ISSN:0976-4933
Journal of Progressive Science
Vol.05, No.01, pp 63-76 (2014)

# Semi-symmetric h-recurrent Finsler connection of $m^{th}$-root metric

**B. N. Prasad*, S. K. Tiwari and C. P. Maurya****
**C-10, Suraj Kund Colony, Gorakhpur (U.P.), India**
Email - baijnath_prasad2003@yahoo.co.in
****Department of Mathematics**
**K.S. Saket P.G. College, Ayodhya, Faizabad (U.P.), India**
Email - sktiwarisaket@yahoo.com, chandraprakashmaurya74@gmail.com

## Abstract

*The purpose of this paper is to obtain the semi-symmetric h-recurrent Finsler connection of $m^{th}$-root metric*

## 1. Introduction

The theory of $m^{th}$-root metrics has been first developed by Shimada (1979) as an interesting example of Finsler metric, immediately following Matsumoto and Numata's (1979) theory of cubic metric . By introducing the regularity of the metric various fundamental quantities as a Finsler metric has been found by Matsumoto and Okubo (1995). In particular, they found the Carton connection of a Finsler space with $m^{th}$-root metric.

Yashuda (1986) was introduced various connections in Finsler space. He specially dealt with TM connection which is a generalization of Carton connection in some sense. Hashiguchi (1975) and Matsumoto (1986) introduced the notion of generalized Cartan's connection C□(T) and generalized Berwald connection B□(T) respectively which are Finsler connection with (h) h-torsion. T. A.Wagner connection is a generalized Cartan connection in which (h) h-torsion is semi-symmetric. A Wagner space is a Finsler space in which Wagner connection is linear where as generalized Berwald space is a Finsler space in which generalized Berwald connection is linear. Matsumoto (1982) has discussed the Wagner's generalized Berwald space of dimension two where as Numata (1984) has considered the generalized Berwald space of G-scalar curvature.

Prasad *et.al.*, (2009) introduced the Wagner connection of Finsler space, with $m^{th}$-root metric which is h-metrical. In the present paper we are concerned with semi-symmetric h-recurrent Finsler connection with $m^{th}$-root metric.

## 2. Preliminaries

The $m^{th}$-root Finsler metric L (x, y) of an n-dimensional differentiable manifold $M^n$ is first defined by H. Shimada (1979) as

$$\{L(x, y)\}^m = a_{i_1 i_2 \cdots i_m}(x)\, y^{i1} y^{i2} \ldots\ldots.. y^{im}$$

where the coefficients $a_{i_1 i_2 \cdots i_m}(x)$ are components of a symmetric tensor field covariant of order m. Consequently, the second root metric is, of course, a Riemannian metric and we shall restrict $m > 2$ throughout the paper. The third and fourth root metrics are especially interesting and have the well known name:

$L^3 = a_{ijk}(x)\, y^i y^j y^k$ Cubit metric (Antonelli, *et al.,* 1993 ) and Matsumoto and Numata (1979)

$L^4 = a_{hijk}(x)\, y^h y^i y^j y^k$ Quartic metric (Roxburgh,1992)

We shall sketch some fundamental point of the theory of Finsler space $F^n = (M^n, L(x, y))$ with $m^{th}$-root metric L(x, y) for the later use.

Let us first define the tensor $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$ as follows:

(a) $L^{m-1} a_i = a_{i i_2 \cdots i_m}(x)\, y^{i2} y^{i3} \ldots\ldots.. y^{im}$ (2.1)

(b) $L^{m-2} a_{ij} = a_{i j i_3 \cdots i_m}(x)\, y^{i3} \ldots\ldots\ldots. y^{im}$

(c) $L^{m-3} a_{ijk} = a_{i j k i_4 \cdots i_m}(x)\, y^{i4} \ldots\ldots\ldots. y^{im}$ .

Then the normalized supporting element $l_i = \dot{\partial}_i L$ , the angular metric tensor $h_{ij} = L(\dot{\partial}_i \dot{\partial}_j L)$ , the fundamental metric tensor $g_{ij} = \dfrac{\dot{\partial}_i \dot{\partial}_j L^2}{2}$, and $g_{ijk} = \dfrac{\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2}{4}$, of $F^n$ are written as

(a) $l_i = a_i$, $h_{ij} = (m - 1)(a_{ij} - a_i a_j)$, (2.2)

(b) $g_{ij} = (m - 1)\, a_{ij} - (m - 2)\, a_i a_j$

(c) $2Lg_{ijk} = (m - 1)(m - 2)(a_{ijk} - a_{ij} a_k - a_{jk} a_i - a_{ki} a_j - 2a_i a_j a_k)$.

We have the following relations among $a_i$, $a_{ij}$ and $a_{ijk}$

(a) $a_i y^i = L, a_{ij} y^j = La_i$, $a_{ijk} y^k = La_{ij}$, (2.3)

(b) $(a_{ij} - a_i a_j) y^j = 0$, $(a_{ijk} - a_{ij} a_k) y^k = 0$,

(c) $(a_{ijk} - a_{ij} a_k) y^i = L(a_{jk} - a_j a_k)$,

(d) $L\,(\dot{\partial}_k a_{ij}) = (m-2)(a_{ijk} - a_{ij}\, a_k)$.

Let us call $a_{ij}$ (x, y) the basic tensor, because this played an important role in the papers of Matsumoto and Numata (1979) and Shimada (1979). The metric L is called regular if the basic tensor has non-vanishing determinant. Throughout our theories of $m^{th}$-root metrics we should suppose the regularity of the metrics.

Let $a_{ij}$ (x, y) be reciprocal tensor of $a_{ij}$ (x, y) and we define

$a^i = a^{ih}\, a_h$, $\quad a^k_{ij} = a^{kh} a_{hij}$ (2.4)

The reciprocal $g^{ij}$ (x, y) of the fundamental metric tensor $g_{ij}$ (x, y) and $l^i = y^i/L = g^{ih}\, l_h$ are written as

$(m \square 1)\, g^{ij} - a^{ij} + (m - 2)\, a^i\, a^j$, $\quad l^i = a^i$. (2.5)

Hence from (2.3)(a) we have

$$a_i a^i = 1, \quad a_{ij} a^j = a_i, \quad a^h_{ij} a_h = a_{ij}.$$

A Finsler connection of ($M^n$, L) is a triad $(F^i_{jk}, N^i_k, g^i_{jk})$ of a v- connection $F^i_{jk}$, a non-linear connection $N^i_k$ and a vertical connection $g^i_{jk}$ (Matsumoto, 1986). If a Finsler connection is given the h and v-covariant derivatives of any tensor field $K^i_j$ are defined by

$$K^i_{j|k} = d_k\, K^i_j + K^r_j\, F^i_{rk} - K^i_r\, F^r_{jk} \tag{2.6}$$

$$K^i_j|_k = \dot{\partial}_k\, K^i_j + K^r_j\, g^i_{rk} - K^i_r\, g^r_{jk} \tag{2.7}$$

where

$$d_k = \partial_k - N^r_k \dot{\partial}_r, \quad \partial_k = \frac{\partial}{\partial x^k}, \quad \dot{\partial}_k = \frac{\partial}{\partial y^k}.$$

The deflection tensor field $D^i_j$, the (h) h-torsion tensor field $T^i_{jk}$ and the (v) v-torsion tensor field $S^i_{jk}$ of a Finsler connection are given by

(a) $D^i_j = y^k F^i_{kj} - N^i_j = y^i_{|j}$ (2.8)

(b) $T^i_{jk} = F^i_{jk} - F^i_{kj}$

(c) $S^i_{jk} = g^i_{jk} - g^i_{kj}$.

When a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection. In this paper we shall use Cartan's connection only which will be denoted by $C\Gamma = (\Gamma^{*i}_{jk},\ G^i_j,\ g^i_{jk})$ and is uniquely determined from the metric function L by the following five axioms

($C_1$) The connection is h-metrical i.e., $g_{ij|k} = 0$

($C_2$) The connection v-metrical i.e., $g_{ij}|_k = 0$

($C_3$) The deflection tensor field $D^i_j$ vanishes,

($C_4$) The torsion tensor field $T^i_{jk}$ vanishes,

($C_5$) The torsion tensor field $S^i_{jk}$ vanishes.

Hashiguchi (1975) replaced the condition ($C_3$) by some weaker condition and determined a Finsler connection with the given deflection tensor field. He has also determined uniquely a Finsler connection by replacing the condition ($C_4$). A generalized Cartan's connection is a Finsler connection satisfying the conditions ($C_1$), ($C_2$), ($C_3$) and ($C_5$) while a Wagner connection with respect to $S_j$ is a Finsler connection satisfying the axioms ($C_1$), ($C_2$), ($C_3$), ($C_5$) and (*$C_4$) The connection is semi-symmetric i.e.,

$$T^i_{jk} = \delta^i_j S_k - \delta^i_k S_j.$$

The Wagner connection will be denoted by

$$W\Gamma = (F^i_{jk},\ N^i_k,\ g^i_{jk})$$

and are given by

$$\text{(a)}\quad F^i_{jk} = \Gamma^{*i}_{jk} + L^2(S^i_{jkh} + g^i_{jr}\ g^r_{kh})S^h + (y^i g_{jkh} - y_j g^i_{kh} - y_k g^i_{jh})S^h \quad (2.9)$$

$$+ g^i_{kh} S_0 + g_{jk} S^i - \delta^i_k S_j,$$

$$\text{(b)}\quad N^i_j = G^i_j - L^2 g^i_{jr} S^r - y_j S^i - \delta^i_j S_0$$

$$\text{(c)}\quad g^i_{jk} = \frac{1}{2} g^{ih}\, \dot{\partial}_h\, g_{jk}$$

where $S^i = g^{ij} S_j$, $S^i_{jkh}$ is the v-curvature tensor of C□ and '0' denotes the contraction with $y^i$ .

## 3. h-recurrent connection with deflection and torsion

Let the Finsler connection is h-recurrent i.e., $g_{ij|k} = b_k g_{ij}$, for some covariant vector $b_k$, we have to notice that some formula have the style different from the ones familiar to us. For example the h-recurrency of the metric tensor $g_{ij}$ gives the formula

$$g_{ijk|l} = g^h_{ik|l}\, g_{hj} + b_l\, g_{ijk}\,. \tag{3.1}$$

**Theorem (3.1).** Given a non linear connection $N^i_k$ a semi- symmetric Finsler (1, 2) tensor field $T^i_{jk}$ and covariant vector fields $b_k$ and $S_k$ in a Finsler space there exists a unique Finsler connection $(F^i_{jk},\ N^i_k,\ g^i_{jk})$ satisfying the axioms $(C_2)$, $(C_5)$ and

$(C'_1)$ The connection is h-recurrent that is $g_{ij|k} = b_k g_{ij}$,

$(C'_3)$ The non linear connection is the given $N^i_k$ ,

$(C'_4)$ The (h) h-torsion tensor field $T^i_{jk}$ is semi-symmetric i.e.,

$$T^i_{jk} = \delta^i_j S_k - \delta^i_k S_j\,.$$

**Proof :** From the axioms $(C_2)$ and $(C_5)$ it follows that the vertical connection $g^i_{jk}$ is the same as Cartan's vertical connection given by (2.9) (c).

From the axiom $(C'_1)$ we have

$$\partial_k g_{ij} - N^m_k \dot{\partial}_m g_{ij} - g_{mj} F^m_{ik} - g_{im} F^m_{jk} = b_k g_{ij}\,.$$

Applying Christoffel process to the above equation and using axiom $(C'_4)$, we get

$$F^i_{jk} = Y^i_{jk} - (g^i_{km} N^m_j + g^i_{jm} N^m_k - g^{hi} g_{jkh} N^m_h) - \frac{1}{2}(b_j \delta^i_k + b_k \delta^i_j - b^i g_{jk}) \tag{3.2}$$

$$+\ g_{jk} S^i - \delta^i_k S_j\,.$$

In view of (3.2) and axiom $(C'_3)$ it is clear that the Finsler connection $(F^i_{jk}, N^i_k,\ g^i_{jk})$ is uniquely determined from the metric function L and from the given vector fields $b_k$ and $S_k$.

For the above connection the deflection tensor field $D^i_k$ is obtained by contraction of (3.2).

$$D^i_k = G^i_k + 2g^i_{km}G^m - g^i_{km}N^m_0 - N^i_k - \frac{1}{2}(b_0\delta^i_k + b_k y^i - b^i y_k) + y_k S^i - \delta^i_k S_0 . \quad (3.3)$$

Contracting (3.3) with $y^k$, we get

$$N^i_0 = 2G^i - D^i_0 - b_0 y^i + \frac{1}{2}b^i L^2 + L^2 S^i - y^i S_0 . (3.4)$$

Substituting the value of $N^i_0$ in (3.3) and using the fact that $g^i_{jk}\, y^j = 0$, we get

$$N^i_k = G^i_k - g^i_{km}(L^2 S^m - y^m S_0 - D^m_0 + \frac{1}{2}b^m L^2) + (y_k S^i - \delta^i_k S_0 - D^i_k)$$

$$-\frac{1}{2}(b_0\delta^i_k + b_k y^i - b^i y_k) .$$

Hence we have the following:

**Theorem (3.2).** Given a Finsler (1, 1) tensor field $D^i_k$ covariant vector fields $b_k$ and $S_k$ in a Finsler space there exists a unique Finsler connection $(F^i_{jk}, N^i_k, g^i_{jk})$ satisfying the axioms $(C'_1)$, $(C_2)$, $(C'_4)$, $(C_5)$ and $(C''_3)$ The deflection tensor field the given $D^i_k$ .

The v-connection $F^i_{jk}$ is given by (3.2), in which the non-linear connection is given by

$$N^i_k = G^i_k - g^i_{km} B^m_0 + B^i_k \quad (3.5)$$

where

$$B^i_k = y_k S^i - \delta^i_k S_0 - D^i_k - \frac{1}{2}(b_0\delta^i_k + b_k y^i - b^i y_k) . (3.6)$$

The vertical connection is given by (2.9) (c).

As a special case of the above theorem if we impose the axiom $(C_3)$ instead of $(C''_3)$, the $B^i_k$ in (3.6) become

$$B^i_k = y_k S^i - \delta^i_k S_0 - \frac{1}{2}(b_0\delta^i_k + b_k y^i - b^i y_k) \quad (3.7)$$

and we have the following

**Theorem (3.3).** Given covariant vector fields $b_k$ and $S_k$ in a Finsler space there exists a unique Finsler connection $(F^i_{jk}, N^i_k, g^i_{jk})$ satisfying the axioms ($C'_1$), ($C_2$), ($C'_4$) and ($C_5$). These corfficients are given by (3.2), (2.9) (c) and

$$N^i_k = G^i_k - g^i_{km}(L^2 S^m - y^m S_0 - \frac{1}{2} b^m L^2) + y_k S^i - \delta^i_k S_0 - \frac{1}{2}(b_0 \delta^i_k + b_k y^i - b^i y_k) . \quad (3.8)$$

If we assume that $B^i_k = 0$, equation (3.5) reduces to $N^i_k = G^i_k$, conversely if $N^i_k = G^i_k$ then from (3.5) we get $B^i_k = g^i_{km} B^m_0$ which after contracting with $y^k$ and using the fact that $g^i_{km}\, y^k = 0$, we get $B^m_0 = 0$ hence $B^i_k = 0$. Thus we have the following which gives the semi-symmetric Finsler connection with deflection.

**Theorem (3.4).** Given covariant vector fields $b_k$ and $S_k$ in a Finsler space there exists a unique Finsler connection $(F^i_{jk}, N^i_k, g^i_{jk})$ satisfying the axioms ($C'_1$), ($C_2$), ($C'_4$) ($C_5$) and

($C'''_3$) The non-linear connection $N^i_k$ is the one given by E Cartan.

The coefficient $F^i_{jk}$ are given by:

$$F^i_{jk} = Y^i_{jk} - (g^i_{km} G^m_j + g^i_{jm} G^m_k - g^{hi}\, g_{jkm}\, G^m_h) - \frac{1}{2}(b_j \delta^i_k + b_k \delta^i_j - b^i g_{jk}) + g_{jk} S^i - \delta^i_k S_j . \quad (3.9)$$

The deflection tensor field $D^i_k$ is expresed as

$$D^i_k = y_k S^i - \delta^i_k S_0 - \frac{1}{2}(b_0 \delta^i_k + b_k y^i - b^i y_k) . \quad (3.10)$$

## 4. Semi-symmtric h-recurrent Finsler connection of $m^{th}$-root metric

First of all we remember equation (2.2)(b) giving the fundamental tensor $g_{ij}$ of a $m^{th}$-root Finsler space $F^n$. This tensor is different from the intrinsic metric tensor $a_{ij}$ in the regular $F^n$. Since

$L^2(x, y) = g_{ij}(x, y)\, y^i y^j = a_{ij}(x, y)\, y^i y^j$. (4.1)

$F^n$ is regarded as a generalized metric space of line element in the Moor sense (1956), because there is generally no such a function M (x, y) that $a_{ij}$ is given by

$$a_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j M^2 .$$

In view point of (4.1) it seems natural to us to consider the problem determining a semi-symmetric Finsler connection based on the intrinsic metric tensor $a_{ij}(x, y)$.

**Theorem (4.1).** In a regular Finsler space $F^n$ with $m^{th}$-root metric, a semi-symmetric Finsler connection $^*C\Gamma = ({}^*F^i_{jk}, {}^*N^i_k, {}^*g^i_{jk})$ is uniquely determined from the intrinsic metric tensor $a_{ij}$ (x, y) by the following five axioms:

($F_1$) It is intrinsic h-metrical i.e., $a_{ij}|_{*k} = 0$,

($F_2$) It is intrinsic v-metrical i.e., $a_{ij}|_{*k} = 0$,

($F_3$) Its deflection tensor field $^*D^i_j$ vanishes i.e., $^*N^i_j = y^k\, {}^*F^i_{kj}$,

($F_4$) The (h) h-torsion tensor field $^*T^i_{jk}$ is semi-symmetric with respect to $p_j$ i.e.,

$$^*T^i_{jk} = \delta^i_k\, p_k - \delta^i_k\, p_j .$$

($F_5$) It is v-symmetric i.e., $^*S^i_{jk} = 0$.

**Proof.** The axioms ($F_2$) and ($F_5$) leads us immediately to

$$^*g^i_{jk} = \frac{1}{2} a^{ir} (\dot{\partial}_k\, a_{jr} + \dot{\partial}_j\, a_{kr} - \dot{\partial}_r\, a_{jk}) \; .(4.2)$$

That is the coefficients $^*g^i_{jk}$ are Christoffel symbols constructed from $a_{ij}$ (x, y) with respect to $y^i$ . Differetiating (2.2)(b) by $y^k$ and using (2.2)(a), we get

$$\dot{\partial}_k\, a_{ij} = \frac{2}{m-1} g_{ijk} + \frac{m-2}{L(m-1)} (h_{ik} l_j + h_{jk} l_i) \; .(4.3)$$

Thus (4.2), (4.3) and (2.5) give the relation

$$^*g^i_{jk} = g^i_{jk} + \frac{m-2}{L(m-1)} h_{jk} l^i \; (4.4)$$

Thus the vertical connection $^*g^i_{jk}$ is determined uniquely from L.

Secondly we shall find $^*F^i_{jk}$ in terms of $F^i_{jk}$ which are uniquely obtained from metric function L by the axioms quoted in theorem (3.4).

Let $D^i_{jk} = {}^*F^i_{jk} - F^i_{jk}$

where $F^i_{jk}$ is given by (3.2).

Since $F^i_{jk} - F^i_{kj} = \delta^i_j\, S_k - \delta^i_k\, S_j$, therefore axiom ($F_4$) gives

$$D^i_{jk} - D^i_{kj} = \delta^i_j\, P_k - \delta^i_k\, P_j \; (4.5)$$

where $P_i = p_i - S_i$.

From the axioms ($F_3$) and ($C_3$), it follows that

$$D^i_{0k} = {}^*N^i_k - N^i_k .$$

Since $g_{ij|k} = b_k\, g_{ij}$, we have $a_{i|k} = \frac{1}{2} b_k\, a_i$, equation (2.2)(b) leads to $a_{ij|k} = b_k\, a_{ij}$ with respect to the connection $F^i_{jk}$ given in theorem (3.4).

By Virtue of $a_{ij|*k} = 0$ and $a_{ij|k} = b_k\, a_{ij}$, we may write

$$\partial_r\, a_{ij}\, D^r_{0k} + D_{ijk} + D_{jik} = b_k a_{ij} \quad (4.6)$$

where $D_{ijk} = a_{rj}\, D^r_{ik}$. Applying Christoffel process to (4.6) and using (4.3) and (4.5), we get

$$D_{ijk} + \frac{1}{m-1}[g^r_{ij}\, D_{0rk} + g^r_{jk} D_{0ri} - g^r_{ki}\, D_{0rj}] + \frac{m-2}{2L(m-1)}[l_i (D_{0jk} - D_{0kj}) \quad (4.7)$$

$$+ l_k (D_{0ji} - D_{0ij}) + l_j (D_{0ki} - D_{0ik}) - \frac{m-2}{L^2(m-1)}(l_i\, l_j\, D_{00k} + l_j\, l_k D_{00i} - l_k\, l_i\, D_{00j})$$

$$+ a_{jk} P_i - a_{ik} P_j - \frac{1}{2}(b_k\, a_{ij} + b_i\, a_{jk} - b_j\, a_{ik}) = 0 .$$

Contraction of (4.7) by $y^i$ yields,

$$D_{0jk} + \frac{1}{m-1} g^r_{jk} D_{0r0} - \frac{m-2}{L^2(m-1)} l_j\, l_k D_{000} + \frac{m-2}{2(m-1)}(D_{0jk} - D_{0kj}) \quad (4.8)$$

$$+ \frac{m-2}{2L(m-1)}[l_j (D_{0k0} - D_{00k}) + l_k (D_{0j0} + D_{00j})]$$

$$+ a_{jk} P_0 - La_k P_j - \frac{1}{2}(b_k L\, a_j + b_0\, a_{jk} - b_j\, La_k) = 0 .$$

Contraction of (4.8) by $y^k$ and use of (2.3) (a) yields

$$\frac{(2m-3)}{m-1} D_{0j0} - \frac{(m-2)}{L(m-1)} l_j\, D_{000} + L\, a_j P_0 - L^2 P_j - b_0\, La_j + \frac{1}{2} b_j\, L^2 = 0 .(4.9)$$

Further contraction of above equation by $y^j$ gives $D_{000} = \frac{1}{2} b_0 L^2$ hence (4.9) reduces to

$$\frac{2m-3}{m-1} D_{0j0} = L^2 P_j - L\, a_j P_0 + \frac{(m-2)}{2(m-1)} b_0 L l_j + b_0\, La_j - \frac{1}{2} b_j\, L^2 .(4.10)$$

Putting the value of $D_{0j0}$ from (4.10) in (4.8), we get

$$D_{0jk}+\frac{L^2}{2m-3}P_r\,g^r_{jk}+\frac{m-2}{2(m-1)}(D_{0jk}-D_{0kj})+\frac{m-2}{2L(m-1)}[l_j\{\frac{m-1}{2m-3}L^2P_k \quad (4.11)$$

$$-\frac{m-1}{2m-3}Ll_kP_0-D_{00k}\}+l_k\{\frac{m-1}{2m-3}L^2P_j-\frac{m-1}{2m-3}Ll_jP_0+D_{00j}\}]+a_{jk}P_0-La_kP_j$$

$$-\frac{1}{2(2m-3)}g^r_{jk}b_rL^2+\frac{(m-2)}{4(2m-3)}(2l_j\,l_kb_0+l_kb_jL-l_jb_kL)-\frac{1}{2}(b_kLa_j+b_0a_{jk}-b_jLa_k)\,.$$

Contraction of (4.11) with $y^j$ leads to

$$D_{00k}=\frac{1}{2}[b_kL^2-\frac{m-2}{2m-3}b_0Ll_k]\,.$$

Thus (4.11) reduces to

$$\frac{3m-4}{2(m-1)}D_{0jk}-\frac{m-2}{2(m-1)}D_{0kj}+\frac{L^2}{2m-3}P_r\,g^r_{jk}+\frac{m-2}{2(m-3)}[L(l_j\,P_k+l_k\,P_j)-2l_j\,l_k\,P_0]$$

$$+a_{jk}P_0-La_kP_j-\frac{1}{2(2m-3)}g^r_{jk}b_rL^2+\frac{1}{2}[\frac{m-2}{2m-3}l_j\,l_k+a_{jk}]b_0$$

$$+\frac{(m-2)^2L}{4(m-1)(2m-3)}[b_k\,l_j-b_j\,l_k]-\frac{1}{2}(b_kLa_j-b_jLa_k)\,. \quad (4.12)$$

Consequently (4.7) yields

$$D_{ijk}=\frac{L^2}{(m-1)(2m-3)}[g^r_{ij}\,g^h_{rk}+g^r_{jk}\,g^h_{ri}-g^r_{ki}\,g^h_{rj}]P_n+\frac{1}{m-1}P_0\,g_{ijk}-\frac{1}{(2m-3)}LP_r$$

$$[g^r_{ij}\,l_k+g^r_{jk}\,l_i]+\frac{1}{m-1}Ll_jg^r_{ik}P_r-\frac{m-2}{2m-3}l_i\,l_kP_j+\frac{m-2}{L(m-1)}a_{ik}l_j\,P_0$$

$$-\frac{(m-2)^2}{L(m-1)(2m-3)}l_i\,l_jl_kP_0+a_{ik}P_j-a_{jk}P_i-\frac{m-2}{(2m-3)}[l_i\,l_jP_k+l_jl_kP_i]$$

$$-\frac{1}{2(m-1)}b_0g_{ijk}+\frac{m-2}{2(m-1)(2m-3)}[\frac{b_hL^2}{2m-3}(g^r_{ij}\,g^h_{rk}+g^r_{jk}\,g^h_{ri}-g^r_{ik}\,g^h_{rj})$$

$$-b_rL^2(g^r_{ij}\,l_k+g^r_{jk}\,l_i-g^r_{ik}\,l_j)-\frac{(3m-5)}{(2m-3)}b_rL(g^r_{ij}\,l_k+g^r_{jk}\,l_i-g^r_{ik}\,l_j)]$$

$$-\frac{1}{2(m-1)(2m-3)}b_hL^2(g^r_{ij}\,g^h_{rk}+g^r_{jk}\,g^h_{ri}-g^r_{ik}\,g^h_{rj})+\frac{(7m^2-20m+14)}{4(m-1)^2(2m-3)}b_rL\times$$

$$(l_k g^r_{ij} + l_i g^r_{jk} - l_j g^r_{ik}) - \frac{(m-2)}{2(2m-3)}[L(l_i l_j b_k + l_j l_k b_i - l_i l_k b_j) + \frac{b_r L}{2m-3}(g^r_{jk} l_i + g^r_{ij} l_k)$$

$$+\frac{(3m-5)}{(2m-3)}(l_i l_j b_k + l_j l_k b_i - l_i l_k b_j)] - \frac{(m-2)}{2L(m-1)}\Big[\frac{1}{2m-3} g^r_{ik} l_j b_r L^2$$

$$-\frac{(m-2)}{(2m-3)} l_i l_j l_k b_0 + b_0 l_j a_{ik}\Big] + \frac{(m-2)}{2L^2(m-1)}[(l_i l_j b_k + l_j l_k b_i - l_i l_k b_j)$$

$$-\frac{3(m-2)}{(2m-3)} b_0 L l_i l_j l_k] + \frac{1}{2}(b_k a_{ij} + b_i a_{jk} - b_j a_{ik}).$$

This equation determines the value of ${}^*F^i_{jk}$ in terms of $F^i_{jk}$ and consequently ${}^*N^i_j$ in terms of $N^i_j$

## 5. Semi-symmetric h-recurrent Wagner connection with m$^{th}$-root metric

We shall find the semi-symmetric h-recurrent Wagner connection WT of a Finsler space $F^n$ with m$^{th}$-root metric. Since $g^i_{jk}$ is immediately given (2.2)(c) and (2.9)(c), we shall only deal with $F^i_{jk}$ and $N^i_j$.

On account of axioms ($C_1$), ($C_3$) and ($C'_4$) the quantities $F^i_{jk}$, $N^i_j$ are determined by three conditions

(a) $a_{ij|k} = b_k a_{ij}$ (b) $N^i_j = y^k F^i_{kj}$ (5.1)

(c) $F^i_{jk} - F^i_{kj} = \delta^i_j S_k - \delta^i_k S_j$.

It follows from (2.3)(d) that (5.1)(a) is written as

$$\partial_k a_{ij} - \frac{(m-2)}{L}(a_{ijr} - a_{ij} a_r)N^r_k - a_{rj} F^r_{ik} - a_{ir} F^r_{jk} = b_k a_{ij}. (5.2)$$

Let us apply Christoffel process to (5.2). If we put

$$2f_{ijk} = \partial_k a_{ij} + \partial_i a_{jk} - \partial_j a_{ik}.$$

Then in view of (5.1) (c), we have

$$a_{jr} F^r_{ki} = f_{ijk} - \frac{m-2}{2L}(a_{ijr} - a_{ij} a_r)N^r_k - \frac{m-2}{2L}(a_{jkr} - a_{jk} a_r)N^r_i \quad (5.3)$$

$$+\frac{m-2}{2L}(a_{kir} - a_{ki} a_r)N^r_j + a_{ik} S_j - a_{ij} S_k - \frac{1}{2}(b_k a_{ij} + b_i a_{jk} - b_j a_{ik}).$$

Contraction of (5.3) by $y^k$ and use of (5.1) (b) gives

$$a_{jr}N^r_i = f_{ij0} - \frac{m-2}{2L}(a_{ijr} - a_{ij}a_r)N^r - \frac{m-2}{2}(a_{jr} - a_j a_r)N^r_i \quad (5.4)$$

$$+\frac{m-2}{2}(a_{ir} - a_i a_r)N^r_j + La_iS_j - a_{ij}S_0 - \frac{1}{2}(b_0a_{ij} + b_iLa_j - b_jLa_i)$$

where we have put $2N^r = y^kN^r_k$. Further contraction of (5.4) by $y^i$ and use of (2.3)(b) and (2.3)(c) gives

$$2(m-1)a_{jr}N^r = f_{0j0} + 2(m-2)a_rN^ra_j + L^2S_j - La_jS_0 - b_0La_j + \frac{1}{2}b_jL^2 \quad .(5.5)$$

If we contract (5.5) by $y^j$, then (2.3)(a) leads to

$$2La_rN^r = f_{000} - \frac{1}{2}b_0 L^2 \quad .(5.6)$$

Hence (5.5) can be written as

$$2(m-1)a_{jr}N^r = f_{0j0} + \frac{m-2}{L}f_{000}a_j + L^2S_j - La_jS_0 - \frac{1}{2}(mb_0La_j - b_jL^2). \quad (5.7)$$

On the other hand transvecting (5.4) by $y^j$, we get from (2.3)(b) and (2.3)(c)

$$La_rN^r_i = f_{i00} - \frac{1}{2}b_i L^2 \quad .(5.8)$$

We substitute (5.6), (5.7) and (5.8) in (5.4) using the notation $a^k_{ij}$ defined by (2.4), we get (5.4) of the following form

$$\frac{m}{2}a_{jr}N^r_i - \frac{m-2}{2}a_{ir}N^r_j = f_{ij0} - \frac{m-2}{2L(m-1)}f_{0r0}a^r_{ij} + \frac{m-2}{2(m-1)L^2}f_{000}a_{ij} \quad (5.9)$$

$$+\frac{m-2}{2L}(f_{i00}a_j - f_{j00}a_i) - \frac{(m-2)L}{2(m-1)}a^r_{ij}S_r - \frac{m}{2(m-1)}a_{ij}S_0 + La_iS_j$$

$$-\frac{(m-2)L}{4(m-1)}a^r_{ij}b_r - \frac{m}{4}\left[\frac{1}{m-1}b_0a_{ij} + a_jb_iL - a_ib_jL\right].$$

Let us sum (5.9), the equation obtained from it by multiplying $\frac{m}{m-2}$ and interchanging i and j. Then we final obtain

$$\frac{2(m-1)}{m-2}a_{ir}N^r_j = f_{ij0} + \frac{m}{m-2}f_{ji0} + \frac{1}{L^2}f_{000}a_{ij} - \frac{1}{L}(f_{i00}a_j - f_{j00}a_i + f_{0r0}a^r_{ji}) \quad (5.10)$$

$$-La^r_{ji}S_r - \frac{m}{m-2}a_{ij}S_0 + La_iS_j + \frac{m}{m-2}La_jS_i - \frac{1}{2}La^r_{ij}b_r$$

$$-\frac{m}{2(m-2)}(b_0a_{ij} + a_ib_jL - a_jb_iL).$$

If we define

$$A_{ijk}(x,y) = (\partial_k a_{ijk_3......k_m} + \partial_i a_{jkk_3......k_m} - \partial_j a_{ikk_3......k_m})y^{k_3}......y^{k_m},$$

then we have

$$2L^{m-2}f_{ijk} = A_{ijk} - \frac{m-2}{mL^2}(a_{ij}A_{k00} + a_{jk}A_{i00} - a_{ki}A_{j00}).$$

Therefore equation (5.10) may be written as

$$\frac{2(m-1)}{m-2}a_{ir}N^r_j = \frac{1}{2(m-2)L^{m-2}}\{(m-2)A_{ij0} + mA_{ji0}\} \tag{5.11}$$

$$-\frac{1}{2mL^{m-1}}a^r_{ij}\{(m-2)A_{r00} + mA_{0r0}\} - La^r_{ji}S_r - \frac{m}{m-2}a_{ij}S_0 + La_iS_j$$

$$+\frac{mL}{m-2}a_jS_i - \frac{1}{2}La^r_{ij}b_r - \frac{m}{2(m-2)}(b_0a_{ij} + a_ib_jL - a_jb_iL).$$

Also equation (5.7) may be written as

$$2(m-1)a_{jr}N^r = \frac{1}{2L^{m-2}}(A_{0j0} + \frac{m-2}{m}A_{j00}) + L^2S_j - La_jS_0 - \frac{1}{2}(mb_0La_j - b_jL^2). \tag{5.12}$$

Summarizing the above results we have the following

**Theorem (5.1).** The coefficients $(F^i_{jk}, N^i_k, g^i_{jk})$ of the semi-symmetric h-recurrent Wagner connection with mth-root metric are given (5.5), (5.10) or (5.11) and (2.2)(c) respectively where $N^i = \frac{1}{2}N^i_r y^r$ is given (5.7) or (5.12).

## References

1. Antonelli, P.L., Ingarden, R.S. and Matsumoto, M. (1993). The theory of sprays and Finsler spaces with applications in Physics and Biology, Kluwer Academic Publications, Dordrecht/Boston/Landon
2. Hashiguchi, M. (1975). On Wagner's generalized Berwald space, *J. Korean Math Soc*., 12:51
3. Matsumoto, M. and Numata S. (1979). On Finsler space with a cubic metric *Tensor N. S*., 33:153.
4. Matsumoto, M. (1986).Foundations of Finsler geometry and special Finsler spaces, Kasieisha Press, *Saikawa Otsu* 520, Japan
5. Matsumoto, M. (1981). Berwald connections with (h) h-torsion and generalized Berwald spaces, *Tensor, N. S*., 35:223
6. Matsumoto, M. (1982). On Wagner's generalized Berwald spaces of dimension two, *Tensor, N. S*., 36:303
7. Matsumoto, M. and Okubo, K.(1995). Theory of Finsler space with mth root metric : Connections and main scalars, *Tensor, N. S*., 56:93
8. Moor, A. (1956). Entwicklung einer Geometric derallgemeinen metrischen Linienelementraume, *Acta Sci. Math.* (Szeged), 17: 85
9. Numata, S. (1984).On generalized Berwald spaces of G-scalar curvature, *Tensor, N. S*., 4: 200
10. Prasad, B. N., Pandey, T. N. and Srivatava, Ekta (2009).Wagner connection of a Finsler space with $m^{th}$-root metric, *Bull. Cal. Math. Soc*., 187:196
11. Roxburgh, I. W. (1972).Poset Newtonian tests of quaetic metric theories of gravity, *Report on Math. Phys*., 31:171
12. Shimada, H. (1979). On Finsler spaces with the metric, *Tensor, N. S*., 83:365
13. Yashuda, H. (1986). A study of connections in a Finsler spaces, Ann. Rep. *Asahikawa Med. Coll*. 7:101

**Received on 18.02.2014 and accepted on 23.05.2014**