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A semi-symmetric non-metric η - recurrent connection on a Riemannian manifold

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Abstract

We define a linear connection on a Riemannian manifold which is semi-symmetric non metric η- recurrent connection and obtain some properties of curvature tensor, Ricci tensor, scalar curvature tensor and concircular curvature tensor with respect to semi-symmetric non-metric η - recurrent connection.

Introduction

Let M^n be an n–dimensional Riemannian manifold of class C^∞ endowed with a Riemannian metric g and let D be the Levi-Civita connection on M^n. Let $\overline{D}$ be a linear connection defined on M^n. The torsion tensor $\overline{T}(X, Y)$ of $\overline{D}$ is given by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y], \qquad (1.1)$$

where X and Yare arbitrary vector fields. If the torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = (a - b)\,[\eta(X)\,Y - \eta(Y)\,X], \qquad (1.2)$$

where η is a 1-form, then $\overline{D}$ is called semi-symmetric connection Friedmann and Schouten (1924). The connection $\overline{D}$ is called a non-metric connection Agashe and Chafle (1992) if

$$(\overline{D}_X g)\,(Y,Z) = -2a\eta(X)\,g(Y,Z) - b\eta(Y)\,g(X,Z) - b\eta(Z)\,g(X,Y). \qquad (1.3)$$

A semi symmetric non metric connection with torsion tensor $\overline{T}(X, Y) = (a - b)\,[\eta(X)Y - \eta(Y)X]$ is defined as a semi symmetric non-metric η - recurrent connection De and Ghosh (1994) if

$$(\overline{D}_X \eta)\,(Y) = A(X)\,\eta(Y), \qquad (1.4)$$

for arbitrary vector fields X and Y, when A is non zero 1-form and Q is a vector field satisfying g(X,Q) = A(X).In section 2 of the present paper as expression for the curvature tensor $\overline{R}(X, Y, Z)$, the Ricci tensor $\overline{Ric}(Y, Z)$ and the scalar curvature $\overline{r}$ of the connection$\overline{D}$ have been deduced. In section 3, a necessary and sufficient condition has been deduced for the Ricci tensor of the semi-symmetric non-metric η -recurrent connection $\overline{D}$ to be symmetric. Also a necessary and sufficient condition has been deduced for the Ricci tensor $\overline{D}$ to be skew symmetric. In the last section it is proved that concircular curvature tensors for $\overline{D}$ and D are equal under certain condition. Also it is shown that if the curvature tensor of $\overline{D}$ vanishes then the manifold is concircularly flat.

1. Semi-symmetric non-metric η recurrent connection

It is known Chaturvedi and Pandey (2009) that for a semi-symmetric non-metric connection $\overline{D}$ is given by

$$\overline{D}_X Y = \overline{D}_X Y + a\eta(X)\, Y + b\eta(Y)\, X, \quad (2.1)$$

where $g(X, \xi) = \eta(X)$for every vector field X.

Let $$\overline{R}(X,Y,Z) = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z, \quad (2.2)$$

and

$$R(X,Y,Z) = D_X D_Y\, Z - D_Y D_X\, Z - D_{[X,Y]} Z, \quad (2.3)$$

be the curvature tensors of the connection $\overline{D}$ and D respectively.

Form (2.1), (2.2) and (2.3), we have

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + a[(D_X\eta)\,(Y) - (D_Y\eta)\,(X)]\, Z + b[(D_X\eta)\,(Z)\, Y - (D_Y\eta)\,(Z)\, X] + b^2[\eta(Y)\,\eta(Z)\, Z - \eta(X)\,\eta(Z)\, Y]. \quad (2.4)$$

Form (1.4), we get

$$(D_X\eta)\,(Y) = (a + b)\,[\eta(X)\,\eta(Y)] + A(X)\,\eta(Y)]\,. \quad (2.5)$$

In view of (2.4) and (2.5), we get

$$\overline{R}(X,Y,Z) = R(X,Y,Z) - aA(Y)\,\eta(X)\, Z - aA(X)\,\eta(Y)\, Z - b(a + b)\,\eta(Y)\,\eta(Z)\, X - bA(Y)\,\eta(Z)\, X + b(a + b)\,\eta(X)\,\eta(Z)\, Y + bA(X)\,\eta(Z)\, Y + b^2\eta(Y)\,\eta(Z) X - b^2\eta(X)\,\eta(Z)\, Y. \quad (2.6)$$

Rearranging (2.6), we get

$$\overline{R}(X,Y,Z) = R(X,Y,Z) - a[A(X)\,\eta(Y) - A(Y)\,\eta(X)]\, Z + b[A(X)\, Y - A(Y)\, X]\,\eta(Z) + ab[\eta(X)\, Y - \eta(Y)\, X]\,\eta(Z). \quad (2.7)$$

From (2.7), we get

$$'\overline{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) + a\big[A(X)\eta(Y) - A(Y)\eta(X)\big]\, g(Z,W) + b\big[A(X)g(Y,W) - A(Y)g(X,W)\big]\eta(Z) + ab\,[\eta(X)g(Y,W) - \eta(Y)g(X,W)]\eta(Z),$$

where$g(\overline{R}(X,Y,Z),W = {}'\overline{R}(X,Y,Z,W),$ (2.8)

and

$$g(R(X,Y,Z)W = {}'\overline{R}(X,Y,Z,W).$$

Let $\overline{R}ic(X, Y)$ and $Ric(X, Y)$ be the Ricci tensors of the connection $\overline{D}$ and D respectively. Further let $\overline{r}$ and r be the scaler curvature of the connection $\overline{D}$ and D respectively. Now putting E_i (i = 1, 2, … n) for X and Y in (2.8), where E_i is an orthonormal basis of the tangent space at a point, $1 \leq i \leq n$. We have

$$\bar{R}ic(Y,Z) = Ric(Y,Z) + aA(Z)\,\eta(Y) - \{a + b(n-1)\}\,A(Y)\,\eta(Z) - ab(n-1)\,\eta(Y)\,\eta(Z) \quad (2.9)$$

Again putting E_i for Y and Z in (2.9), we get

$$\bar{r} = r\,b\,(n-1)\,A(\xi) - ab\,(n-1)\,\eta(\xi).$$

Thus the curvature tensor $\bar{R}(X, Y, Z)$, Ricci tensor $\bar{R}ic$ and the scalar curvature $\bar{r}$ of $\bar{D}$ is given by (2.8), (2.9) and (2.10) respectively and hence we can state the following theorem :

Theorem (2.1) Curvature tensor $\bar{R}(X, Y, Z)$ of $\bar{D}$ on a Riemannian manifold satisfies the following relation

$$'\bar{R}(X,Y,Z,W) + {}'R(Y,X,Z,W) = 0,$$

and

$$\bar{R}(X,Y,Z,W) + {}'R(Y,Z,X,W) + {}'R(Z,X,Y,W) = a\big[\{A(X)\eta(Y) - A(Y)\eta(X)\}\,Z + A(Y)\eta(Z) - A(Z)\eta(Y)\}\,X + \{A(Z)\eta(X) - A(X)\eta(Z)\}\,Y\big] + b[\{A(X)Y - A(Y)X\}\eta(Z) + \{A(Y)Z - A(X)Z\}\eta(X) + \{A(Z)\,X - A(X)\,Z\}\,\eta(Y)] + ab[\{\eta(X)Y - \eta(Y)X\}\eta(Z) + \{\eta(Y)Z - \eta(Z)Y\}\eta(X) - \{\eta(Z)X - \eta(X)Z\}\eta(Y)]\,.$$

(i) A necessary and sufficient condition for the Ricci tensor of the semi-symmetric non metric η-recurrent connection $\bar{D}$ to be symmetric is that the relation $A(Z)\eta(Y) = A(Y)\eta(Z)$ holds, provided $2a + b(n-1) \neq 0$.

(ii) A necessary and sufficient condition for the Ricci tensor of the semi-symmetric non-metric η-recurrent connection $\bar{D}$ to be skew symmetric is that Ricci tensor of the Levi-Civita connection D is given by

$$Ric(Y,Z) = ab\,(n-1)\eta(Y)\,\eta(Z) - \left(\frac{b(n-1)}{2}\right)\big[A(Z)\eta(Y) - A(Y)\eta(Z)\big].$$

2. Concircular curvature tensor of a Riemannian manifold with respect to semi-symmetric non-metric η-recurrent connection $\bar{D}$.

Let concircular curvature tensor of a Riemannian manifold with respect to the Levi-Civita connection is given by Mishra (1984)

$$V(X,Y,Z) = R(X,Y,Z) - \frac{r}{n(n-1)}\big[g(Y,Z)X - g(X,Z)Y\big]. \quad (3.1)$$

Analogous to this definition, we define concircular curvature tensor with respect to semi symmetric non-metric η- recurrent connection $\bar{D}$ is given by

$$\bar{V}(X,Y,Z) = \bar{R}(X,Y,Z) - \frac{\bar{r}}{n(n-1)}\,[g(Y,Z)\,X - g(X,Z)\,Y] \quad (3.2)$$

In view of (2.7), (2.10). (3.1) and (3.2) we get

$$\bar{V}(X,Y,Z) = V(X,Y,Z) + a\big[A(X)\eta(Y) - A(Y)\eta(X)\big]\,Z + b\big[A(X)Y - A(Y)X\big]\eta(Z) + ab[\eta(X)Y - \eta(Y)X]\eta(Z) + \frac{b}{n}[A(\xi) + a\eta(\xi)]\big[g(Y,Z)X - g(X,Z)Y\big], \quad (3.3)$$

If

$$a\big[A(X)\eta(Y) - A(Y)\eta(X)\big]Z + b\big[A(X)Y - A(Y)X\big]\eta(Z) + ab[\eta(X)Y - \eta(Y)X]\eta(Z) + \frac{b}{n}[A(\xi) + a\eta(\xi)]\big[g(Y,Z)X - g(X,Z)Y\big] = 0. \quad (3.4)$$

Then from (3.3), we get

$$\bar{V}(X,Y,Z) = V(X,Y,Z) \quad (3.5)$$

Conversely if (3.5) holds then from (3.3), we get (3.4).Hence we can state the following theorem:

Theorem (3.1): If a Riemannian manifold admits a semi-symmetric non-metric η-recurrent connection $\bar{D}$ then a necessary and sufficient condtion for the concircular curvature tensor V of the manifold with respect to Levi-civita connection D and the concircular curvature tensor $\bar{V}$ of the manifold with respect to semi-symmetric non metric η-recurrent connection$\bar{D}$ to be equal is that

$$a[A(X)\eta(Y) - A(Y)\eta(X)]Z + b[A(X)Y - A(Y)X]\eta(Z) + ab[\eta(X)Y - \eta(Y)X]\eta(Z) + \frac{b}{n}[A(\xi) + a\eta(\xi)][g(Y,Z)X - g(X,Z)Y = 0$$

If $\bar{R}ic(X,Y) = 0$ then $\bar{r} = 0$.

Then from (3.2), we get

$$\bar{V}(X,Y,Z) = V(X,Y,Z). \quad (3.6)$$

Using (3.6) in (3.3) we get

$$\bar{R}(X,Y,Z) = V(X,Y,Z) + a\big[A(X)\eta Y - A(Y)\eta(X)\big]Z + b\big[A(X)Y - A(Y)X\big]\eta(Z) - ab[\eta(X)Y - \eta(Y)X]\eta(Z) + \frac{b}{n}[A(\xi) + a\eta(\xi)].[g(Y,Z)X - g(X,Z)Y]. \quad (3.7)$$

Again from (3.4) and (3.7), we have

$$\bar{R}(X,Y,Z) = V(X,Y,Z). \quad (3.8)$$

Converse part is also true.

Hence we have the following theorem:

Theorem (3.2): If the Ricci tensor of semi-symmetric non-metric η-recurrent connection $\bar{D}$ in a Riemannian manifold vanishes then curvature tensor with respect to $\bar{D}$ is equal to the concircular curvature tensor of the manifold if and only if

$$a\big[A(X)\eta(Y) - A(Y)\eta(X)\big]Z + b\big[A(X)Y - A(Y)X\big]\eta(Z) + ab.[\eta(X)Y - \eta(Y)X]\eta(Z) + \frac{b}{n}[A(\xi) + \eta(\xi)a].\big[g(Y,Z)X - g(X,Z)Y\big] = 0.$$

Let us assume that

$$\bar{R}(X,Y,Z) = \big[g(Y,Z)X - g(X,Z)Y\big] + a\big[A(Y)\eta(X) - A(X)\eta(Y)\big]Z - b\big[A(X)Y - A(Y)X\big]\eta(Z) + ab[\eta(Y)X - \eta(X)Y]\eta(Z). \quad (3.9)$$

In view of (2.7) and (3.9) gives

$$R(X,Y,Z) = g(Y,Z)X - g(X,Z)Y. \quad (3.10)$$

Contraction of (3.10) gives

$$Ric(Y, Z) = (n - 1)\, g(Y, Z). \tag{3.11}$$

Further contraction of (3.11) gives

$$R = n(n - 1). \tag{3.12}$$

In view of (3.10), (3.12) and (3.1), we get

$$V(X, Y, Z) = 0. \tag{3.13}$$

Hence we have the following theorem:

Theorem (3.3): If the Riemannian manifold admits a semi-symmetric non-metric η-recurrent connection $\overline{D}$ whose curvature tensor is of the form (3.9) then the manifold is concircularly flat.

From (3.3), we get

$$\overline{V}(Y, Z, Z) = V(Y, X, Z) + a[A(Y)\eta(X) - A(X)\eta(Y)]\, Z + b[A(Y)X - A(X)Y]\eta(Z) + ab[\eta(Y)X - \eta(X)Y]\eta(Z) + \frac{b}{n}[A(\xi) + a\eta(\xi)].[g(X, Z)Y - g(Y, Z)X]. \tag{3.14}$$

From (3.3), (3.14) and using the fact we get

$$\overline{V}(X, Y, Z) + \overline{V}(Y, X, Z) = 0\,. \tag{3.15}$$

Cyclic rotation of (3.3), we get

$$\overline{V}(X, Y, Z) + \overline{V}(Y, X, Z) + \overline{V}(Z, X, Y) = [a - b][\{A(X)\eta(Y) - A(Y)\eta(X)\}\, Z + \{A(Y)\eta(Z) - A(Z)\eta(Y)\}\, X + \{A(Z)\eta(X) - A(X)\eta(Z)\}\, Y]. \tag{3.16}$$

From (3.15) and (3.16) we have the following theorem:

Theorem (3.4): The concircular curvature tensor with respect to semi symmetric non-metric η-recurrent connection $\overline{D}$ in a Riemannian manifold satisfies the following algebraic properties :

$$\overline{V}(X, Y, Z) + \overline{V}(Y, X, Z) = 0,$$

and

$$\overline{V}(X, Y, Z) + \overline{V}(Y, Z, X) + \overline{V}(Z, X, Y) = 0 \text{ iff } A(X)\, \eta(Y) - A(Y)\, \eta(X) = 0.$$

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