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On Einstein–Sasakian holomorphically conformal bi-recurrent and bi-symmetric spaces

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Abstract

Tachibana (1967), studied on the Bochner curvature tensor. Sinha and Singh (1971), studied on Kaehlerian spaces with recurrent Bochner curvature. Singh (1973), studied on a Kaehlerian space with recurrent holomorphic projective curvature tensor. Singh (1979), studied on Einstein – Kaehlerian Conharmonic recurrent space. Negi and Rawat (1997) studied theorems on Kaehlerian spaces with recurrent and symmetric Bochner curvature tensor. Rawat and Prasad (2008), studied and defined some recurrent and symmetric properties in an almost Kaehlerian space. Rawat and Uniyal (2010), studied on infinitesimal conformal and projective transformations of K-space and Kaehlerian recurrent space. Further, Rawat, Kumar and Uniyal (2012), studied on hyperbolically Kaehlerian bi-recurrent and bi-symmetric spaces. In the present paper, we have studied on Einstein-Sasakian holomorphically conformal bi-recurrent and bi-symmetric spaces and several theorems have been established and proved therein. The necessary and sufficient condition for an Einstein – Sasakian space to be Einstein-Sasakian bi-recurrent and bi-symmetric has been investigated.

Key Words- Einstein space, Sasakian space, Bi-recurrent space, bi-symmetric space, conformal curvature

1. Introduction

An n-dimensional Sasakian space " S_n" (or, normal contact metric space) is a Riemannian space, which admits a unit killing vector field η^i satisfying

$$\nabla_i \nabla_j \eta_k = \eta_j g_{ik} - \eta_k g_{ij} \quad \ldots(1.1)$$

It is well known that the Sasakian space S_n is orientable and odd dimensional. Also, we know that an n-dimensional Kaehlerian space K_n is a Riemannian space, which admits a structure tensor field F_i^h satisfying (Yano,1965)

$$F_j^h F_h^i = -\delta_j^i, \quad \ldots (1.2)$$

$$F_{ij} = - F_{ji}, \quad (F_{ij} = F_i^a g_{ij}), \quad \ldots (1.3)$$

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and

$$F^h_{i,j} = 0, \qquad \ldots (1.4)$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

Thus both S_n and K_n are Riemannian spaces satisfying the properties of a Riemannian spaces.

The Riemannian curvature tensor field R^h_{ijk} is given by

$$R^h_{ijk} = \partial_i \left\{ {h \atop j\ k} \right\} - \partial_j \left\{ {h \atop i\ k} \right\} + \left\{ {h \atop i\ q} \right\} \left\{ {q \atop j\ k} \right\} - \left\{ {h \atop j\ q} \right\} \left\{ {q \atop i\ k} \right\}, \qquad \ldots (1.5)$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the scalar curvature in S_n are respectively given by

$$R_{ij} = R^l_{lij} \qquad \text{and} \qquad R = R_{ij}\, g^{ij}$$

It is well known that these tensors satisfies the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \qquad \ldots (1.6)$$

$$R_{,i} = 2\, R_{i,a} \qquad \ldots (1.7)$$

$$F^a_i\ R_{aj} = -R_{ia}\, F^a_j \qquad \ldots (1.8)$$

and

$$F^a_i\, R^j_a = R^a_i\, F^j_a \qquad \ldots (1.9)$$

The holomorphically conformal (Bochner), holomorphically projective and Conharmonic curvature tensors are respectively given by

$$K^h_{ijk} = R^h_{ijk} + \frac{1}{(n+4)}\left(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i + g_{ik}\,R^h_j - g_{jk}\,R^h_i + S_{ik}\,F^h_j - S_{jk}\,F^h_i + F_{ik}\,S^h_j - F_{jk}\,S^h_i + 2S_{ij}\,F^h_k + 2F_{ij}\,S^h_k\right) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right), \qquad \ldots (1.10)$$

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}\left(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i + S_{ik}\,F^h_j - S_{jk}\,F^h_i + 2S_{ij}\,F^h_k\right), \qquad \ldots (1.11)$$

and

$$T^h_{ijk} = R^h_{ijk} + \frac{1}{(n+4)}\left(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i + g_{ik}\,R^h_j - g_{jk}\,R^h_i + S_{ik}\,F^h_j - S_{jk}\,F^h_i + F_{ik}\,S^h_j - F_{jk}\,S^h_i + 2S_{ij}\,F^h_k + 2F_{ij}\,S^h_k\right), \qquad \ldots (1.12)$$

where,

$$S_{ij} = F^a_i R_{aj}\,.$$

Einstein – Space- Einstein space is defined as a space, which is homogeneous relative to the Ricci-tensor R_{ij} . That is to say, if $R_{ij} = \lambda g_{ij}$ at every point of a space, then that space is called Einstein space. Inner multiplication by g^{ij} shows that

$$R = \lambda n \quad \text{or,} \quad \lambda = \frac{R}{n} \quad i.e. \quad \frac{R_{ij}}{g_{ij}} = \frac{R}{n}$$

Consequently $$R_{ij} = \frac{R}{n} g_{ij} \qquad \ldots (1.13)$$

Hence, the space is an Einstein –space, if $R_{ij} = \frac{1}{n} R g_{ij}$, at every point of the space.

Let us suppose that a Sasakian space is an Einstein –one, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij} ; \quad R_{,a} = 0 \qquad \ldots (1.14)$$

at every point of the space . From which, we obtain

$$R_{ij,a} = 0, \qquad S_{ij,a} = 0 \qquad \text{and} \qquad S_{ij} = \frac{R}{n} F_{ij} . \qquad \ldots (1.15)$$

We shall call an Einstein-Sasakian space or, in brief, by an E-S* space.

If the Sasakian space is Einstein one, then the Bochner curvature tensor, Projective curvature tensor and Conharmonic curvature tensor are respectively reduces to the forms

$$* K^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)} \left(g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k \right), \qquad \ldots (1.16)$$

$$* P^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)} \left(g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k \right), \qquad \ldots (1.17)$$

and

$$* T^h_{ijk} = R^h_{ijk} + \frac{2R}{n(n+4)} \left(g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k \right), \qquad (1.18)$$

Remark (1.1) - From (1.16) and (1.17), it is clear that in an E-S* space $* K^h_{ijk}$ and $* P^h_{ijk}$ coincides.

Definition (1.1) - A Sasakian space is a space of constant holomorphic sectional curvature , if the tensor $* K^h_{ijk}$ given by (1.16) vanishes identically.

Let R_{hijk} be the component of the Riemannian curvature tensor. We define a bi-recurrent space as a non-flat Riemannian space V_n , the Riemannian curvature tensor of which satisfies the relation of the form

$$R_{hijk,ab} = \lambda_{ab} R_{hij} \qquad \ldots (1.19)$$

where λ_{ab} is a non-zero tensor of second order, called the tensor of recurrence or, recurrence tensor .

Definition (1.2) - A Sasakian space is said to be Sasakian bi-recurrent space, if the curvature tensor satisfy the condition

$$R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0, \quad \ldots \quad (1.20)$$

for some non-zero tensor λ_{ab} .

The space is said to be Sasakian Ricci bi-recurrent (or, semi bi-recurrent), if it satisfies the condition

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0, \quad \ldots \quad (1.21)$$

Multiplying (1.21) by g^{ij} , we get

$$R_{,ab} - \lambda_{ab} R = 0. \quad \ldots \quad (1.22)$$

Remark (1.2)- From (1.20) and (1.21), it follows that every Sasakian bi-recurrent space is Sasakian – Ricci bi-recurrent, but the converse is not necessarily true.

An immediate consequence of (1.19) and Bianchi identity

$$R_{hijk,ab} + R_{hika,jb} + R_{hiaj,kb} = 0$$

Gives for a bi-recurrent space

$$\lambda_{ab} R_{hijk} + \lambda_{jb} R_{hika} + \lambda_{kb} R_{hiaj} = 0 . \quad \ldots \quad (1.23)$$

In the case

$$R_{hijk,ab} = 0$$

(1.19) and (1.23) are satisfied for $\lambda_{ij} = 0$ and the space may or may not satisfy (1.23) for some non-zero tensor λ_{ij} .

2. Einstein –Sasakian bi-recurrent spaces

Definition (2.1) - An E-S* space satisfying the condition

$$* K^h_{ijk,ab} - \lambda_{ab} * K^h_{ijk} = 0, \quad \ldots \quad (2.1)$$

for some non-zero tensor λ_{ab} , is called an E – S* space with bi-recurrent Bochner curvature tensor.

Definition (2.2) - An E – S* space satisfying the condition

$$* P^h_{ijk,ab} - \lambda_{ab} * P^h_{ijk} = 0, \quad \ldots \quad (2.2)$$

for some non-zero tensor λ_{ab} , is called an E – S* space with bi-recurrent holomorphically projective curvature tensor.

Definition (2.3) - An E – S* space satisfying the condition

$$* \quad T^h_{ijk,ab} - \lambda_{ab} * T^h_{ijk} = 0, \quad (2.3)$$

for some non-zero tensor λ_{ab} , is called an E – S* space with bi - recurrent Conharmonic Curvature tensor.

Theorem (2.1) - A necessary and sufficient condition for an E – S* space to be E – S* space with bi - recurrent holomorphically Conformal (Bochner) curvature tensor is that the space be Sasakian bi- recurrent .

Proof: Differentiating (1.16) covariantly with respect to x^a , and again differentiating the result thus obtained covariantly w.r.to x^b , we have

$$* K^h_{ijk,ab} = R^h_{ijk,ab} + \frac{R_{,ab}}{n(n+2)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right), \quad \ldots \quad (2.4)$$

Multiplying (1.16) by λ_{ab} and subtracting the result so obtained from (2.4), we get

$$K^h_{ijk,ab} - \lambda_{ab} * K^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}\, R^h_{ijk} + \frac{(R_{,ab} - \lambda_{ab}\, R)}{n(n+2)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right), \quad \ldots \quad (2.5)$$

If the space is an E – S* space with bi-recurrent Bochner curvature tensor, then the above relation reduces to (1.20), which shows that the space is Sasakian bi-recurrent. Hence the condition is necessary.

Conversely, let the space be Sasakian bi-recurrent, then (2.5) in view of (1. 20) gives

$$* K^h_{ijk,ab} - \lambda_{ab} * K^h_{ijk} = 0,$$

This shows that the space is Einstein – Sasakian space with bi-recurrent Bochner curvature tensor. Hence the condition is sufficient. This completes the proof of the theorem.

Theorem (2.2)- A necessary and sufficient condition for an E – S* space to be an E – S* Conharmonic bi-recurrent is that the space be an E – S* space with bi-recurrent holomorphically conformal (Bochner) curvature tensor.

Proof- Differentiating (1.18) , covariantly with respect to x^a , and again differentiating the result thus obtained covariantly w.r.to x^b , we obtain

$$* T^h_{ijk,ab} = R^h_{ijk,ab} + \frac{2R_{,ab}}{n(n+4)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right), \quad \ldots \quad (2.6)$$

Multiplying (1.18) by λ_{ab} and subtracting the result thus obtained from (2.6) , we have

$$* \quad T^h_{ijk,ab} - \lambda_{ab} * T^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}\, R^h_{ijk} + \frac{2(R_{,ab} - \lambda_{ab}\, R)}{n(n+4)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right), \quad \ldots \quad (2.7)$$

Now, using the fact $R_{,ab} - \lambda_{ab}\ R = 0,$ the above equation (2.7) reduces to the form

$$*T^h_{ijk,ab} - \lambda_{ab} * T^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}\, R^h_{ijk} \quad \ldots \quad (2.8)$$

From (2.5) and (2.8) , we get

$$*T^h_{ijk,ab} - \lambda_{ab} * T^h_{ijk} =* K^h_{ijk,ab} - \lambda_{ab} * K^h_{ijk}\ , \quad \ldots \quad (2.9)$$

If the space is an E – S* Conharmonic bi-recurrent, then (2.9) in view of (2.3) gives

$$*K^h_{ijk,ab} - \lambda_{ab} * K^h_{ijk} = 0, \quad \ldots (2.10)$$

which shows that the space is E – S* space with bi-recurrent holomorphically Conformal (Bochner) curvature tensor. Hence the condition is necessary

Conversely, let the space be an E – S* space with bi-recurrent holomorphically Conformal (Bochner) curvature tensor, then (2.9) in view of (2.1), Reduces to

$$* T^h_{ijk,ab} - \lambda_{ab} * T^h_{ijk} = 0,$$

Which shows that the space is an E – S* Conharmonic bi-recurrent space. Hence, the condition is sufficient . This completes the proof of the theorem

Theorem (2.3) - A necessary and sufficient condition for an E – S* space to be an Sasakian bi-recurrent space is that the scalar curvature be equal to zero.

Proof-Suppose that an E – S* space is Sasakian bi-recurrent space. Making use of equations (1.14), (1.15) and (1.18) in (2.3), we get

$$R^h_{ijk,ab} = \lambda_{ab}\ [R^h_{ijk} + \frac{2R}{n(n+4)}\Big(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\Big)], \quad \ldots \quad (2.11)$$

Since an E – S* space is Sasakian bi recurrent, then (2.11) reduces to

$$\frac{2R}{n(n+4)}\Big(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\Big) = 0, \quad \ldots \quad (2.12)$$

which gives R = 0 , i.e. the scalar curvature is zero.

Conversely, if an E – S* space satisfies R = 0 , then (2.11) , reduces to

$$R^h_{ijk,ab} - \lambda_{ab}\,R^h_{ijk} = 0,$$

which gives that the space is Sasakian bi-recurrent . This completes the proof of the theorem.

Theorem (2.4)- If a bi-recurrent space be Einstein, then the Ricci-curvature tensor vanishes.

Proof: Considering (1.23), transvecting by $g^{hk}\,g^{ij}$, we get

$$\lambda_{ab}\,R - \lambda_{jb}\,g^{ij}\,R_{ia} - \lambda_{kb}\,g^{hk}\,R_{ha} = 0,$$

i.e. $$\lambda_{ab}\,R - 2\lambda_{jb}\,g^{ij}\,R_{ia} = 0 \quad \ldots \quad (2.13)$$

Let a bi-recurrent space be an Einstein one. Then making use of (1.13) in (2.13) , we obtain

$$\lambda_{ab}\,R - 2\lambda_{jb}\,g^{ij}\,\frac{R}{n}\,g_{ia} = 0,$$

Whence $$(n - 2)\ \lambda_{ab}\,R = 0.$$

Since $\lambda_{ab} \neq 0$ and $n > 2$, $R = 0$, which is equivalent in an Einstein space to saying that $R_{ij} = 0$.

This completes the proof

3. EINSTEIN- SASAKIAN BI – SYMMETRIC SPACES

Definition (3.1) -A Sasakian space satisfying the relation

$$R^{h}_{ijk,ab} = 0, \quad \text{or , equivalently} \quad R_{ijkl,ab} = 0, \qquad \ldots \quad (3.1)$$

is said to be Sasakian bi- symmetric space and it is called Ricci -bi-symmetric (or, semi -bi-symmetric) , if it satisfies

$$R_{ij,ab} = 0, \qquad \ldots \quad (3.2)$$

Multiplying (3.2) by g^{ij}, we have

$$R_{,ab} = 0. \qquad \ldots \quad (3.3)$$

Remark (3.1)- From (3.1) and (3.2) , it follows that every Sasakian bi-symmetric space is Sasakian Ricci - bi-symmetric, but the converse is not necessarily true

Definition (3.2) - An E – S* space satisfying the relation

$$K^{h}_{ijk,ab} = 0, \quad \text{or , equivalently} \quad {}^{*}K_{ijkl,ab} = 0, \qquad \ldots \quad (3.4)$$

is called an E – S* space with bi-symmetric Bochner curvature tensor .

Definition (3.3) - An E – S* space satisfying the relation

$$^{*}P^{h}_{ijk,ab} = 0, \quad \text{or , equivalently} \quad {}^{*}P_{ijkl,ab} = 0, \qquad \ldots \quad (3.5)$$

is called an E – S* space with bi-symmetric holomorphically projective curvature tensor.

Definition (3.4) - An E – S* space satisfying the relation

$$^{*}T^{h}_{ijk,ab} = 0, \quad \text{or , equivalently} \quad {}^{*}T_{ijkl,ab} = 0, \qquad \ldots \quad (3.6)$$

is called an E – S* space with bi-symmetric Conharmonic curvature tensor

Theorem (3.1) - A necessary and sufficient condition for an E – S* space to be E – S* space with bi-symmetric holomorphically conformal (Bochner) curvature tensor is that the space be Sasakian bi-symmetric.

Proof- Sasakian bi-symmetric space and E – S* space with bi-symmetric holomorphically Conformal (Bochner) curvature tensor is given by (3.1) and (3.4).

Therefore, the statement of the above theorem follows in view of (3.1), (3.2), (3.4) and (2.4).

Theorem (3.2) - A necessary and sufficient condition for an E – S* space to be an E – S* Conharmonic bi - symmetric is that the space be an E – S* space with bi-symmetric holomorphically conformal (Bochner) curvature tensor .

Proof- The E- S* Conharmonic bi-symmetric space and E –S* space with bi-symmetric holomorphically conformal (Bochner) curvature tensor are given by (3.6) and (3.4).

Therefore, the statement of the above theorem follows in view of (3.4), (3.6) and (2.6).

Theorem (3.3)-A necessary and sufficient condition for an E – S* space to be a Sasakian bi-symmetric space is that the scalar curvature be equal to zero.

Proof- The Sasakian bi-symmetric space is given by (3.1). The statement of the above theorem follows in view of (3.1), (3.2), (2.1) and (2.12).

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