}/\{(I-K)LC\} = (K_{,3})/(I-K)$.

From **case (II),** we have $H_{,\alpha} = I_{,\alpha} = 0$ for $\alpha = 2, 3$ and h-connection vectors are such that

(a) h_1 is arbitrary, $h_2 = 0$, $h_3 = (IK_{,2})/\{K(H-2K)\}$, $h_4 = 0$,

(b) J_1 is arbitrary, $J_2 = \{(LC)_{,4}\}/(LC)$, $J_3 = 0$, $J_4 = (K_{,2})/\{(H-2K)$ (3.5)

(c) k_1 is arbitrary, $k_2 = 0$, $k_3 = \{I(LC)_{,4}\}/\{(I-K)LC\}$, $k_4 = 0$.

From **case (III),** we have $K_{,\alpha} = I_{,\alpha} =0$ for $\alpha = 3, 4$, $H_{,3} = 0$ and h-connection vectors are such that

(a) h_1 is arbitrary, $h_2 = 0$, $h_3 = (I_{,2})/(H-2I)$, $h_4 = 0$,

(b) J_1 is arbitrary, $J_2 = 0$, $J_3 = 0$, $J_4 = (KI_{,2})/\{I(H-2I)\}=(K_{,2})/(H-2K)$, (3.6)

(c) k_1, k_2, k_3, k_4 are arbitrary.

From **case (IV),** we have $K_{,\alpha} = H_{,\alpha} = 0$ for $\alpha = 2,4$, $I_{,4} = 0$ and h-connection vectors are such that

(a) h_1 is arbitrary, $h_2 = \{(LC)_{,3}\}/(LC)$, $h_3 = (H-2I)^{-1}I_{,2}$, $h_4 = 0$,

(b) J_1 is arbitrary, $J_2 = 0$, $J_3 = 0$, $J_4 = (KI_{,2})/\{I(H-2I)\}$, (3.7)

(c) k_1 is arbitrary, $k_2 = 0$, $k_3 = 0$, $k_4 = \{K(LC)_{,3}\}/\{(I-K)LC = (K_{,3})/(I-K)$.

From **case (V),** we have $H_{,\alpha} = I_{,\alpha} = K_{,\alpha}= 0$ for $\alpha = 2, 3, 4$ and h-connection vectors are such that

(a) h_1, h_3 are arbitrary and $h_2 = h_4 = 0$, (3.8)

(b) J_1, J_4 are arbitrary and $J_2 = J_3= 0$,

(c) k_i is arbitrary.

Summarizing above results, we have the following :

THEOREM 3.2 Let F^4 be a four-dimensional C^h-symmetric Finsler space with non-zero constant unified main scalar. If the main scalars are such that $J = H' = I' =K' = J' = 0$, then

(I) for $H \neq 2I \neq 2K$, the h-connection vectors vanishes iff $I_{,2}=0$ whereas k_1, J_1, and h_1 are arbitrary,

(II) for $H = 2I \neq 2K$, the h-connection vectors vanishes iff $K_{,2}=0$ whereas k_1, J_1, h_1 are arbitrary,

(III) for $H \neq 2I = 2K$, the h-connection vectors vanishes iff $I_{,2}=0$ whereas h_1, J_1 and k_i are arbitrary,

(IV) for $H = 2K \neq 2I$, the h-connection vectors vanishes iff $I_{,2}=0$ whereas h_1, J_1, and k_1 are arbitrary,

(V) for $H = 2I = 2K$, we have $h_i = h_1 l_i + h_3 n_i$, $J_i = J_1 l_i + J_4 p_i$, k_i are arbitrary.

REMARK- Here the question arises. Does four-dimensional C^h-symmetric C- reducible Finsler space is C-reducible? A four-dimensional C^h-symmetric C-reducible Finsler space is C-reducible. In above theorem we have supposed that $J = H' = I' = K' = J'=0$. These conditions also hold in C-reducible Finsler space (Prasad, Chaubey and Patel, 2007). Besides C-reducible Finsler space, there are also some special Finsler space in which these conditions hold which is given below :

EXAMPLE 1. A Finsler space of dimension n (n>2) is called C-reducible if Cijk is

Written as (Matsumoto, 1986).

$$C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij})/(n + 1), \quad (3.9)$$

where h_{ij} is the angular metric tensor. Since $\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}$ are scalar components of

h_{ij} with respect to the Miron's frame $\{e^i_{(\alpha)}\}$ of F^4, therefore in terms of scalar components equation (3.9) can be written for a four-dimensional C-reducible

Finsler space as

$$C_{\alpha\beta\gamma} =[LC\{\delta_{2\alpha}(\delta_{\beta\gamma}-\delta_{1\beta}\delta_{1\gamma})+\delta_{2\beta}(\delta_{\gamma\alpha}-\delta_{1\gamma}\delta_{1\alpha})+\delta_{2\gamma}(\delta_{\alpha\beta}-\delta_{1\alpha}\delta_{1\beta})\}]/5 \quad (3.10)$$

Using the notation given in (2.4), the above equation gives

$$J = H' = I' = K' = J' = 0 , \qquad H = 3I = 3K= (3/5)LC \quad (3.11)$$

If the unified main scalar is constant, LC is constant . Therefore, H, I, K are constant and we have the following result :

THEOREM 3.3- In a four-dimensional C-reducible Finsler space with non-zero constant unified main scalar, the main scalars H, I and K are non-zero constants and all the remaining scalars vanish.

Using equation (3.11) and theorem (3.3) in eqation (3.3), we obtain $h_\alpha = J_\alpha = 0$ for $\alpha = 2, 3, 4$. Hence, we have the following :

THEOREM 3.4 - In a four-dimensional C^h-symmetric C-reducible Finsler space with non-zero constant unified main scalar, the h-connection vectors h_i and Ji vanish identically if $h_1 = J_1 = 0$.

EXAMPLE-2 If C_{ijk} of a Finsler space of dimension n>2 is written in the form

$$C_{ijk} = p(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1) + (q/C^2)C_iC_jC_k \quad (3.12)$$

where p+q = 1 . If p and q are constants, it is called semi C-reducible Finsler space with constant coefficients (Matsumoto and Numata, 1980) . In terms of scalar components the equation (3.12) can be written for a four-dimensional space as

$$C_{\alpha\beta\gamma} = [pLC\{ \delta_{2\alpha}(\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}) + \delta_{2\beta}(\delta_{\gamma\alpha} - \delta_{1\gamma}\delta_{1\alpha}) + \delta_{2\gamma}(\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\}]/5 + qLC\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma} . \quad (3.13)$$

Using the notation given in (2.4), the equation (3.13) gives

$$J = H' = I' = K' = J' = 0, \quad H = LC\{(3p/5) + q \}, \quad I = K = pLC/5 \quad (3.14)$$

THEOREM 3.5 In a four-dimensional semi C-reducible Finsler space with constant coefficient and constant unified main scalar, the main scalars H, I and K are non-zero constant and all the remaining main scalars vanish .

Using equation (3.14) and theorem (3.5) in equation (3.3), we get $h_\alpha = J_\alpha = 0$

for $\alpha = 2, 3, 4$. Hence we have the following :

Theorem 3.6 In a four-dimensional C^h-symmetric semi C-reducible Finsler space with constant coefficient and constant unified main scalar, the h-connection vectors h_i and J_i vanish identically if $h_1 = J_1 = 0$.

Example 3. An n-dimensional (n>2) Finsler space is called C2-like (Matsumoto and Numata, 1980) if $C_{ijk} = C_iC_jC_k/C^2$. Thus for a four-dimensional C2-like Finsler space, we have

$$H = LC, \quad I=K=J=H'=I'=K'=J'=0.$$

Hence, we have the following -

THEOREM 3.7 - In a four-dimensional C2-like Finsler space all the main scalars vanish except the main scalar H that is equal to the unified main scalar LC.

4. The results reducible to three dimensional Finsler space

In three-dimensional Finsler space there are only three main scalars H, I, J and one h-connection vectors h_i. Therefore putting $K=H'=I'=K'=J'=0$ and $J_i=k_i=0$ in (2.14), we get

$$C_{1\beta\gamma,\delta} = 0, \quad C_{222,\delta} = H_{,\delta}+3Jh_\delta, \quad C_{223,\delta} = -J_{,\delta} + (H-2I)h_\delta,$$

$$C_{233,\delta} = I_{,\delta}-3Jh_\delta, \quad C_{333,\delta} =J_{,\delta}+3Ih_\delta.$$

This equation is same as equation (29.13) of the book (Matsumoto,1986, page 195) for three dimensional Finsler space. Besides this equation (2.16) takes the form $-J_{,2} + (H-2I)h_2 = H_{,3}+3Jh_3$, $I_{,2} - 3Jh_2 = -J_{,3} + (H-2I)h_3$, $J_{,2} + 3Ih_2 = I_{,3} - 3Jh_3$.

For three dimensional Finsler space theorem(3.3) and (3.4) can be written as:

THEOREM 4.1 In a three-dimensional C- reducible Finsler space with non-zero constant unified main scalar, the main scalars H and I are constants given by 3LC/4 and LC/4 whereas J=0 (Matsumoto, 1973, Ikeda, 1991).

THEOREM 4.2 In a three-dimensional C^h-symmetric C-reducible Finsler space with non-zero constant unified main scalar, the h-connection vectors vanish identically if h_1=0.

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