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Conformal Exponential Change of Finsler Metric

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Abstract

The purpose of the present paper is to find the necessary condition under which a conformal exponential change becomes a projective change. We have also found the conditions under which conformal exponential change of Douglas space becomes a Douglas space.

Keywords- Conformal exponential change, Projective change, Finster space, Douglas space.

1. Introduction

Let $F^n = (M^n, L)$ be a Finster space equipped with the fundamental function L (x, y) on the smooth manifold M^n. Let $\beta = b_i(x) y^i$ be one-form of the manifold M^n, then $L \to \bar{L}e^{\beta/\bar{L}}$, $\bar{L} = e^{\sigma} L$ is called conformal exponential change of Finster metric. If we write $L^* \to \bar{L}e^{\beta/\bar{L}}$ and $F^{*n} = (M^n, L^*)$, then the Finsler space F^{*n} is said to be obtained from F^n by a conformal exponential change. The quantities corresponding to F^{*n} are denoted by putting star on those quantities.

2. Preliminaries

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and respectively and write $L_i = \partial_i L$, $L_{ij} = \partial_i \partial_i L$, $L_{ijk} = \partial_i \partial_i \partial_k L$. Then $h_{ij} = LL_{ij}$ = angular metric tensor of F^n.

The geodesics of F^n are given by the system of differential equations

$$\frac{d^2x^i}{ds^2} + G^i\left(x, \frac{dx}{ds}\right) = 0$$

where $G^i(x, y)$ are positively homogenous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \quad F = \frac{L^2}{2} \tag{2.1}$$

where g^{ij} is the inverse of $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$.

The well known Berwald connection $G\Gamma = (G^i_{jk}, G^i_j)$ of the Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by Park and Lee (2001)

$$G^i_j = \dot{\partial}_j G^i, \qquad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Cartan's connection is constructed from the metric function L by the following five axioms Park and Lee (2001)

(i) $g_{ij|k} = 0$, (ii) $g_{ij}|_k = 0$ (iii) $F^i_{jk} = F^i_{kj}$ (iv) $F^i_{0k} = G^i_k$ (v) $G^i_{jk} = G^i_{kj}$

where $_{|k}$ and $|_k$ denote h- and v-covariant derivatives with respect to $C\Gamma$. The h-covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are zero.

We shall denote

$$2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}. \tag{2.2}$$

3. Conformal Exponential Change of Finsler Metric

The conformal exponential change of Finsler metric L is given by

$$L^* = \bar{L}e^{\beta/\bar{L}} \tag{3.1}$$

where $\beta(x, y) = b_i(x)\, y^i$.

We may put

$$G^{*i} = G^i + D^i. \tag{3.2}$$

Then $G^{*i}_j = G^i_j + D^i_j$ and $G^{*i}_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_{jk} = \dot{\partial}^i_j D_k$ and $D^i_j = \dot{\partial}_j D^i$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equations $L_{ij|k} = 0$ and $L^*_{ij|k} = 0$ in F^n and F^{*n} respectively (Mosmoto, 1974). i.e.,

(a) $$\partial_k L_{ij} - L_{ijr}G^r_k - L_{ir}F^r_{jk} - L_{jr}F^r_{ik} = 0, \tag{3.3}$$

(b) $$\partial_k L^*_{ij} - L^*_{ijr}G^{*r}_k - L^*_{rj}F^{*r}_{ik} - L^*_{ir}F^{*r}_{jk} = 0.$$

Since $\partial_i\beta = b_i$, $\dot{\partial}_i L = e^{\sigma} L_i$ from (3.1), we have

(a) $$L_i^* = \left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}e^{\beta/\bar{L}}\,L_i + e^{\beta/\bar{L}}\,b_i \tag{3.4}$$

(b) $$L_{ij}^* = \left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}e^{\beta/\bar{L}}\,L_{ji} + e^{\beta/\bar{L}}\left[\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}L_iL_j - \frac{\beta e^{\sigma}}{\bar{L}^2}(L_ib_j + L_jb_i) + \frac{1}{\bar{L}^2}b_ib_j\right]$$

(c) $$\partial_j L_i^* = \left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}e^{\beta/\bar{L}}\,(\partial_j L_i) + e^{\beta/\bar{L}}\left[\left\{\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}L_i - \frac{\beta e^{2\sigma}}{\bar{L}^2}\right\}(\partial_j L)\right.$$

$$\left. + \left\{\frac{\beta e^{\sigma}}{\bar{L}^2}L_i - \frac{1}{\bar{L}}b_i\right\}(\partial_j\beta) + (\partial_j b_i) + \left\{\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_i - \frac{\beta}{\bar{L}}b_i\right\}\sigma_j\right]$$

(d) $$\partial_k L_{ij}^* = \left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}e^{\beta/\bar{L}}\,(\partial_k L_{ij}) + e^{\beta/\bar{L}}\left[\left\{\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}L_{ij} - \frac{e^{3\sigma}\beta^2(\beta+3\bar{L})}{\bar{L}^5}L_iL_j\right.\right.$$

$$\left. + \frac{e^{2\sigma}\beta(\beta+2\bar{L})}{\bar{L}^4}(L_ib_j + L_jb_i) - \frac{(\beta+\bar{L})}{\bar{L}^3}b_ib_j\right\}(\partial_k L)$$

$$+ \left\{\frac{-\beta e^{\sigma}L_{ij}}{\bar{L}^2} + \frac{e^{2\sigma}\beta(\beta+2\bar{L})}{\bar{L}^4}L_iL_j - \frac{(\beta+\bar{L})e^{\sigma}}{\bar{L}^3}(L_ib_j + L_jb_i) + \frac{1}{\bar{L}^2}b_ib_j\right\}(\partial_k\beta)$$

$$+ \left\{\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}L_j - \frac{\beta e^{\sigma}}{\bar{L}^2}b_j\right\}(\partial_k L_i) + \left\{\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}L_i - \frac{\beta e^{\sigma}}{\bar{L}^2}b_i\right\}(\partial_k L_j)$$

$$- \left\{\frac{\beta e^{\sigma}}{\bar{L}^2}L_j - \frac{1}{\bar{L}}b_j\right\}(\partial_k b_i) - \left\{\frac{\beta e^{\sigma}}{\bar{L}^2}L_i - \frac{1}{\bar{L}}b_i\right\}(\partial_k b_j)$$

$$+ \left\{\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_{ij} - \frac{(\beta+\bar{L})e^{2\sigma}\beta^2}{\bar{L}^4}L_iL_j - \frac{(\beta+\bar{L})}{\bar{L}^2}b_ib_j - \frac{(\beta+\bar{L})}{\bar{L}^4}L_iL_j\right.$$

$$\left.\left. + \frac{\beta(\beta+\bar{L})\,e^{\sigma}}{\bar{L}^3}(L_ib_j + L_jb_i)\right\}\sigma_k\right]$$

(e) $$L_{ijk}^* = \left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}e^{\beta/\bar{L}}\,L_{ijk} + e^{\beta/\bar{L}}\left[\frac{\beta^2 e^{2\sigma}}{\bar{L}^3}(L_iL_{jk} + L_jL_{ik} + L_kL_{ij})\right.$$

$$-\frac{\beta e^{\sigma}}{\overline{L}^2}(L_{ij}b_k + L_{jk}b_i + L_{ik}b_j) - \frac{\beta^2 e^{2\sigma}(\beta+3\overline{L})}{\overline{L}^5} L_i L_j L_k$$

$$+\frac{\beta(\beta+2\overline{L})e^{2\sigma}}{\overline{L}^4}(L_i L_j b_k + L_j L_k b_i + L_k L_i b_j) - \frac{(\beta+\overline{L})e^{\sigma}}{\overline{L}^3} \times$$

$$\left.(L_i b_j b_k + L_j b_i b_k + L_k b_i b_j) + \frac{1}{\overline{L}^2} b_i b_j b_k \right].$$

From (3.2) and (3.3) (b), we have

$$\partial_k L^*_{ij} - L^*_{ijr}(G^r_k + D^r_k) - L^*_{rj}(F^r_{ik} + {}^cD^r_{ik}) - L^*_{rr}(F^r_{jk} + {}^cD^r_{jk}) = 0$$

where $F^{*i}_{jk} - F^i_{jk} = {}^cD^i_{jk}$.

Substituting the values of $\partial_k L^*_{ij}$, L^*_{ir} and L^*_{ijr} from (3.4), using (3.3) (a) and contracting the result thus obtained by y^k, we get

$$2\left[\left(1-\frac{\beta}{\overline{L}}\right)e^{\sigma} L_{ijr} + \frac{\beta^2 e^{2\sigma}}{\overline{L}^3}(L_i L_{jr} + L_j L_{ir} + L_r L_{ij}) - \frac{\beta e^{\sigma}}{\overline{L}^2}(L_{ij}b_r + L_{jr}b_i + L_{ri}b_j) \right. \quad (3.5)$$

$$-\frac{\beta^2(\beta+3\overline{L})e^{3\sigma}}{\overline{L}^5} L_i L_j L_r + \frac{\beta(\beta+2\overline{L})e^{2\sigma}}{\overline{L}^4}(L_i L_j b_r + L_i L_r b_i + L_r b_i b_j)$$

$$\left.-\frac{(\beta+\overline{L})e^{\sigma}}{\overline{L}^3}(L_i b_j b_r + L_j b_i b_j + L_r b_i b_j) + \frac{1}{\overline{L}^2} b_i b_j b_r \right] D^r$$

$$+\left\{\left(1-\frac{\beta}{\overline{L}}\right)e^{\sigma} L_{ir} + \frac{\beta^2 e^{2\sigma}}{\overline{L}^3} L_r L_j - \frac{\beta e^{\sigma}}{\overline{L}^2}(L_j b_r + L_r b_j) + \frac{1}{\overline{L}} b_j b_r \right\} D^r_i$$

$$+\left\{\left(1-\frac{\beta}{\overline{L}}\right)e^{\sigma} L_{jr} + \frac{\beta^2 e^{2\sigma}}{\overline{L}^3} L_r L_i - \frac{\beta e^{\sigma}}{\overline{L}^2}(L_i b_r + L_r b_i) + \frac{1}{\overline{L}} b_i b_r \right\} D^r_j$$

$$+\left\{\frac{\beta e^{\sigma}}{\overline{L}^2} L_j - \frac{1}{\overline{L}} b_j \right\}(r_{i0} + s_{i0}) + \left\{\frac{\beta e^{\sigma}}{\overline{L}^2} L_i - \frac{1}{\overline{L}} b_i \right\}(r_{j0} + s_{j0})$$

$$+\left\{\frac{\beta e^{\sigma}}{\overline{L}^2} L_{ij} - \frac{\beta(\beta+2\overline{L})e^{2\sigma}}{\overline{L}^4} L_i L_j + \frac{(\beta+\overline{L})e^{\sigma}}{\overline{L}^3}(L_i b_j + L_j b_i) - \frac{1}{\overline{L}^2} b_i b_j \right\} r_{00}$$

$$+\left\{\left(1-\frac{\beta}{\overline{L}}+\frac{\beta^2}{\overline{L}^2}\right)e^{\sigma} L_{ij} - \frac{\beta^2(\beta+\overline{L})e^{2\sigma}}{\overline{L}^4} L_i L_j - \frac{(\beta+\overline{L})}{\overline{L}^2} b_i b_j\right.$$

$$+\frac{\beta e^{\sigma}(\beta+\bar{L})}{\bar{L}^3}(L_i b_j + L_j b_i)\Bigg\}\sigma_0 = 0,$$

where '0' stands for contraction with y^k, viz. $r_{j0} = r_{jk}\, y^k$, $r_{00} = r_{ij}\, y^i y^j$ and we have used the fact that $D^i_{jk} y^k = {}^cD^i_{jk} y^k = D^i_j$ (Mosmoto,1986)

Next, we deal with $\partial_j L^*_i - L^*_{ir} G^{*r}_j - L^*_r F^{*r}_{ij} = 0$, then

$$\partial_j L^*_i - L^*_{ir}(G^r_j + D^r_j) - L^*_r(F^r_{ij} + {}^cD^r_{ij}) = 0 \,. \text{--} \tag{3.6}$$

Putting the values of $\partial_j L^*_i$, L^*_{ir} and L^*_r from (3.4) in (3.6), using equation $L^*_{ij} =$ $\partial_j L_i - L_{ir} G^r_j - L_r F^r_{ij} = 0$ and rearranging the terms, we get

$$b_{i|j} = \left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma} L_{ir} + \frac{\beta^2 e^{2\sigma}}{\bar{L}^3} L_r L_i - \frac{\beta e^{\sigma}}{\bar{L}^2}(L_i b_r + L_r b_i) + \frac{1}{\bar{L}} b_i b_r\right\} D^r_j$$

$$+\left\{\frac{\beta e^{\sigma}}{\bar{L}^2} L_i - \frac{1}{\bar{L}} b_i\right\}(r_{j0} + s_{j0}) + \left[\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma} L_r + b_r\right]{}^cD^r_{ij}$$

$$-\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma} L_i - \frac{\beta}{\bar{L}} b_i\right]\sigma_j,$$

which after using (2.2) gives

$$2r_{ij} = \left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma} L_{ir} + \frac{\beta^2 e^{2\sigma}}{\bar{L}^2} L_r L_i - \frac{\beta^2 e^{\sigma}}{\bar{L}^2}(L_i b_r + L_r b_i) + \frac{1}{\bar{L}} b_i b_r\right\} D^r_j \tag{3.7}$$

$$+\left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma} L_{jr} + \frac{\beta^2 e^{2\sigma}}{\bar{L}^2} L_r L_j - \frac{\beta^2 e^{\sigma}}{\bar{L}^2}(L_j b_r + L_r b_j) + \frac{1}{\bar{L}} b_j b_r\right\} D^r_i$$

$$+\left\{\frac{\beta e^{\sigma}}{\bar{L}^2} L_i - \frac{1}{\bar{L}} b_i\right\}(r_{j0} + s_{j0}) + \left\{\frac{\beta e^{\sigma}}{\bar{L}^2} L_j - \frac{1}{\bar{L}} b_j\right\}(r_{i0} + s_{i0})$$

$$+2\left[\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma} L_r + b_r\right]{}^cD^r_{ij} - \left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma} L_i - \frac{\beta}{\bar{L}} b_i\right]\sigma_j$$

$$-\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma} L_j - \frac{\beta}{\bar{L}} b_j\right]\sigma_i$$

and

$$2s_{ij}=\left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}L_{ir}+\frac{\beta^2e^{2\sigma}}{\bar{L}^2}L_rL_i-\frac{\beta^2e^{\sigma}}{\bar{L}^2}(L_ib_r+L_rb_i)+\frac{1}{\bar{L}}b_ib_r\right\}D^r_j \tag{3.8}$$

$$+\left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}L_{jr}+\frac{\beta^2e^{2\sigma}}{\bar{L}^2}L_rL_j-\frac{\beta^2e^{\sigma}}{\bar{L}^2}(L_jb_r+L_rb_j)+\frac{1}{\bar{L}}b_jb_r\right\}D^r_i$$

$$+\left\{\frac{\beta e^{\sigma}}{\bar{L}^2}L_i-\frac{1}{\bar{L}}b_i\right\}(r_{j0}+s_{j0})+\left\{\frac{\beta e^{\sigma}}{\bar{L}^2}L_j-\frac{1}{\bar{L}}b_j\right\}(r_{i0}+s_{i0})$$

$$-\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_i-\frac{\beta}{\bar{L}}b_i\right]\sigma_j+\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_j-\frac{\beta}{\bar{L}}b_j\right]\sigma_i .$$

Subtracting (3.7) from (3.5) and contracting the resulting equation by y^i, we obtain

$$2\left\{-\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}L_{jr}-\frac{\beta^2e^{2\sigma}}{\bar{L}^3}L_rL_j+\frac{\beta e^{\sigma}}{\bar{L}^2}(L_jb_r+L_rb_j)-\frac{1}{\bar{L}}b_jb_r\right\}D^r \tag{3.9}$$

$$+\left\{\frac{1}{\bar{L}}b_j-\frac{\beta e^{\sigma}}{\bar{L}^2}L_j\right\}r_{00}+2r_{j0}=2\left\{\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}L_r+b_r\right\}D^r_j-(\bar{L}-\beta)\sigma_j .$$

Contracting (3.9) by y^i, we get

$$2(\bar{L}-\beta)e^{\sigma}L_rD^r+2\bar{L}b_rD^r=\bar{L}r_{00}+\frac{1}{2}\bar{L}(\bar{L}-\beta)\,\sigma_0 . \tag{3.10}$$

Subtracting (3.8) from (3.5) and contracting the resulting equation by y^j, we get

$$\left[\left(1-\frac{\beta}{\bar{L}}\right)e^{\sigma}L_{ir}+\frac{\beta^2e^{2\sigma}}{\bar{L}^3}L_iL_r-\frac{\beta e^{\sigma}}{\bar{L}^2}(L_ib_r+L_rb_i)+\frac{1}{\bar{L}}b_ib_r\right]D^r \tag{3.11}$$

$$=s_{i0}+\frac{1}{2}\left(\frac{1}{\bar{L}}b_i-\frac{\beta e^{\sigma}}{\bar{L}^2}L_i\right)r_{00}+\frac{1}{2}\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_i-\frac{\beta}{\bar{L}}b_i\right]\sigma_0-\frac{1}{2}(\bar{L}-\beta)\sigma_i .$$

In view of $LL_{ir}=g_{ir}-L_iL_r$, the equation (3.1) can be written as

$$\frac{\bar{L}-\beta}{\bar{L}^2}e^{2\sigma}g_{ir}D^r+\left\{\frac{(\beta^2+\beta\bar{L}-\bar{L}^2)e^{2\sigma}}{\bar{L}^3}L_i-\frac{\beta e^{\sigma}}{\bar{L}^2}b_i\right\}L_rD^r \tag{3.12}$$

$$+\left(\frac{1}{\bar{L}}b_i-\frac{\beta e^{\sigma}}{\bar{L}^2}L_i\right)b_rD^r=s_{i0}+\frac{1}{2}\left(\frac{1}{\bar{L}}b_i-\frac{\beta e^{\sigma}}{\bar{L}^2}L_i\right)r_{00}$$

$$+\frac{1}{2}\left[\left(1-\frac{\beta}{\bar{L}}+\frac{\beta^2}{\bar{L}^2}\right)e^{\sigma}L_i-\frac{\beta}{\bar{L}}b_i\right]\sigma_0-\frac{1}{2}(\bar{L}-\beta)\sigma_i.$$

Contracting (3.12) by $b^i=g^{ij}b_j$, we get

$$-2\bar{L}tb_rD^r+2\beta e^{\sigma}tL_rD^r=2\bar{L}^4s_0+\bar{L}(b^2\bar{L}^2-\beta^2e^{2\sigma})r_{00} \quad (3.13)$$

$$+\beta\bar{L}[\bar{L}^2(e^{2\sigma}-b^2)-\beta e^{2\sigma}(\bar{L}-\beta)]\sigma_0-\bar{L}^4(\bar{L}-\beta)\sigma_b$$

where $t=\beta(\beta+\bar{L})e^{2\sigma}-\bar{L}^2(e^{2\sigma}+b^2)$, $s_0=s_{r0}b^r$ and $\sigma_ib^i=\sigma_b$.

The equation (3.10) and (3.13) constitute the system of algebraic equations in L_rD^r and D^r whose soluation is given by

$$L_rD^r=\frac{1}{2te^{\sigma}}.A \quad (3.14)$$

and

$$b_rD^r=\frac{-(\bar{L}-\beta)A}{2\bar{L}t}+\frac{1}{2}r_{00}+\frac{(\bar{L}-\beta)\sigma_0}{4} \quad (3.15)$$

where $A=2\bar{L}^3s_0-e^{2\sigma}\bar{L}(\bar{L}-\beta)r_{00}-\bar{L}^3(\bar{L}-\beta)\sigma_b$

$$+\beta\{\bar{L}^3(e^{2\sigma}-b^2)-\beta e^{2\sigma}(\bar{L}-\beta)\}\sigma_0+\frac{1}{2}t(\bar{L}-\beta)\sigma_0.$$

Contracting (3.12) by g^{ij} and putting the values of b_rD^r and L_rD^r from (3.14) and (3.15) respectively, we get

$$D^i=\frac{(\bar{L}-2\beta)A}{2\bar{L}t(\bar{L}-\beta)}y^i+\frac{e^{-2\sigma}\bar{L}A}{2t(\bar{L}-\beta)}b^i+\frac{e^{-2\sigma}\bar{L}^2}{(\bar{L}-\beta)}s_0^i-\frac{e^{-2\sigma}\bar{L}^2\sigma^i}{2} \quad (3.16)$$

$$+\frac{1}{4\bar{L}(\bar{L}-\beta)}\{(2\bar{L}^2-\beta\bar{L}+\beta^2)y^i-\bar{L}^2(\bar{L}+\beta)e^{-2\sigma}b^i\}\sigma_0,$$

where $l^i=\frac{y^i}{L}$.

Proposition (3.1). The difference tensor $D^i=G^{*i}-G^i$ of exponential change of Finsler metric is given by (3.16).

4. Projective Change of Finsler Metric

The Finsler space F^{*n} is said to be projective to Finsler spece F^n if every geodesic of F^n is transformed to a geodesic of F^{*n}. It is well known that the change $L \to L^*$ is projective if $G^{*i} = G^i + P(x,y)\,y^i$, where P(x, y) is a homogeneous scalar function of degree one in y^i, called projective factor (Matsumoto, 1992).

Thus from (3.2) it follows that $L \to L^*$ is projective if $D^i = Py^i$. Now we consider that the conformal exponential change $L \to L^* = \bar{L}e^{\beta/\bar{L}}$ is projective.

Then from equation (3.16), we have

$$Py^i = \frac{(\bar{L}-2\beta)A}{2\bar{L}t(\bar{L}-\beta)}y^i + \frac{e^{-2\sigma}\bar{L}A}{2t(\bar{L}-\beta)}b^i + \frac{e^{-2\sigma}\bar{L}^2 s_0^i}{(\bar{L}-\beta)} - \frac{e^{-2\sigma}\bar{L}^2\sigma^i}{2} \tag{4.1}$$

$$+\frac{1}{4\bar{L}(\bar{L}-\beta)}\{(2\bar{L}^2-\beta\bar{L}+\beta^2)y^i - \bar{L}^2(\bar{L}+\beta)e^{-2\sigma}b^i\}\sigma_0.$$

Contracting (4.1) by $y_i(=g_{ij}y^j)$ and using the fact that $s_0^i y_i = 0$ and $y^i y_i = L^2$, we get

$$P = \frac{1}{2\bar{L}t}A. \tag{4.2}$$

Putting the value of P from (4.2) in (4.1), we get

$$\frac{\beta A}{2\bar{L}t(\bar{L}-\beta)}y^i = \frac{e^{-2\sigma}\bar{L}A}{2t(\bar{L}-\beta)}b^i + \frac{e^{-2\sigma}\bar{L}^2 s_0^i}{(\bar{L}-\beta)} - \frac{e^{-2\sigma}\bar{L}^2\sigma^i}{2} \tag{4.3}$$

$$+\frac{1}{4\bar{L}(\bar{L}-\beta)}\{(2\bar{L}^2-\beta\bar{L}+\beta^2)y^i - \bar{L}^2(\bar{L}+\beta)e^{-2\sigma}b^i\}\sigma_0.$$

Theorem (4.1). The conformal exponential change of a Finsler space is projective if the condition (4.3) hold and the projective factor P is given by $P = \frac{A}{2\bar{L}t}$.

5. Conformal Exponential Change of Douglas Space

The Finsler space F^n is called a Douglas space if and only if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i (Park and Lee, 2001). We shall write hp(r) are the homogeneous polynomial in y^i of degree r.

If we write $B^{ij} = D^i y^j - D^j y^i$ then from (3.16), we get

$$B^{ij} = \frac{e^{-2\sigma}\bar{L}A}{2t(\bar{L}-\beta)}(b^i y^j - b^j y^i) + \frac{e^{-2\sigma}\bar{L}^2}{(\bar{L}-\beta)}(s_0^i y^j - s_0^j y^i) - \frac{e^{-2\sigma}\bar{L}^2}{2}(\sigma^i y^j - \sigma^j y^i). \tag{5.1}$$

If a Douglas space is transformed to the Douglas space by a conformal exponential change (3.1), then B^{ij} must be hp (Motsumoto, 1996) and vice-versa.

Theorem (5.1) The conformal exponential change of Douglas space is a Douglas space iff B^{ij} given by (5.1) is hp (Motsumoto, 1996).

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