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Study on Sasakian space with recurrent and symmetric Bochner CurvatureTensorof P^{th} order

K. S. Rawat and Sandeep Chauhan
Department of Mathematics
H.N.B.Garhwal University Campus
Badshahi Thaul, Tehri (Garhwal)-249199
Uttarakhand, India.
Email: drksrawathnbgu@gmail.com

Abstract

Okumura (1962), studied some remarks on space with a certain contact structure. Singh (1971), studied on Kaehlerian spaces with recurrent Bochner curvature tensor. Negi and Rawat (1994), studied some bi-recurrence and bi-symmetric properties in a Kaehlerian space. Rawat (2002), studied on Geometry of locally product almost Tachibana space. Rawat and Dobhal (2009), studied on Einstein–Kaehlerian s-recurrent space. Rawat and Kumar (2009), studied on curvature collineations in a Tachibana recurrent space. Further, Rawat and Prasad (2010), studied on holomorphically projectively flat parabolically Kaehlerian space.In the present paper; we have been studied on Sasakian space with recurrent and symmetric Bochner curvature tensor of p^{th} order. Several theorems also have been established and proved therein.

Key Words: Sasakian space, recurrent space, symmetric space, Bochner curvature tensor.

1. **Introduction:** An n-dimensional Sasakian space "S_n" (or, normal contact metric space) is a Riemannian space, which admits a unit killing vector field η^isatisfying (Okumura,1962) :

$$\nabla_i\nabla_j\eta_k = \eta_j g_{ik} - \eta_k g_{ij} \tag{1.1}$$

It is well known that the Sasakian space is orientable and odd dimensional. Also, we know that an n- dimensional Kaehlerian space " K_n" is a Riemannian space which admits structure tensor field F_i^h satisfying (Yano, 1965) the following conditions:

$$F_i^h F_h^i = -\delta_j^i, \tag{1.2}$$

$$F_{ij} = -F_{ji}, \; (F_{ij} = F_i^a g_{aj}) \tag{1.3}$$

and

$$F_{i,j}^{h} = 0, \tag{1.4}$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

Thus, both S_n and K_n are Riemannian spaces satisfying the properties of a Riemannian space.

The Riemannian curvature tensor field R_{ijk}^{h} is given by

$$R_{ijk}^{h} = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ il \end{Bmatrix}\begin{Bmatrix} l \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix}, \tag{1.5}$$

where $\partial_j = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes real local coordinates.
The Ricci- tensor and scalar curvature in S_n are respectively given by

$$R_{ij} = R_{aij}^{a} \quad \text{and} \quad R = R_{ij}g^{ij}$$

If we define a tensor

$$S_{ij} = F_i^a R_{aj}, \tag{1.6}$$

Then, we have

$$S_{ij} = - S_{ji}, \tag{1.7}$$

$$F_i^a = -S_{ia}F_j^a \tag{1.8}$$

and

$$F_i^a S_{jk,a} = R_{ji,k} - R_{ki,j} \tag{1.9}$$

It has been verified by (Yano [4]), that the metric tensor g_{ij} and the Ricci tensor denoted by R_{ij} are hybrid in I and j . Therefore, we get

$$g_{ij} = g_{sr}F_i^s F_j^r \tag{1.10}$$

and

$$R_{ij} = R_{sr}F_i^s F_j^r \tag{1.11}$$

The Bochner curvature tensor with respect to local coordinate system is given by

$$K_{hijk} = R_{hijk} - \frac{1}{(n+2)}\left(R_{ij}g_{hk} + R_{hi}g_{jk} + g_{ij}R_{hk} + g_{hi}R_{jk}\right) + \frac{R}{2(n+1)(n+2)}\left(g_{ij}g_{hk} + g_{hi}g_{jk}\right) \tag{1.12}$$

If we put

$$L_{ij} = R_{ij} - \frac{R}{4(n+1)}g_{ij} \tag{1.13}$$

and

$$M_{ij} = F_i^a L_{aj} = S_{ij} - \frac{R}{4(n+1)} F_{ij} \tag{1.14}$$

Then (1.12), in view of (1.13) reduces to the form

$$K_{hij} \ = \ R_{hijk} - \frac{1}{(n+2)} \ (L_{ij}g_{hk} + \ L_{hi}g_{jk} + L_{hk}g_{ij} + \ L_{jk}g_{hi} \tag{1.15}$$

2. Sasakian Recurrent Spaces of p^th Order

Definition (2.1): A Sasakian space with Riemannian curvature tensor is said to be Sasakian
recurrent space of p^th order, if it satisfies

$$R^h_{ijk,ab....p} - \lambda_{ab....p} R^h_{ijk} = 0, \tag{2.1}$$

For some non-zero recurrence tensor $\lambda_{ab...p}$.

The space is said to be Sasakian Ricci-recurrent space of p^th order, if it satisfies the condition

$$\mathrm{R}_{\mathrm{ij,ab....p}} - \lambda_{ab....p} R_{ij} = 0, \tag{2.2}$$

Multiplying (2.2) by g^{ij}, we have

$$R_{,ab...p} - \lambda_{ab...p} R = 0. \tag{2.3}$$

Remark (2.1): From (2.2), it follows that every Sasakian recurrent space of p^th order is Ricci recurrent space of p^th order, but the converse is not necessarily true

Definition (2.2): A Sasakian space satisfying the condition

$$K_{hijk,ab.....p} - \lambda_{ab....p} K_{hijk} = 0. \tag{2.4}$$

for some non-zero tensor $\lambda_{ab....p}$, will be called a Sasakian space with recurrent Bochner curvature tensor of p^th order.

Note: Now in whole calculation we will take α in place of **ab…….p** for convenience.

Theorem (2.1) : If a Sasakian space satisfies any two of the following Properties:

(i) The space is Sasakian recurrent space of p^th order,
(ii) The space is Sasakian Ricci-recurrent space of p^th order,

(iii)The space is a Sasakian space with recurrent Bochner curvature tensor of p^{th} order, it must also satisfy the third.

Proof : Sasakian recurrent space of p^{th} order, Sasakian Ricci-recurrent space of p^{th} order and a Sasakian space with recurrent Bochner curvature tensor of p^{th} order are respectively characterized by (2.1), (2.2) and (2.4).

Differentiating (1.15) covariantly w. r. to x^α, we get

$$K_{hijk,\alpha} = R_{hijk,\alpha} - \frac{1}{(n+2)}(L_{ij,\alpha}g_{hk} + L_{hi,}g_{jk} + L_{hk,\alpha}g_{ij} + L_{jk,\alpha}g_{hi}) \qquad (2.5)$$

Multiplying (1.15) with λ_α and subtracting the result thus obtained from (2.5), we have

$$K_{hijk,\alpha} - \lambda_\alpha K_{hijk} = R_{hijk,\alpha} - \lambda_\alpha R_{hijk} - \frac{1}{(n+2)} [(L_{ij,\alpha} - \lambda_\alpha L_{ij})g_{hk} + (L_{hi,\alpha} - \lambda_\alpha L_{hi})g_{jk} + (L_{hk,\alpha} - \lambda_\alpha L_{hk})g_{ij} + (L_{jk,\alpha} - \lambda_\alpha L_{jk})g_{hi}] \qquad (2.6)$$

The statement of the above theorem follows in view of (1.13), (2.1), (2.2), (2.4) and (2.6).

Theorem (2.2): The necessary and sufficient condition for a Sasakian space to be Sasakian Ricci-recurrent space of p^{th} order is that

$$K_{hijk,\alpha} - \lambda_\alpha K_{hijk} = R_{hijk,\alpha} - \lambda_\alpha R_{hijk} .$$

Proof: Let the Sasakian space be Sasakian Ricci-recurrent space of p^{th} order, then the relation (2.2) is satisfied.

Since the space is Ricci-recurrent space, then the equation (2.6) in view of (2.1) reduces to

$$K_{hijk,\alpha} - \lambda_\alpha K_{hijk} = R_{hijk,\alpha} - \lambda_\alpha R_{hijk} . \qquad (2.7)$$

Conversely, if in a Sasakian space, equation (2.7) is satisfied, then from (2.6), we have

$$(L_{ij,\alpha} - \lambda_\alpha L_{ij})g_{hk} + (L_{hi,\alpha} - \lambda_\alpha L_{hi})g_{jk} + (L_{hk,\alpha} - \lambda_\alpha L_{hk})g_{ij} + (L_{jk,\alpha} - \lambda_\alpha L_{jk})g_{hi} = 0. \qquad (2.8)$$

Which yields with the help of (1.13)

$$R_{ij,\alpha} - \lambda_\alpha R_{ij} = 0.$$

i.e., the space is Sasakian Ricci-recurrent space of p^{th} order.

Theorem (2.3): Every Sasakian recurrent space of p^{th} order is a Sasakian space with recurrent Bochner curvature tensor of p^{th} order.

Proof: If the space is Sasakian recurrent space of p^{th} order, equations (2.1) and (2.2) are satisfied, then (2.6) in view of (2.1), (2.2) and (1.13) reduces to

$$K_{hijk,\alpha} - \lambda_\alpha K_{hijk} = 0$$

which shows that the space will also be Sasakian space with recurrent Bochner curvature tensor of p^{th} order.

3. Sasakian Symmetric Spaces of p^{th} Order

Definition (3.1): A Sasakian space is said to be Sasakian symmetric space of pth order, if it

Satisfies the following relation

$$R^h_{ijk,\alpha} = 0, \text{ or, equivalently } R_{ijkl,\alpha} = 0 \tag{3.1}$$

Obviously, a Sasakian symmetric space of p^{th} order is Sasakian Ricci-symmetric space of p^{th} order, if it satisfies the condition

$$R_{ij,\alpha} = 0, \tag{3.2}$$

Multiplying the above equation (3.2) by g^{ij} , we get

$$R_{,\alpha} = 0. \tag{3.3}$$

Definition (3.2): A Sasakian space satisfying the relation

$$K_{hijk,\alpha} = 0, \text{or, equivalently } K^h_{ijk,\alpha} = 0, \tag{3.4}$$

is called a Sasakian space with symmetric Bochner curvature tensor of p^{th} order.

Theorem (3.1): If a Sasakian space satisfies any two of the following Properties:

(i) The space is Sasakian symmetric space of p^{th} order,
(ii) The space is Sasakian Ricci-symmetric space of p^{th} order,
(iii) The space is a Sasakian space with symmetric Bochner curvature tensor of p^{th} order, it must also satisfy the third.

Proof: Sasakian symmetric space of p^{th} order, Sasakian Ricci- symmetric space of p^{th} order and a Sasakian space with symmetric Bochner Curvature tensor of p^{th} order are respectively characterized by the equations (3.1), (3.2) and (3.4).

Therefore, the statement of the above theorem follows in view of equations (1.13), (2.5), (3.1), (3.2) and (3.4).

Theorem (3.2) : The necessary and sufficient condition for a Sasakian space to be Sasakian Ricci-symmetric space of p^{th} order is that

$$K_{hijk,\alpha} = R_{hijk,\alpha} \tag{3.5}$$

Proof : Let the space be Sasakian Ricci-symmetric space of p^{th} order , then the relation (3.2) is satisfied.

The statement of the above theorem follows in view of equations (1.13), (2.5), (3.2) and (3.4).

Conversely, in a Sasakian space equation (3.5) is satisfied, then from (2.5), we have

$$L_{ij,\alpha}g_{hk} + L_{hi,\alpha}g_{jk} + L_{hk,\alpha}g_{ij} + L_{jk,\alpha}g_{hi} = 0 \tag{3.6}$$

which yields with the help of (1.13)

$$R_{ij,\alpha} = 0,$$

i.e., the space is Sasakian Ricci-symmetric space of p^{th} order.

Theorem (3.3): Every Sasakian symmetric space of p^{th} order is a Sasakian space with symmetric Bochner curvature tensor of p^{th} order.

Proof: If the space is Sasakian symmetric space of p^{th} order, equations (3.1) and (3.2) are satisfied, and (2.5), in view of (1.13), (3.1) and (3.2), reduces to

$$K_{hijk,\alpha} = 0 ,$$

which shows that the space will also be Sasakian space with symmetric Bochner curvature tensor of p^{th} order.

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