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Recurrence relation for product moments of modified weibull distribution under progressive censoring

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Abstract
Modified Weibull distribution introduced by A.M. Sarhan in 2009. Exponential, Weibull and Linear Exponential distribution is particular case of Modified Weibull Distribution. In this paper, we discuss some properties of moments.

Keywords- Modified Waybill distribution, Recurrence Relation and Product Moments

1-Introduction
In many life-testing and reliability experiments, the experimenter might not get the failure times for all experimental units. For example, the study may have to be terminated due to lack of funds, or individuals leaves a clinical trial due to some other reasons, or In an industrial experiment, units may break accidentally and we are don't want to expose more unit for long time due to high cost or other reasons. Some time we remove the units as in pre planned way. Our main objective is to save time and cost of the experiment. Data obtained from such experiments are called censored data. In this paper we deal with general scheme of progressively Type-II right censoring scheme. In this scheme n units are placed on an experiment and observe m (<n) failure. At the time of the first failure, R_1 of the remaining n-1 surviving units are randomly withdrawn from the experiment, R_2 of the, $n-2-R_1$ surviving units are randomly withdrawn from the experiment, and so on. Finally at the last time m-th failure all the remaining, $n-m-R_1 \ldots . -R_m$ surviving units are withdrawn. If, $R_1=0, R_2=0, \ldots \mathrm{R_m}$, we get the conventional type-II right censoring scheme. Balakrishnan and Aggarwala (2000) provide a good look on theory, methods and applications of progressive censoring.

In analyzing lifetime data one often uses the exponential, Rayleigh, linear failure rate, Weibull or Generalized Exponential Distribution. These distributions have several desirable properties and nice physical interpretations which enable them to be used frequently, for more details we may refer to Bain (1974), Barlow and Proschan (1981), Lawless (2003), Lin *et al.,* 2006, Miller (1981), Gupta and Kundu (2002) Gupta and Kundu (2001), Shahan and Zaindin (2008). If the failure times are based on an absolutely continuous distribution function F with probability density function f, the joint probability density function of the progressively censored failure times $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ is given by

$$f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \dots, X_{m:m:n}^{R_m}}(x_1, x_2, \dots, x_m) = A_{n:R_1,R_2,\dots,R_{m-1}} \prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i},$$

$$-\infty < x_1 < x_2 < \dots < x_m < \infty \qquad 1.1$$

where $f(.)$ and $F(.)$ are probability density function and cumulative function of the random sample, and

$$A_{n:R_1,R_2,\dots,R_{m-1}} = n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-R_2\dots-R_{m-1}-m+1)$$

$$A_{n:R_1,R_2,\dots,R_{m-1}} = A_{n:\bar{R}_{m-1}}.$$

We discuss some properties in second section. In third section, we find out the recurrence relation of simple and product moments. In fourth section we compare the results in different particular conditions.

2- The Modified Weibull Distribution

The cumulative distribution function of Modified Weibull Distribution is given by

$$F(x;\alpha,\beta,\gamma) = 1 - e^{-(\alpha x + \beta x^{\gamma})} \qquad \alpha > 0, \beta > 0, \gamma > 0 \qquad (2.1)$$

and probability density function of MWD is given by

$$f(x;\alpha,\beta,\gamma) = (\alpha + \beta\lambda x^{\gamma-1}) e^{-(\alpha x + \beta x^{\gamma})}, \alpha > 0, \beta > 0, \gamma > 0 \qquad (2.2)$$

$$f(x;\alpha,\beta,\gamma) = (\alpha + \beta\lambda x^{\gamma-1})(1 - F(x;\alpha,\beta,\gamma)) \qquad (2.3)$$

Moment μ_k of MWD is defined as,

$$\begin{aligned} \mu_k &= E(X^k) \\ &= \int_0^{\infty} x^k f(x;\alpha,\beta,\gamma) dx \\ &= \int_0^{\infty} x^k (\alpha + \beta\gamma x^{\gamma-1}) e^{-(\alpha x + \beta x^{\gamma})} dx \end{aligned}$$

Case I. If $\alpha > 0, \beta > 0$, μ_k is given by

$$\mu_k = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i}{i!} \left[\frac{\Gamma(k + i\gamma + 1)}{\alpha^{k+i\gamma}} + \frac{\beta\lambda \Gamma(k + (i+1)\gamma)}{\alpha^{k+(i+1)\gamma}} \right]$$

Case II. For $\alpha = 0, \beta > 0$, we have

$$\mu_k = \frac{\Gamma\left(\frac{k}{\gamma} + 1\right)}{\beta^{k/\gamma}}$$

Case III. If $\beta = 0$ and $\alpha > 0$, then

$$\mu_k = \frac{\overline{|k+1}}{\alpha^k}$$

Parameter estimation under type II progressive censoring

Joint probability distribution under progressively censored sample given by

$$\begin{aligned}
& f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}}(x_1, x_2, \ldots, x_m) \\
&= A_{n:\bar{R}_{m-1}} \prod_{i=1}^{m} (\alpha + \beta\gamma x_i^{\gamma-1}) e^{-(\alpha x_i + \beta x_i^{\gamma})} \prod_{i=1}^{m} \left[e^{-(\alpha x_i + \beta x_i^{\gamma})} \right]^{R_i} \\
&= A_{n:\bar{R}_{m-1}} \prod_{i=1}^{m} (\alpha + \beta\gamma x_i^{\gamma-1}) e^{-\sum_{i=1}^{m} (R_i+1)(\alpha x_i + \beta x_i^{\gamma})}
\end{aligned} \tag{2.4}$$

Log likelihood function of equation (2.4) given by

$$\ln(f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}}(x_1, x_2, \ldots, x_m))$$

$$= \ln A_{n:\bar{R}_{m-1}} + \sum_{i=1}^{m} \ln((\alpha + \beta\gamma x_i^{\gamma-1})) - \sum_{i=1}^{m} (R_i + 1)(\alpha x_i + \beta x_i^{\gamma})$$

Differentiating with respect to α, β, γ we get,

$$\frac{\partial \ln(f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}}(x_1, x_2, \ldots, x_m))}{\partial \alpha} = \sum_{i=1}^{m} \frac{1}{(\alpha + \beta\gamma x_i^{\gamma-1})} - \sum_{i=1}^{m} (R_i + 1) x_i$$

$$\frac{\partial \ln(f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}}(x_1, x_2, \ldots, x_m))}{\partial \beta} = \sum_{i=1}^{m} \frac{\gamma x_i^{\gamma-1}}{(\alpha + \beta\gamma x_i^{\gamma-1})} - \sum_{i=1}^{m} (R_i + 1) x_i^{\gamma}$$

$$\frac{\partial \ln(f_{X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}}(x_1, x_2, \ldots, x_m))}{\partial \gamma} = \sum_{i=1}^{m} \frac{\beta(x_i^{\gamma-1} + \gamma x_i^{\gamma-1} \ln x_i)}{(\alpha + \beta\gamma x_i^{\gamma-1})} - \sum_{i=1}^{m} \beta(R_i + 1) x_i^{\gamma} \ln x_i$$

Solving the three equations, we can find estimate $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$.

3-Recurrence relations of moments

Single Moments- Let $X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}$ be a progressively type two right censored order statistics with censoring scheme $(R_1, R_2, \ldots, R_m)$ from the modified Weibull distribution, the single moments of the progressively type II censoring can be written as,

$$\mu_{i:m:n}^{(R_1, R_2, \ldots, R_m)^{(k)}} = E[X_{i:m:n}^{(R_1, R_2, \ldots, R_m)^{(k)}}]$$

$$\mu_{i:m:n}^{(R_1, R_2, \ldots, R_m)^{(k)}} = A_{n:\bar{R}_{m-1}} \iint\limits_{0 < x_1 < \ldots < x_m < \infty} \ldots \int f(x_1)[1 - F(x_1)]^{R_1} \ldots f(x_m)[1 - F(x_m)]^{R_m} \, dx_1 \ldots dx_m$$

Case I- For $2 \le m \le n, k \ge 0$, relation between moments are given by,

$$m_{1:m:n}^{(R_1,R_2,...,R_m)^{(k)}} = \frac{a(n-R_1+1)}{k+1} m_{1:m-1:n}^{(R_1+R_2+1,...,R_m)^{(k+1)}} + \frac{bg(n-R_1+1)}{k+g} m_{1:m-1:n}^{(R_1+R_2+1,...,R_m)^{(k+g)}} + \frac{a(R_1+1)}{k+1} m_{1:m:n}^{(R_1,R_2,...,R_m)^{(k+1)}} + \frac{bg(R_1+1)}{k+g} m_{1:m:n}^{(R_1,R_2,...,R_m)^{(k+g)}}$$

Case II- For $2 \le i \le m-1, m \le n$ and $k \ge 0$, relation between moments are given by,

$$\begin{aligned} m_{i:m:n}^{(R_1,R_2,...,R_m)^{(k+g)}} = \frac{k+g}{bg(R_i+1)} \{ m_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}} - \frac{a}{k+1} [(n-R_1-R_2...-R_i-i) m_{i:m-1:n}^{(R_1,...,R_i+R_{i+1}+1,...,R_m)^{(k+1)}} \\ -(n-R_1-R_2...-R_{i-1}-i+1) m_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,...,R_m)^{(k+1)}} + (R_i+1) m_{i:m:n}^{(R_1,R_2,...,R_m)^{(k+1)}}]\} \\ - \frac{1}{(R_i+1)} \{ [(n-R_1-R_2...-R_i-i) m_{i:m-1:n}^{(R_1,...,R_i+R_{i+1}+1,...,R_m)^{(k+g)}} \\ -(n-R_1-R_2...-R_{i-1}-i+1) m_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,...,R_m)^{(k+g)}}]\} \end{aligned}$$

Case III – For $2 \le m \le n$ and $k \ge 0$, relation between moments are given by,

$$\begin{aligned} \mu_{m:m:n}^{(R_1,R_2,...,R_m)^{(k+\gamma)}} = \frac{k+\gamma}{\beta\gamma(R_i+1)} \{ \mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}} - \frac{\alpha}{k+1} [-(n-R_1-R_2...-R_{m-1}-m+1) \mu_{m-1:m-1:n}^{(R_1,...,R_{m-2},R_{m-1}+1)^{(k+1)}} \\ + \mu_{m-1:m-1:n}^{(R_1,...,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} + (R_m+1) \mu_{m:m:n}^{(R_1,R_2,...,R_m)^{(k+1)}}]\} \\ - \frac{1}{(R_m+1)} [(n-R_1-R_2...-R_{m-1}-m+1) \mu_{m:m-1:n}^{(R_1,...,R_{m-1}+R_m+1)^{(k+\gamma)}} \end{aligned}$$

Proof:- Case I- k^{th} moment in progressive censoring is given by

$$\mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}} = A_{n:\bar{R}_{m-1}} \iint ... \int I(x_2) f(x_2)[1-F(x_2)]^{R_2} ... f(x_m)[1-F(x_m)]^{R_m} dx_2 ... dx_m \tag{3.1}$$

where $I(x_2) = \int_0^{x_2} x_1 [1-F(x_1)]^{R_1} f(x_1) dx_1$

Use the result $f(x;\alpha,\beta,\gamma) = (\alpha + \beta\gamma x^{\gamma-1})(1-F(x;\alpha,\beta,\gamma))$ in above equation, we get

$$\Rightarrow I(x_2) = \int_0^{x_2} x_1^k (\alpha + \beta\gamma x^{\gamma-1})[1-F(x_1)]^{R_1+1} dx_1$$

$$= \alpha \int_0^{x_2} x_1^k [1-F(x_1)]^{R_1+1} dx_1 + \beta\gamma \int_0^{x_2} x_1^k x^{\gamma-1} [1-F(x_1)]^{R_1+1} dx_1$$

Using integration by parts, we get.

$$I(x_2) = \alpha[\frac{x_2^k}{k+1}[1-F(x_2)]^{R_1+1}] + \alpha\int_0^{x_2}\frac{x_1^{k+1}}{k+1}(R_1+1)[1-F(x_1)]^{R_1} f(x_1)dx_1$$
$$+\beta\gamma[\frac{x_2^{k+\gamma}}{k+\gamma}[1-F(x_2)]^{R_1+1}] + \beta\gamma\frac{1}{k+\gamma}\int_0^{x_2} x_1^{k+\gamma}(R_1+1)[1-F(x_1)]^{R_1} f(x_1)dx_1$$

Put the value in eq. (3.1) we get,

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = \frac{\alpha(n-R_1+1)}{k+1}\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(k+1)}} + \frac{\beta\gamma(n-R_1+1)}{k+\gamma}\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(k+\gamma)}}$$
$$+\frac{\alpha(R_1+1)}{k+1}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+1)}} + \frac{\beta\gamma(R_1+1)}{k+\gamma}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\gamma)}}$$

$$\Rightarrow \mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\gamma)}} = \frac{k+\gamma}{\beta\gamma(R_1+1)}[\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} - \frac{\alpha}{k+1}\{(n-R_1+1)\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(k+1)}}$$
$$+(R_1+1)\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+1)}}\}] - \frac{(n-R_1+1)}{(R_1+1)}\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(k+\gamma)}}$$

Case II-

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = A_{n:\bar{R}_{m-1}}\iint\dots\int_{0<x_1<\dots<x_m<\infty} f(x_1)[1-F(x_1)]^{R_1}\dots I(x_i)f(x_m)[1-F(x_m)]^{R_m}dx_1\dots dx_m \tag{3.2}$$

where $I(x_i) = \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i}dx_i$

$$I(x_i) = \int_{x_{i-1}}^{x_{i+1}} x_i^k(\alpha+\beta\gamma x_i^{\gamma-1})[1-F(x_i)]^{R_i+1}dx_i = \frac{\alpha}{k+1}[x_{i+1}^{k+1}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+1}[1-F(x_{i-1})]^{R_i+1}]$$
$$+\frac{\alpha}{k+\gamma}[x_{i+1}^{k+\gamma}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\gamma}[1-F(x_{i-1})]^{R_i+1}]$$
$$+\frac{\alpha(R_i+1)}{1+k}\int_{x_{i-1}}^{x_{i+1}} x_i^{k+1}[1-F(x_i)]^{R_i}f(x_i)dx_i$$
$$+\frac{\beta\gamma(R_i+1)}{\gamma+k}\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\gamma}[1-F(x_i)]^{R_i}f(x_i)dx_i$$

Use above result in eq. (3.2), we get,

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}$$
$$= \frac{A_{n:\bar{R}_{m-1}}\alpha}{k+1}\iint\dots\int_{0<x_1<\dots<x_m<\infty} f(x_1)[1-F(x_1)]^{R_1}\dots[x_{i+1}^{k+1}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+1}[1-F(x_{i-1})]^{R_i+1}]\dots f(x_m)[1-F(x_m)]^{R_m}dx_1\dots dx_{i-1}dx_{i+1}\dots dx_m$$

$$+\frac{A_{n:\bar{R}_{m-1}}\beta\gamma}{k+\gamma}\iint\dots\int_{0<x_1<\dots<x_m<\infty} f(x_1)[1-F(x_1)]^{R_1}\dots[x_{i+1}^{k+\gamma}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\gamma}[1-F(x_{i-1})]^{R_i+1}]\dots f(x_m)[1-F(x_m)]^{R_m}dx_1\dots dx_{i-1}dx_{i+1}\dots dx_m$$

$$+\frac{A_{n:\bar{R}_{m-1}}\alpha(R_i+1)}{k+1}\iint\dots\int_{0<x_1<\dots<x_m<\infty} x_i^{k+1}f(x_1)[1-F(x_1)]^{R_1}\dots f(x_m)[1-F(x_m)]^{R_m}dx_1\dots dx_m$$

$$+\frac{A_{n:\bar{R}_{m-1}}\beta\gamma(R_i+1)}{k+\gamma}\iint\dots\int_{0<x_1<\dots<x_m<\infty} x_i^{k+\gamma}f(x_1)[1-F(x_1)]^{R_1}\dots f(x_m)[1-F(x_m)]^{R_m}dx_1\dots dx_m$$

$$\Rightarrow \quad \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k)}$$

$$=\frac{\alpha}{k+1}[(n-R_1-R_2\dots-R_i-i)\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+1)}$$

$$-(n-R_1-R_2\dots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,\dots,R_m)(k+1)}]$$

$$+\frac{\beta\gamma}{k+\gamma}[(n-R_1-R_2\dots-R_i-i)\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+\gamma)}$$

$$-(n-R_1-R_2\dots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,\dots,R_m)(k+\gamma)}]$$

$$+\frac{\alpha(R_i+1)}{k+1}\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+1)}$$

$$+\frac{\beta\gamma(R_i+1)}{k+\gamma}\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+\gamma)}$$

$$\Rightarrow \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+\gamma)}$$

$$=\frac{k+\gamma}{\beta\gamma(R_i+1)}\{\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k)}-\frac{\alpha}{k+1}[(n-R_1-R_2\dots-R_i-i)\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+1)}$$

$$-(n-R_1-R_2\dots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,\dots,R_m)(k+1)}+(R_i+1)\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+1)}]\}$$

$$-\frac{1}{(R_i+1)}\{[(n-R_1-R_2\dots-R_i-i)\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+\gamma)}$$

$$-(n-R_1-R_2\dots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,\dots,R_m)(k+\gamma)}]\}$$

Similarly we can prove the third case.

4-Product moment

For any continuous distribution, we can write the (r,s)th product moment of the progressively type II right censored order statistics form (1.1) as follows

Case-1 For $1\le r<s\le m-1, s-r\ge 2, m\le n$ and $i, j\ge 0$, relation between product moments are given by

$$\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)(i,j+\gamma)}=\frac{j+\gamma}{\beta\lambda(R_s+1)}\{\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)(i,j)}-\frac{\alpha}{j+1}[(n-R_1-R_2\dots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\dots,R_{s-1},R_s+R_{s+1}+1,\dots,R_m)(i,j+1)}$$

$$-(n-R_1-R_2\dots-R_s-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\dots,R_{s-2},R_{s-1}+R_s+1,\dots,R_m)(i,j+1)}+(R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)(i,j+1)}]\}$$

$$-\frac{1}{(R_s+1)}[(n-R_1-R_2\dots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\dots,R_s+R_{s+1}+1,\dots,R_m)(i,j+\gamma)}$$

$$-(n-R_1-R_2\dots-R_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\dots,R_{s-2},R_{s-1}+R_s+1,\dots,R_m)(i,j+\gamma)}]$$

Case 2- For $1\le r<s\le m-1, s-r\ge 2, m\le n$ and $i, j\ge 0$, relation between product moments are given by

$$\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)^{(i+\gamma,j)}}$$
$$=\frac{i+\gamma}{\beta\lambda(R_r+1)}\{\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j)}}-\frac{\alpha}{i+1}[(n-R_1-R_2\dots-R_r-r)\mu_{r,s:m-1:n}^{(R_1,\dots,R_{r-1},R_r+R_{r+1}+1,\dots,R_m)^{(i+1,j)}}$$
$$-(n-R_1-R_2\dots-R_r-r+1)\mu_{r-1,s:m-1:n}^{(R_1,\dots,R_{r-2},R_{r-1}+R_r+1,\dots,R_m)^{(i+1,j)}}+(R_r+1)\mu_{r,s:m:n}^{(R_1,R_2,\dots,R_m)^{(i+1,j)}}]\}$$
$$-\frac{1}{(R_r+1)}[(n-R_1-R_2\dots-R_r-r)\mu_{r,s:m-1:n}^{(R_1,\dots,R_r+R_{r+1}+1,\dots,R_m)^{(i+\gamma,j)}}$$
$$-(n-R_1-R_2\dots-R_{r-1}-r+1)\mu_{r-1,s:m-1:n}^{(R_1,\dots,R_{r-2},R_{r-1}+R_r+1,\dots,R_m)^{(i+\gamma,j)}}]$$

Case 3- For $2\le r\le m-1, m\le n$ and $i, j\ge 0$

$$\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i+\gamma,j)}}$$
$$=\frac{i+\gamma}{\beta\gamma(R_r+1)}\{\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j)}}-\frac{\alpha}{i+1}[(n-R_1-R_2\dots-R_r-r)\mu_{r:m-1:n}^{(R_1,\dots,R_{r-1},R_r+R_{r+1}+1,\dots,R_m)^{(i+j+1)}}$$
$$-(n-R_1-R_2\dots-R_r-r+1)\mu_{r-1,r:m-1:n}^{(R_1,\dots,R_{r-2},R_{r-1}+R_r+1,\dots,R_m)^{(i+1,j)}}+(R_r+1)\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i+1,j)}}]\}$$
$$-\frac{1}{(R_r+1)}[(n-R_1-R_2\dots-R_r-r)\mu_{r:m-1:n}^{(R_1,\dots,R_r+R_{r+1}+1,\dots,R_m)^{(i+\gamma+j)}}$$
$$-(n-R_1-R_2\dots-R_{r-1}-r+1)\mu_{r-1,r:m-1:n}^{(R_1,\dots,R_{r-2},R_{r-1}+R_r+1,\dots,R_m)^{(i+\gamma,j)}}]$$

Case 4- For $1\le r\le m-2, m\le n$ and $i, j\ge 0$

$$\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j+\gamma)}}=\frac{j+\gamma}{\beta\gamma(R_r+1)}\{\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j)}}-\frac{\alpha}{i+1}[(n-R_1-R_2\dots-R_{r+1}-r-1)\mu_{r,r+1:m-1:n}^{(R_1,\dots,R_{r+1}+R_{r+2}+1,\dots,R_m)^{(i,j+1)}}$$
$$-(n-R_1-R_2\dots-R_r-r)\mu_{r:m-1:n}^{(R_1,\dots,R_{r-1},R_r+R_{r+1}+1,\dots,R_m)^{(i,j+1)}}+(R_{r+1}+1)\mu_{r,r+1:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j+1)}}]\}$$
$$+\frac{1}{(R_r+1)}[-(n-R_1-R_2\dots-R_{r+1}-r-1)\mu_{r,r+1:m-1:n}^{(R_1,\dots,R_r+R_{r+1}+1,\dots,R_m)^{(i,j+\gamma)}}$$
$$+(n-R_1-R_2\dots-R_r-r)\mu_{r:m-1:n}^{(R_1,\dots,R_{r-2},R_r+R_{r+1}+1,\dots,R_m)^{(i+\gamma+j)}}]$$

Case 5- For $2\le m\le n$, and $i, j\ge 0$

$$\mu_{1,2:m:n}^{(R_1,R_2,\dots,R_m)^{(i+\gamma,j)}}$$
$$=\frac{i+\gamma}{\beta\gamma(R_1+1)}\{\mu_{1,2:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j)}}-\frac{\alpha}{i+1}[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(i+j+1)}}$$
$$+(R_1+1)\mu_{1,2:m:n}^{(R_1,R_2,\dots,R_m)^{(i+1,j)}}]\}$$
$$+\frac{1}{(R_1+1)}[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,\dots,R_m)^{(i+j+\gamma)}}$$

Proof Case I-

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}$$

$$= A_{n:\bar{R}_{m-1}} \iint \ldots \int_{0<x_1<\ldots x_{s-1}<x_{s+1}\ldots<x_m<\infty} x_r^{\,i} \{ \int_{x_{s-1}}^{x_{s+1}} x_s^j f(x_s)[1-F(x_s)]^{R_s} dx_s \} f(x_1)[1-F(x_1)]^{R_1} \ldots \tag{3.3}$$

$$f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}} f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}} \ldots f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_{s-1} dx_{s+1} \ldots dx_m$$

$$= \alpha \left[\frac{x_s^{j+1}}{j+1} [1-[1-F(x_s)]^{R_s+1}]_{x_{s-1}}^{x_{s+1}} + \int_{x_{s-1}}^{x_{s+1}} \frac{R_s+1}{j+1} x_s^{j+1} f(x_s)[1-F(x_s)]^{R_s} dx_s \right]$$

$$+ \beta\gamma \left[\frac{x_s^{j+\gamma}}{j+\gamma} [1-[1-F(x_s)]^{R_s+1}]_{x_{s-1}}^{x_{s+1}} + \int_{x_{s-1}}^{x_{s+1}} \frac{R_s+1}{j+\gamma} x_s^{j+\lambda} f(x_s)[1-F(x_s)]^{R_s} dx_s \right]$$

Put these values in eq. 3.3, we get,

$$\Rightarrow \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}$$

$$= \frac{\alpha}{j+1} [(n-R_1-R_2\ldots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+1,\ldots,R_m)^{(i,j+1)}}$$

$$-(n-R_1-R_2\ldots-R_s-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,\ldots,R_m)^{(i,j+1)}}]$$

$$+ \frac{\alpha(R_s+1)}{j+1} \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}}$$

$$+ = \frac{\beta\gamma}{j+\gamma} [(n-R_1-R_2\ldots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+1,\ldots,R_m)^{(i,j+\gamma)}}$$

$$-(n-R_1-R_2\ldots-R_s-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,\ldots,R_m)^{(i,j+\gamma)}}]$$

$$+ \frac{\beta\gamma(R_s+1)}{j+\lambda} \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\gamma)}}$$

$$\Rightarrow \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\gamma)}}$$

$$= \frac{j+\gamma}{\beta\lambda(R_s+1)} \{ \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} - \frac{\alpha}{j+1} [(n-R_1-R_2\ldots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+1,\ldots,R_m)^{(i,j+1)}}$$

$$-(n-R_1-R_2\ldots-R_s-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,\ldots,R_m)^{(i,j+1)}} + (R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}}]\}$$

$$- \frac{1}{(R_s+1)} [(n-R_1-R_2\ldots-R_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_s+R_{s+1}+1,\ldots,R_m)^{(i,j+\gamma)}}$$

$$-(n-R_1-R_2\ldots-R_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,\ldots,R_m)^{(i,j+\gamma)}}]$$

Similarly we can prove other result.

Conclusion

Liner failure rate distribution, exponential distribution, Rayleigh distribution and Weibull distribution are the particular case of Modified Weibull distribution. These distributions are very useful in survival analysis. Moments and product moments are required in different situations. In many real life situations progressive censored data have obtained. Recurrence relation of moment and product moment are useful. Hence we can obtain the recurrence relation for these distributions as particular cases.

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