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Geometry of semi pseudo-Ricci symmetric spacetimes under Gray's decomposition

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Abstract

In this paper, we characterize semi-pseudo Ricci symmetric spacetimes thriving with Gray's decomposition as well as generalized Robertson-Walker spacetimes. For semi pseusdo–Ricci symmetric spacetimes, we determine the form of the Ricci tensor in all $O(x)$-invariant subspaces provided by Gray's decomposition of the gradient of the Ricci tensor. In three cases we obtain that the Ricci tensor is in the form of perfect fluid and in one case the spacetime becomes a generalized Robertson-Walker spacetimes. Finally, it is shown that a semi-pseudo Ricci symmetric generalized Robertson-Walker spacetime is a perfect spacetime.

Keywords and Phrases: Pseudo Ricci-symmetric, conformal curvature tensor, spacetime,Gray's decomposition

1. Introduction

Lorentzian geometry is the mathematical framework that supports some of the most important theories in GR and string theory. From a purely mathematical point of view, a Lorentzian manifold M is a smooth manifold endowed with a symmetric non degenerate bilinear form g, called the metric of signature $(-,+,+,\ \dots+)$ that is, index of g is one. In general, a Lorentzian manifold (M^n, g) may not have a globally timelike vector field. If (M^n, g) admits a globally timelike vector fields, it is called a time-oriented Lorentzian manifold, physically known as spacetime. Several authors have investigated spacetime in various ways, such as (Blaga, 2020; Chaubey, Suh ad De, 2020; Duggal and Sharma, 2005; Hazmetal, 2023) and also various others.

A semi-pseudo Ricci-symmetric (Tarafdar etal. 1995) and denoted by $(SPRS)_n$, if the Ricci tensor Ric of the type (0, 2) of the manifold is non-zero and satisfies the relation

$$(D_X\, Ric)\,(Y,Z) = \eta(Y)\; Ric(X,Z) + \eta(Z)\; Ric(X,Y), \tag{1.1}$$

where D denotes the covariant differentiation with respect to the metric g, η is a non-zero one form and ρ is a vector field equivalent to

$$\eta(X) = g(X,\rho), \tag{1.2}$$

for all X. The manifold reduces to a Ricci-symmetric manifold if 1-form η is zero. Several authors have investigated semi-pseudo Ricci-symmetric manifold.

Changing X and Y in (1.1) and then subtracting these two equations, we obtain

$$(D_X\,Ric)\,(Y,Z) - (D_Y\,Ric)\,(X,Z) = \eta(Y)\,Ric(X,Z) + \eta(X)\,Ric(Y,Z). \tag{1.3}$$

Contracting with respect to Y and Z in (1.3), we get

$$dr(X) = 2Ric(X,\rho) - 2r\eta(X). \tag{1.4}$$

The conformal curvature tensor in a Lorentzian manifold (M^n, g), $n > 3$, is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{1.5}$$

where Q is the Ricci operator defined by $Ric\,(X,Y) = g(QX,Y)$ and r is the scalar curvature.

A Lorentzian manifold M of dimension $n \geq 3$ endowed with the Lorentzian metric g defined by

$$ds^2 = -(dt^2) + \psi^2(t)\, g^*_{lm}(x)\, d^l_x\, d^m_x,$$

where t is the time and $g^*_{lm}(x)$ is the metric tensor of a Riemannian manifold M^*, is a generalized Robertson-Walker (briefly, GRW) is a spacetime. In other words, a GRW spacetime is the warped product $I \times \psi^2 M^*$, where I is an open interval of the real line, ψ is a smooth warping function such that $\psi > 0$ and M^* is an $(n-1)$-dimensional Riemannian manifold (Alias, Romero and Sanchez, 1995).

Lorentzian manifolds with the Ricci tensor

$$Ric(X,Y) = \ddot{\alpha}\, g(X,Y) + \ddot{\beta}\, \eta(X)\, \eta(Y), \tag{1.6}$$

where $\ddot{\alpha}$ and $\ddot{\beta}$ are scalars and ρ is a unit timelike vector field corresponding to the one-form η, are called perfect fluid spacetime (PFS).

The energy momentum tensor (EMT) $\mathcal{T}$ for a PFS has the following form (Neill, 1983)

$$\mathcal{T}(X,Y) = \ddot{p}\, g(X,Y) + (\ddot{p} + \ddot{\sigma})\, \eta(X)\, \eta(Y), \tag{1.7}$$

where $\ddot{\sigma}$ and $\ddot{p}$ represents energy density (ED) and isotropic pressure (I.P).

Einstein field equation (EFE) without cosmological constant is as follows:

$$Ric(X,Y) - \frac{r}{2}\, g(X,Y) = k\, \mathcal{T}(X,Y), \tag{1.8}$$

where k is the gravitational constant.

The present paper is organized as follows:

After introduction, in section two, we investigate all the seven cases of Gray's decomposition of $(SPRS)_n$. The study of $(SPRS)_n$ with GRW spacetime is presented in section 3.

2. Gray's decomposition and Semi-pseudo Ricci symmetric spacetimes

Considering the action of the orthogonal group on the space of the tensors with the symmetries of the covariant derivative of the Ricci curvature, Gray decomposed such space into irreducible components

(Gray, 1978). Gray proposed that the covariant derivative of the Ricci tensor, that is $D.Ric$, can be decomposed into $O(x)$-invariant terms. According to him, the covariant derivative of the Ricci tensor can be converted into $O(x)$-invariant term as follows (Mantica, Molinari, Suh and Shenawy, 2019)

$$(D_X\, Ric)\,(Y,Z) = R(X,Y)Z + \frac{n\,(D_X\, r)}{(n-1)(n+2)}\, g(Y,Z) + \frac{(n-2)(D_Y\, r)}{2(n-1)(n+2)}\, g(X,Z) + \frac{(n-2)(D_Z\, r)}{2(n-1)(n+2)}\, g(X,Y), \quad (2.1)$$

for all vector fields X, Y, Z and $R(X,Y)Z = R(X,Z)Y$ is a tensor with zero trace that can be written as a sum of its orthogonal components:

$$R(X,Y)Z = \frac{1}{3}[R(X,Y)Z + R(Y,Z)X + R(Z,X)Y] + \frac{1}{3}[R(X,Y)Z - R(Y,X)Z] + \frac{1}{3}[R(X,Y)Z - R(Z,X)Y]. \quad (2.2)$$

The decompositions (2.1) and (2.2) yield $O(x)$-invariant subspace, which is characterized by linear invariant equations in $(D_X\, Ric)\,(Y,Z)$.

Therefore, the relation between $(D_X\, Ric)\,(Y,Z)$ and the divergence of the conformal curvature tensor C can be given by the equation

$$(div\, C)(X,Y)Z = \left(\frac{n-3}{n-2}\right)[R(X,Y)Z - R(Y,X)Z]. \quad (2.3)$$

The subspace in Gray's decomposition are as follows:

(i) The trivial subspace is given by $(D_X Ric)\,(Y,Z) = 0$.

(ii) The subspace $\mathcal{T}$ is characterized by $R(X,Y)Z = 0$, i.e,

$$(D_X Ric)\,(Y,Z) = \frac{n\,(D_X\, r)}{(n-1)(n+2)}\, g(Y,Z) + \frac{(n-2)(D_Y\, r)}{2(n-1)(n+2)}\, g(X,Z) + \frac{(n-2)(D_Z\, r)}{2(n-1)(n+2)}\, g(X,Y). \quad (2.4)$$

Manifolds satisfying equation (2.4) are called Sinyukov manifolds (Sinyukov, 1979) & (Formella, 1989).

(iii) The orthogonal complements $\mathcal{T}$ are characterized by

$$(D_X\, Ric)\,(Y,Z) + (D_Y\, Ric)\,(Z,X) + (D_Z\, Ric)\,(X,Y) = 0, \quad (2.5)$$

which yields that the scalar curvature r is constant. Also, the Ricci tensor is killing tensor (Tachibana, 1969) if equation (2.5) satisfied.

(iv) In the subspaces $\mathcal{B}$ and $\mathcal{B}'$ the Ricci tensor is of Codazzi type i.e,

$$(D_X\, Ric)\,(Y,Z) = (D_Y\, Ric)\,(X,Z). \quad (2.6)$$

(v) The Ricci tensor fulfills the following cyclic condition in the subspace $\mathcal{T} \oplus \mathcal{A}$,

$$(D_X\, Ric)\,(Y,Z) + (D_Y\, Ric)\,(Z,X) + (D_Z\, Ric)\,(X,Y) = 2\frac{dr(X)}{(n+2)}\, g(Y,Z) + 2\frac{dr(Y)}{(n+2)}\, g(Z,X) + 2\frac{dr(Z)}{(n+2)}\, g(X,Y), \quad (2.7)$$

that is, the Ricci tensor is conformal killing (Rani, Edgar and Barnes, 2003)

(vi) The Ricci tensor fulfills the following Codazzi condition in the subspace $\mathcal{T} \oplus \mathcal{B}$,

$$(D_X\, Ric)\,(Y,Z) - (D_Y\, Ric)\,(X,Z) = \frac{dr(X)}{2(n-1)}\, g(Y,Z) - \frac{dr(Y)}{2(n-1)}\, g(X,Z), \tag{2.8}$$

which given $div\, C = 0$.

(vii) In the subspace $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Let us consider each of these seven cases separately.

Case (i): The trivial subspace $(D_X\, Ric)\,(Y,Z) = 0$.

Theorem 2.1. An $(SPRS)_n$ spacetime does not belong to the trivial subspace.

Proof: Since $(D_X\, Ric) = 0$, then from the definition of $(SPRS)_n$ the one-form η must vanish at any point of the manifold, which contradicts the definition of $(SPRS)_n$.

Case (ii): The subspace $\mathcal{T}$ where $R(X,Y)Z = 0$.

Theorem 2.2. If an $(SPRS)_n$ spacetimes belongs to the subspace $\mathcal{T}$, then the spacetime is a perfect fluid spacetime.

Proof: The Ricci tensor satisfies the relation $\tilde{R}(X,Y)Z = 0$ in the subspace $\mathcal{T}$ and hence from the relation (2.3) we obtain $div\, C = 0$.

Thus, we have

$$(D_X\, Ric)\,(Y,Z) - (D_Z\, Ric)\,(X,Y) = \frac{1}{2(n-1)}[\, dr(X)\, g(Y,Z) - dr(Z)\, g(X,Y)]. \tag{2.9}$$

Using (1.1) and (1.4) in (2.9), we get

$$\eta(Z)\, Ric(Y,X) - \eta(X)\, Ric(Z,Y) = \frac{r}{n-1}\,[\eta(Z)\, g(X,Y) - \eta(X)\, g(Y,Z) - \frac{1}{n-1}[Ric(Z,\rho)g(X,Y) - Ric(X,\rho)g(Y,Z)]. \tag{2.10}$$

Now, putting $X = Y = \rho$ in (2.10) and using $\eta(\rho) = -1$, we have

$$Ric(Z,\rho) = -\frac{n}{n-2}\, t.\eta(Z), \tag{2.11}$$

where $t = Ric\,(\rho,\rho)$.

Again, putting ρ for X in (2.10) and using (2.11), we get

$$Ric(Y,Z) = \left(\frac{r+t}{n-1}\right)\, g(Y,Z) + \left[\frac{r(n-2)+n^2 t}{(n-1)(n-2)}\right]\, \eta(Y)\, \eta(Z). \tag{2.12}$$

This implies that an $(SPRS)_n$ spacetime is a perfect fluid spacetime PFS.

Case (iii): The subspace $\mathcal{A}$ is characterized by the condition (2.5).

Theorem 2.3: If an $(SPRS)_n$ spacetime belongs to the subspace $\mathcal{A}$, then the associated one form are $\eta(X) = 0$.

Proof: From (1.1) and (2.5), we have

$$\eta(X)\, Ric(Z,Y) + \eta(Y)\, Ric(X,Z) + \eta(Z)\, Ric(Y,X) = 0, \tag{2.13}$$

Walkar's Lemma (Walker, 1950) is now listed as below:

Lemma: If α_{ij}, β_j are numbers satisfying $a_{ij} = a_{ji}, \alpha_{ij}\,\beta_k + \alpha_{jk}\,\beta_i + \alpha_{ki}\,\beta_j = 0$ for $i, j, k = 1,2, \dots, n$, then either all α_{ij} are zero or all β_i are zero.

As $Ric(X, Y) \neq 0$, then according to Walker's Lemma from (2.1), we have $\eta(X) = 0$.

Case (iv): In this subspace, the Ricci tensor is of Codazzi type.

Theorem 2.4: If a $(SPRS)_n$ spacetime belongs to the subspace $\mathcal{B}$ and $\mathcal{B}'$, then the spacetime is a Ricci simple spacetime.

Proof: If a $(SPRS)_n$ belongs to $\mathcal{B}$ and $\mathcal{B}'$, then

$$(D_X Ric)(Y, Z) = (D_Y Ric)(X, Z). \tag{2.14}$$

Using (1.1) in (2.14), we get

$$\eta(X) Ric(Y, Z) = \eta(Y) Ric(X, Z). \tag{2.15}$$

Putting ρ for X in above equation, we get

$$Ric(Y, Z) = -\eta(Y) Ric(\rho, Z), \tag{2.16}$$

Contracting Y and Z in (2.15), we obtain

$$\eta(X)\, r = Ric(X, \rho). \tag{2.17}$$

Using (2.17) in (2.16), we get

$$Ric(Y, Z) = -r.\eta(Y)\eta(Z),$$

which implies that the spacetime is Ricci simple (Mantica and Molinan, 2017).

It is known that

$$(div\ C)(X, Y)Z = \left(\frac{n-3}{n-2}\right)[\{(D_X Ric)(Y, Z) - (D_Y Ric)(X, Z)\} - \frac{1}{2(n-1)}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\}], \tag{2.18}$$

but in one case Ricci tensor satisfies (2.14). Hence from (2.14) and (2.18) gives $div\ C = 0$, provided scalar curvature is constant.

Mantica, Suh and De, (2016) proved the following theorem:

Theorem A: If an n-dimensional Lorentzian manifold $(M^n\ g)$, $n > 3$, with the Ricci tensor is of the form $Ric(X, Y) = -r\,\eta(X)\,\eta(Y)$ satisfies the curvature condition $div\ C = 0$, then $(M^n\ g)$ is a GRW spacetime.

Theorem 2.5: If an $(SPRS)_n$ spacetime belongs to the class $\mathcal{B}$ and $\mathcal{B}'$, then the spacetime becomes a GRW spacetime.

Proof: Due to Theorem (2.4) and Theorem A, we can prove theorem (2.5).

Case (v): In this subspace, the Ricci tensor satisfies (2.7). Mantica etal. (2019) proved that the subspace $\mathcal{T} \oplus \mathcal{A}$ and $\mathcal{T}$ are equivalent. In this circumstances, we reached $div\ C = 0$. Therefore, the result is the same as in theorem (2.2).

Case (vi): Let $(SPRS)_n$ belong to $\mathcal{T} \oplus \mathcal{B}$. In this case, we get $div\ C = 0$. So, we can state the same result as in theorem (2.2).

Case (vii): In the subspace $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Theorem 2.6: If an $(SPRS)_n$ spacetime belongs to the subspace $\mathcal{A} \oplus \mathcal{B}$, then the velocity vector field ρ is an eigen vector corresponding to the eigen value r.

Proof: Since the scalar curvature tensor r is constant, then equation (1.4) gives $Ric\ (X, \rho) = r\ g(X, \rho)$. This proves the proof.

3. $(SPRS)_n$ GRW spacetimes

In this section we characterize semi-pseudo Ricci symmetric GRW spacetimes. Mantica and Molinari (2017) proved that a Lorentzian manifold of dimension $n \geq 3$ is a GRW spacetime if and only if it admits a unit timelike torse forming vector field. That is $\nabla_k u_j = \phi(g_{kj} + u_k u_j)$.

Theorem 3.1: A $(SPRS)_n$ GRW spacetime is a perfect fluid spacetime.

Proof: We assume that the $A\ (SPRS)_n$ spacetime be a GRW spacetime. Then we have

$$(D_X\, \eta)(Y) = \psi\ [g(X,Y) + \eta(X)\, \eta(Y)] \text{ and } Ric(X,\rho) = \mu\, g(X,\rho), \tag{3.1}$$

for some smooth function $\psi \neq 0$ and μ on M.

Now,

$$(D_X\, Ric)(Y,\rho) = X\ Ric\ (Y,\rho) - Ric\ (D_X Y,\rho) - Ric(Y, D_X \rho). \tag{3.2}$$

Using (3.1) in (3.2), we get

$$(D_X\, Ric)(Y,\rho) = X(\mu)\, \eta(Y) + \mu\, \psi\, g(X,Y) - \psi\, Ric(Y,X), \tag{3.3}$$

where $X(\mu) = g(X, grad\mu)$.

Combining equations (1.1) and (3.1), we get

$$(D_X\, Ric)(Y,\rho) = \mu\, \eta(X)\eta(Y) + \eta(\rho)Ric(X,Y). \tag{3.4}$$

Comparing equations (3.3) and (3.4), we obtain

$$\mu\, \eta(X)\eta(Y) + \eta(\rho)Ric(X,Y) = X(\mu)\, \eta(Y) + \mu\, \psi\, g(X,Y) - \psi\, Ric(Y,X). \tag{3.5}$$

Setting ρ for Y in above equation and using $\eta(\rho) = -1$, we find

$$X(\mu) = -\mu\, \eta(\rho)\, \eta(X) + \mu\, \eta(X). \tag{3.6}$$

Contracting X and Y in (3.5), we get

$$\rho(\mu) + n\, \mu\, \psi - r\, \psi = -\mu + \eta(e)r. \tag{3.7}$$

Equation (3.6) and (3.7) gives

$$\mu = \frac{r[\eta(\rho)+\psi)}{n\psi - 1}. \tag{3.8}$$

From (3.1) and (3.8), we have

$$Ric\ (X,\rho) = \frac{r[\eta(\rho)+\psi]}{(n\psi-1)}\ g(X,\rho). \tag{3.9}$$

This means that ρ is an eigen vector corresponding to the eigen vector $\frac{r[\eta(\rho)+\psi]}{(n\psi-1)}$.

In view of equation (3.5) and (3.6), we get

$$Ric(X,Y) = \frac{\mu}{\psi+\eta(\rho)}\ [\psi\ g(X,Y) - \eta(X)\,\eta(Y)\,\eta(\rho)]. \tag{3.10}$$

Putting the value of μ from (3.8) in (3.10), we get

$$Ric\ (X,Y) = \frac{r}{n\psi-1}[\psi\ g(X,Y) - \eta(X)\,\eta(Y)\,\eta(\rho)]. \tag{3.11}$$

Moreover, if $r = 0$ then (3.8) gives $\mu = 0$.

But μ can not be zero, $r \neq 0$. Hence a $(SPRS)_n$ GRW spacetime is a perfect fluid spacetime (PFS).

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