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Some properties of α −cosymplectic manifolds

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Abstract
The aim of the present paper is to study α −cosymplectic manifolds that satisfy certain curvature condition on $\tilde{Q}$ −curvature tensor.

Keywords -α −cosymplectic manifolds, $\tilde{Q}$ −curvature tensor and η −Einstien manifold.

Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. We denoted by D the covariant differentiation with respect to the Riemannian metric g. Then we have

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where

$$'R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The Ricci tensor Ric of (M^n, g) be defined as

$$Ric(X,Y) = trac\{Z \to R(X,Y)Z\}.$$

Locally Ric is given by

$$Ric(X,Y) = \sum_{i=1}^{n} 'R\,(X, E_i\,, Y, E_i),$$

where $\{E_1, E_2, \dots, E_n\}$ is locally orthonormal frames field and X, Y, Z, W are vector fields on M, the Ricci operator Q is a tensor of type (1,1) on M^n defined by

$$g(QX,Y) = Ric(X,Y),$$

for all vector field on M^n.

Let (M^n, g), $n > 3$, be connected Riemannian manifold of class C^∞ and D be its Riemannian connection. The Weyl conformal curvature tensor C Yano and Kon (1984) and , $\tilde{Q}$ −curvature tensor Montica and Sah (2013) of (M^n, g) are defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[\,Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)Y\,] + \frac{r}{(n-1)(n-2)}[g(Y,Z)QX - g(X,Z)Y], \quad (1.1)$$

and

$$\tilde{Q}(X,Y)Z = R(X,Y)Z - \frac{\Psi}{(n-1)}[g(Y,Z)X - g(X,Z)Y], \quad (1.2)$$

where Ψ is arbitrary scalar function and $\Psi = \frac{r}{n}$, then $\tilde{Q}$ −ccurvature tensor reduces to concircular curvature tensor Yano and Kon (1984), r is the scalar curvature tensor.

Let C be the Conformal curvature tensor of M^n. The tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi\left(T_p(M^n) \oplus \mathrm{L}(\xi_p)\right)$, where $\mathrm{L}(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have a map:

$$C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus \mathrm{L}(\xi_p)$$

It may be natural to consider the following particular cases:

(1) $C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \mathrm{L}(\xi_p)$, that is, the projection of the image of C in$\phi(T_p(M^n))$ is zero.

(2) $C: T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n))$that is, the projection of the image of C in $\mathrm{L}(\xi_p)$is zero.

(3) $C: T_p(M^n) \times T_p(M^n) \times \phi(T_p(M^n)) \to \mathrm{L}(\xi_p)$ i.e, when C is restricted to $T_p(M^n) \times T_p(M^n) \times \phi\left(T_p(M^n)\right) \times \phi\left(T_p(M^n)\right)$, the projection of the image of C in$\phi(T_p(M^n))$ is zero. This condition is equivalent to Cabrerizo etal (1999),

$\phi^2 C(\phi X, \phi Y)\phi Z = 0$ if only if

$$g(C(\phi X,\phi Y)\phi Z,\phi W)=0. \tag{1.3}$$

A differentiable $(M^n,g),\ n>3$, satisfying the condition (1.3) is called ϕ −Conformally flat manifold studied by many authors such as Zhen (1992), Zhen etal (1997), Özgür (2003) and Ö zt ü rk (2013) in K-contact manifold, LP-Sasakain manifold and α −cosymplectic manifolds.

In this section, after introduction and preliminaries, in section3, it is show that α −cosymplectic manifold be constant curvature such that $\alpha\neq 0$. A necessary and sufficient condition for $\xi-\tilde{Q}$-flat manifold is to obtained in section 4. In final section it is proved that $\phi-\tilde{Q}$ flat manifold is an η −Einstein manifold.

2. Preliminaries

Let M^n be an dimensional differentiable manifold with a triplet $(\phi,\ \xi,\eta)$ where ϕ is (1,1)-tensor field, ξ is a vector field, η is a 1-form on M^n such that

$$\eta(\xi)=1, \qquad \phi^2X=-X+\eta(X)\xi, \tag{2.1}$$

Which implies

$$\phi\xi\ =\ 0,\quad \eta o\phi=0 rank\ \phi=(n-1) \tag{2.2}$$

If M^n admits a Riemannian metric g such that

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\qquad g(X,\xi)=\eta(X), \tag{2.3}$$

Then M^n is said to be admit almost contact structure (ϕ,ξ,η,g).On such a manifold, the fundamental 2-form Φ is defined by $\Phi(X,Y)=g(\phi X,Y),\forall\ X,Y\ \in TM$.

An almost contact metric manifold (M,ϕ,ξ,η,g) is said to be almost cosymplectic if $d\eta=0$ and $d\Phi=0$, where d is the exterior differential operator. The product of almost K$\ddot{a}$ehlerian manifolds and real line $\mathbb{R}$ or the S^1 circle are the simplest examples of almost cosymplectic manifolds (M,ϕ,ξ,η,g) is said to be normal if the Nijenhuis tensor

$$N(X,Y)=[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]+\phi^2[X,Y]+2d\eta(X,Y)\xi,$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifold is called cosymplectic manifold. It is well known that an almost contact metric structure is cosymplectic if and only if both $d\eta$ and $d\Phi$ vanish. An almost contact metric manifold M^n is said to be almost α −Kenmotsu manifold if $d\eta=0$ and $d\Phi=2\alpha\eta\hat{}\Phi,\alpha$ being a

non-zero real constant. It may br noted that almost α −Kenmotsu structures are related to some special local Conformal deformation of almost cosymplectic structures.

In order it treat these two classes in a unified way, we have a new notion of an almost α −cosymplectic manifold for every any real number α that is defined as$d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$.

A normal almost α −cosymplectic manifold is called an α −cosymplectic manifold. An α − cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α − Kenmotsu under the condition $\alpha \neq 0$ for $\in \mathbb{R}$. on such an α − cosymplectic manifold, we have

$$(D_X\phi)(Y) = \alpha[\, g(\phi X,\ Y)\xi + \ \eta(Y)\ \phi X] \,, \tag{2.4}$$

for $\alpha \in \mathbb{R}$ on M^n.

In α −cosymplectic manifold, the following relation holds:

$$R(X,Y)\xi = \alpha^2[\eta(\mathrm{X})Y - \ \eta(\mathrm{Y})X], \tag{2.5}$$
$$R(\xi,X)Y = \alpha^2[\eta(\mathrm{Y})X - g(X,Y)\xi] \,, \tag{2.6}$$

$$\mathrm{D_X}\,\xi = -\alpha\phi X, \tag{2.7}$$

$$R(\xi,X)\xi = \alpha^2[X - \eta(\mathrm{X})\xi] \,, \tag{2.8}$$

$$g(R(\xi,X)Y,\xi) = \alpha^2[\eta(\mathrm{X})\eta(\mathrm{Y}) - g(X,Y)] \,, \tag{2.9}$$

$$Ric(X,\xi) = -\alpha^2(\mathrm{n}-1)\eta(\mathrm{X}) \,, \tag{2.10}$$

$$Ric(\phi X,\phi\mathrm{Y}) = Ric(X,Y) + \alpha^2(\mathrm{n}-1)\eta(\mathrm{X})\eta(\mathrm{Y}) \,, \tag{2.11}$$

$$Q\xi = -\alpha^2(n-1)\xi, \tag{2.12}$$
$$Ric(\xi,\xi) = -\alpha^2(\mathrm{n}-1) \,, \tag{2.13}$$

for any vector fields X and Y and $\alpha \in \mathbb{R}$.

Kenmotsu manifolds have been studied by another Jun, De and Pathak (2005) and he obtained the above results for $\alpha = 1$.

An α −cosymplectic manifold $M^n$is said to be Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(X,\mathrm{Y}) = \lambda g(X,Y), \tag{2.14}$$

where λ is constant and it is η-Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(X,\mathrm{Y}) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X and Y, where λ_1 and λ_2 are function on M^n.

3. $\tilde{Q}$ − curvature tensor α − cosymplectic manifold.

Let us consider α −cosymplectic manifold satisfying the following condition

$$\tilde{Q}(X,Y)Z = 0. \tag{3.1}$$

Then from (1.2) and (3.1), we get

$$R(X,Y)Z = \frac{\Psi}{(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.2}$$

Putting ξ for Z in (3.2) and using (2.3) and (2.5), we get

$$\alpha^2[\eta(\mathrm{X})Y - \eta(\mathrm{Y})X] = \frac{\Psi}{(n-1)}[\eta(\mathrm{Y})X - \eta(\mathrm{X})Y]. \tag{3.3}$$

From (3.3), we get

$$[\eta(\mathrm{X})Y - \eta(\mathrm{Y})X][(n-1)\alpha^2 + \Psi] = 0. \tag{3.4}$$

Putting ξ for X in (3.4), we get

$$(\phi^2 Y)[(n-1)\alpha^2 + \Psi] = 0. \tag{3.5}$$

Hence from (3.5), we see that

Either $\phi^2 Y = 0$ or $[(n-1)\alpha^2 + \Psi] = 0$.

But $\phi^2 Y \neq 0$. Hence$(n-1)\alpha^2 + \Psi = 0$, $\alpha \neq 0$. (3.6)

Hence from (3.2) and (3.6), we get

$$R(X,Y)Z = \alpha^2[(g(Y,Z)X - g(X,Z)Y]. \tag{3.7}$$

From (3.7), we can state the following theorem:

Theorem (3.1). Let M^n be an α −cosymplectic manifold if the manifold M^n is $\tilde{Q}$-flat then M^n is a manifold of constant curvature such that $\alpha \neq 0$.

Definition (3.1). A α −cosymplectic manifold is said to be $\xi - \tilde{Q}$-flat De and Biswas (2006) if

$$\tilde{Q}(X,Y)\xi = 0. \tag{3.8}$$

Putting ξ for Z in (1.2), we get

$$\tilde{Q}(X,Y)\xi = R(X,Y)\xi - \frac{\Psi}{(n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \tag{3.9}$$

From (2.5) and (3.9), we get

$$\tilde{Q}(X,Y)\xi = \left(\alpha^2 + \frac{\Psi}{(n-1)}\right)[\eta(\mathrm{X})Y - \eta(\mathrm{Y})X]. \tag{3.10}$$

From (3.8) and (3.10), we get

$$\Psi = -(\mathrm{n}-1)\alpha^2, \alpha \neq 0 \tag{3.11}$$

In view of (3.10) and (3.11), we get

$$\tilde{Q}(X,Y)\xi = 0.$$

Hence we state the following theorem:

Theorem (3.2). Let M^n be an α −cosymplectic manifold. Then the manifold be $\xi - \tilde{Q}$-flat if and only if $\Psi = -(n-1)\alpha^2, \alpha \neq 0$.

4. $\phi - \tilde{Q}$ flat α − cosymplectic manifold

In this case we assume that (M^n, g)be $\phi - \tilde{Q}$ flat α − cosymplectic manifold. Then it is easy to see that

$$\phi^2\, g(\tilde{Q}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.1}$$

So by the use of (1.2) and (4.1), we get

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = -\frac{\Psi}{n-1}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)\, g(\phi Y, \phi W)] \tag{4.2}$$

Let $\{E_1, E_2, \dots, E_{n-1}, \xi\}$ be a local orthognormal basis of vector fields in M^n. Using the fact that $\{\phi E_1, \phi E_2, \dots, \phi E_{n-1}, \xi\}$ is also a local orthonormal basis. Putting E_i for X and W in (4.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} {}'R\,(\phi E_i, \phi Y, \phi Z, \phi E_i) = -\frac{\Psi}{n-1}\sum_{i=1}^{n-1}[\, g(\phi Y, \phi\, Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)\, g(\phi Y, \phi e_i)] \tag{4.3}$$

It can be verify that Öztürk (2013)

$$\textstyle\sum_{i=1}^{n-1} {}'R\,(\phi E_i, \phi Y, \phi Z, \phi E_i) = Ric(\phi Y, \phi Z) + \alpha^2 g(\phi Y, \phi Z), \tag{4.4}$$

and

$$\textstyle\sum_{i=1}^{n-1} g\,(\phi e_i, \phi Z)\, g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.5}$$

Therefore, in view of (4.3), (4.4) and (4.5), we get

$$Ric(\phi Y, \phi Z) = \left(\frac{\Psi}{n-1} - \alpha^2\right) g(\phi Y, \phi Z). \tag{4.6}$$

By making the use of (2.3), (2.11) and (4.6), we get

$$Ric(Y, Z) = \left[\frac{\Psi}{n-1} - \alpha^2\right] g(Y, Z) - \left[\frac{\Psi}{n-1} + \alpha^2(n-2)\right]\eta(Y)\eta(Z). \tag{4.6}$$

which show that M^n is an η −Einstein manifold by virtue of (2.15). Hence we have the following theorem:

Theorem (4.1). Let M^n be an n-dimensional $(n > 3)$, $\phi - \tilde{Q}$ flat $\alpha -$ cosymplectic manifold.Then M^n is an $\eta -$Einstein manifold where $\lambda_1 = \frac{\Psi}{n-1} - \alpha^2$ and $\lambda_2 = \frac{\Psi}{n-1} + \alpha^2(n-2)$.

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