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Some properties of α −cosymplectic manifolds

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Abstract

The aim of the present paper is to study α −cosymplectic manifolds that satisfy certain curvature condition on $\tilde{Q}$ −curvature tensor.

Keywords -α −cosymplectic manifolds, $\tilde{Q}$ −curvature tensor and η −Einstien manifold.

Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. We denoted by D the covariant differentiation with respect to the Riemannian metric g. Then we have

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where

$$'R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The Ricci tensor Ric of (M^n, g) be defined as

$$Ric(X,Y) = trac\{Z \to R(X,Y)Z\}.$$

Locally Ric is given by

$$Ric(X,Y) = \sum_{i=1}^{n} {'R}(X, E_i, Y, E_i),$$

where $\{E_1, E_2, \ldots, E_n\}$ is locally orthonormal frames field and X, Y, Z, W are vector fields on M, the Ricci operator Q is a tensor of type (1,1) on M^n defined by

$$g(QX,Y) = Ric(X,Y),$$

for all vector field on M^n.

Let (M^n, g), $n > 3$, be connected Riemannian manifold of class C^∞ and D be its Riemannian connection. The Weyl conformal curvature tensor C Yano and Kon (1984) and , $\tilde{Q}$ −curvature tensor Montica and Sah (2013) of (M^n, g) are defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\,[\,Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)Y\,] + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)QX - g(X,Z)Y\big], \tag{1.1}$$

and

$$\tilde{Q}(X,Y)Z = R(X,Y)Z - \frac{\Psi}{(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

where Ψ is arbitrary scalar function and $\Psi = \frac{r}{n}$, then $\tilde{Q}$ −ccurvature tensor reduces to concircular curvature tensor Yano and Kon (1984), r is the scalar curvature tensor.

Let C be the Conformal curvature tensor of M^n. The tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi\left(T_p(M^n) \oplus \mathrm{L}(\xi_p)\right)$, where $\mathrm{L}(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have a map:

$$C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus \mathrm{L}(\xi_p)$$

It may be natural to consider the following particular cases:

(1) $C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \mathrm{L}(\xi_p)$, that is, the projection of the image of C in$\phi(T_p(M^n))$ is zero.

(2) $C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n))$that is, the projection of the image of C in $\mathrm{L}(\xi_p)$is zero.

(3) $C: T_p(M^n) \times T_p(M^n) \times \phi(T_p(M^n)) \to \mathrm{L}(\xi_p)$ i.e, when C is restricted to $T_p(M^n) \times T_p(M^n) \times \phi\left(T_p(M^n)\right) \times \phi\left(T_p(M^n)\right)$, the projection of the image of C in$\phi(T_p(M^n))$ is zero. This condition is equivalent to Cabrerizo etal (1999),

$\phi^2 C(\phi X, \phi Y)\phi Z = 0$ if only if

$$g(C(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{1.3}$$

A differentiable (M^n, g), $n > 3$, satisfying the condition (1.3) is called ϕ −Conformally flat manifold studied by many authors such as Zhen (1992), Zhen etal (1997), Özgür (2003) and Ö zt ü rk (2013) in K-contact manifold, LP-Sasakain manifold and α −cosymplectic manifolds.

In this section, after introduction and preliminaries, in section3, it is show that α −cosymplectic manifold be constant curvature such that $\alpha \neq 0$. A necessary and sufficient condition for $\xi - \tilde{Q}$-flat manifold is to obtained in section 4. In final section it is proved that $\phi - \tilde{Q}$ flat manifold is an η −Einstein manifold.

2. Preliminaries

Let M^n be an dimensional differentiable manifold with a triplet (ϕ, ξ, η) where ϕ is (1,1)-tensor field, ξ is a vector field, η is a 1-form on M^n such that

$$\eta(\xi) = 1, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

Which implies

$$\phi\xi = 0, \quad \eta o\phi = 0 rank\, \phi = (n-1) \tag{2.2}$$

If M^n admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

Then M^n is said to be admit almost contact structure (ϕ, ξ, η, g).On such a manifold, the fundamental 2-form Φ is defined by $\Phi(X, Y) = g(\phi X, Y), \forall X, Y \in TM$.

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. The product of almost K$\ddot{a}$ehlerian manifolds and real line $\mathbb{R}$ or the S^1 circle are the simplest examples of almost cosymplectic manifolds (M, ϕ, ξ, η, g) is said to be normal if the Nijenhuis tensor

$$N(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] + 2d\eta(X, Y)\xi,$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifold is called cosymplectic manifold. It is well known that an almost contact metric structure is cosymplectic if and only if both $d\eta$ and $d\Phi$ vanish. An almost contact metric manifold M^n is said to be almost α − Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant. It may br noted that almost α −Kenmotsu structures are related to some special local Conformal deformation of almost cosymplectic structures.

In order it treat these two classes in a unified way, we have a new notion of an almost α −cosymplectic manifold for every any real number α that is defined as $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$.

A normal almost α −cosymplectic manifold is called an α −cosymplectic manifold. An α − cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α − Kenmotsu under the condition $\alpha \neq 0$ for $\in \mathbb{R}$. on such an α − cosymplectic manifold, we have

$$(D_X\phi)(Y) = \alpha[\, g(\phi X,\ Y)\xi + \eta(Y)\,\phi X], \tag{2.4}$$

for $\alpha \in \mathbb{R}$ on M^n.

In α −cosymplectic manifold, the following relation holds:

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X], \tag{2.5}$$
$$R(\xi,X)Y = \alpha^2[\eta(Y)X - g(X,Y)\xi], \tag{2.6}$$

$$D_X\,\xi = -\alpha\phi X, \tag{2.7}$$

$$R(\xi,X)\xi = \alpha^2[X - \eta(X)\xi], \tag{2.8}$$

$$g(R(\xi,X)Y,\xi) = \alpha^2[\eta(X)\eta(Y) - g(X,Y)], \tag{2.9}$$

$$Ric(X,\xi) = -\alpha^2(n-1)\eta(X), \tag{2.10}$$

$$Ric(\phi X,\phi Y) = Ric(X,Y) + \alpha^2(n-1)\eta(X)\eta(Y), \tag{2.11}$$

$$Q\xi = -\alpha^2(n-1)\xi, \tag{2.12}$$
$$Ric(\xi,\xi) = -\alpha^2(n-1), \tag{2.13}$$

for any vector fields X and Y and $\alpha \in \mathbb{R}$.

Kenmotsu manifolds have been studied by another Jun, De and Pathak (2005) and he obtained the above results for $\alpha = 1$.

An α −cosymplectic manifold M^n is said to be Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(X,Y) = \lambda g(X,Y), \tag{2.14}$$

where λ is constant and it is η-Einstein manifold if its Ricci tensor Ric is of the form

$Ric(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y),$ (2.15)

for any vector fields X and Y, where λ_1 and λ_2 are function on M^n.

3. $\tilde{Q}-$ curvature tensor $\alpha-$ cosymplectic manifold.

Let us consider $\alpha-$cosymplectic manifold satisfying the following condition

$\tilde{Q}(X,Y)Z = 0.$ (3.1)

Then from (1.2) and (3.1), we get

$$R(X,Y)Z = \frac{\Psi}{(n-1)}[g(Y,Z)X - g(X,Z)Y], \quad (3.2)$$

Putting ξ for Z in (3.2) and using (2.3) and (2.5), we get

$$\alpha^2[\eta(X)Y - \eta(Y)X] = \frac{\Psi}{(n-1)}[\eta(Y)X - \eta(X)Y]. \quad (3.3)$$

From (3.3), we get

$$[\eta(X)Y - \eta(Y)X][(n-1)\alpha^2 + \Psi] = 0. \quad (3.4)$$

Putting ξ for X in (3.4), we get

$$(\phi^2 Y)[(n-1)\alpha^2 + \Psi] = 0. \quad (3.5)$$

Hence from (3.5), we see that

Either $\phi^2 Y = 0$ or $[(n-1)\alpha^2 + \Psi] = 0$.

But $\phi^2 Y \neq 0$. Hence$(n-1)\alpha^2 + \Psi = 0, \alpha \neq 0.$ (3.6)

Hence from (3.2) and (3.6), we get

$$R(X,Y)Z = \alpha^2[(g(Y,Z)X - g(X,Z)Y]. \quad (3.7)$$

From (3.7), we can state the following theorem:

Theorem (3.1). Let M^n be an $\alpha-$cosymplectic manifold if the manifold M^n is $\tilde{Q}$-flat then M^n is a manifold of constant curvature such that $\alpha \neq 0$.

Definition (3.1). A $\alpha-$cosymplectic manifold is said to be $\xi - \tilde{Q}$-flat De and Biswas (2006) if

$$\tilde{Q}(X,Y)\xi = 0. \quad (3.8)$$

Putting ξ for Z in (1.2), we get

$$\tilde{Q}(X,Y)\xi = R(X,Y)\xi - \frac{\Psi}{(n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \quad (3.9)$$

From (2.5) and (3.9), we get

$$\tilde{Q}(X,Y)\xi = \left(\alpha^2 + \frac{\Psi}{(n-1)}\right)[\eta(X)Y - \eta(Y)X]. \quad (3.10)$$

From (3.8) and (3.10), we get

$$\Psi = -(n-1)\alpha^2, \alpha \neq 0 \quad (3.11)$$

In view of (3.10) and (3.11), we get

$$\tilde{Q}(X,Y)\xi = 0.$$

Hence we state the following theorem:

Theorem (3.2). Let M^n be an α −cosymplectic manifold. Then the manifold be $\xi-\tilde{Q}$-flat if and only if $\Psi=-(n-1)\alpha^2, \alpha\neq 0$.

4. $\phi-\tilde{Q}$ flat α − cosymplectic manifold

In this case we assume that (M^n, g)be $\phi-\tilde{Q}$ flat α − cosymplectic manifold. Then it is easy to see that

$$\phi^2\, g(\tilde{Q}(\phi X,\phi Y)\phi Z,\phi W)=0, \tag{4.1}$$

So by the use of (1.2) and (4.1), we get

$$g(R(\phi X,\phi Y)\phi Z,\phi W)=-\frac{\Psi}{n-1}[g(\phi Y,\phi Z)g(\phi X,\phi W)-$$

$$g(\phi X,\phi Z)\, g(\phi Y,\phi W)] \tag{4.2}$$

Let $\{E_1,E_2,\dots,E_{n-1},\xi\}$ be a local orthognormal basis of vector fields in M^n. Using the fact that $\{\phi E_1,\phi E_2,\dots,\phi E_{n-1},\xi\}$ is also a local orthonormal basis. Putting E_i for X and W in (4.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1}{}'R\,(\phi E_i,\phi Y,\phi Z,\phi E_i)=-\frac{\Psi}{n-1}\sum_{i=1}^{n-1}[\,g(\phi Y,\phi\, Z)g(\phi e_i,\phi e_i)$$

$$-\,g(\phi e_i,\phi Z)\, g(\phi Y,\phi e_i)] \tag{4.3}$$

It can be verify that Öztürk (2013)

$$\sum_{i=1}^{n-1}{}'R\,(\phi E_i,\phi Y,\phi Z,\phi E_i)=Ric(\phi Y,\phi Z)+\alpha^2 g(\phi Y,\phi Z), \tag{4.4}$$

and

$$\sum_{i=1}^{n-1} g\,(\phi e_i,\phi Z)\, g(\phi Y,\phi e_i)=g(\phi Y,\phi Z). \tag{4.5}$$

Therefore, in view of (4.3), (4.4) and (4.5), we get

$$Ric(\phi Y,\phi Z)=\left(\frac{\Psi}{n-1}-\alpha^2\right)g(\phi Y,\phi Z). \tag{4.6}$$

By making the use of (2.3), (2.11) and (4.6), we get

$$Ric(Y,Z) = \left[\frac{\Psi}{n-1} - \alpha^2\right] g(Y,Z) - \left[\frac{\Psi}{n-1} + \alpha^2(n-2)\right] \eta(Y)\eta(Z). \qquad (5.6)$$

which show that M^n is an η −Einstein manifold by virtue of (2.15). Hence we have the following theorem:

Theorem (4.1). Let M^n be an n-dimensional $(n > 3)$, $\phi - \tilde{Q}$ flat $\alpha -$ cosymplectic manifold.Then M^n is an η −Einstein manifold where $\lambda_1 = \frac{\Psi}{n-1} - \alpha^2$ and $\lambda_2 = \frac{\Psi}{n-1} + \alpha^2(n-2)$.

References

1. Cabrerizo, J.L., Fernandez, L.M., Fernandez, M. and Zhen, G. (1999). The structure of a class of k-contact manifolds, Acta Math. Hungar, 82(4): 331-340.
2. De, U.C. and Biswas S. (2006). A note on ξ −conformally flat contact manifolds, Bull. Malays. Math. Sci. Soc. 29(1): 51-57.
3. Jun, J. B., De, U. C. and Pathak, G. (2005). On Kenmotsu manifold, J. Korean Math. Soc.., 42: 435 – 445.
4. Mantica, C.A. and Suh, Y.J. (2013). Pseudo Ω −symmetric Riemannian manifolds, Int. J. Geom. Methods Mod. Phys. 10(5):
5. Özgür, C. (2003).ϕ-Conformally flat LP-Sasakian manifold, Rad. Mat. 12(1): 99 – 106.
6. Öztürk, H. (2013). Some curvature conditions on $\alpha -$ cosymplectic manifolds, mathematics and Statistics 1(4): 188-195.
7. Yano, K. and Kon, M. (1984). Structures on Manifolds, Series in Pure Math, vol 3, world scientific.
8. Zhen, G. (1992). On conformal symmetric K-contact manifolds, Chinese Quart. J. Math.7: 5-10.
9. Zhen, G., Carbrerizo, J.L., Fernandez, L.M. and Fernandez, M. (1997). On ξ-Conformally flat contact metric manifolds, Indian J. pure Appl. Math. 28, no.6: 725-734.

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