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Some results on Semi-Generalized Recurrent α-cosymplectic manifolds

Archana Singh and J.P. Singh[1]
Department of Mathematics and Computer Science,
Mizoram University, Aizawl, Mizoram-796004, India
Email-Archanamath17@gmail.com, jpsmaths@gmail.com

Abstract
The object of the present paper is to study some geometrical properties of a semi-generalized recurrent α-Cosymplectic manifolds.

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1. Introduction
The idea of recurrentmanifold was introduced by Walker (1950). On the other hand De and Guha (1991) introduced generalized recurrent manifold with the non-zero 1-form A and another non-zero associated 1-form B. Such a manifold has been denoted by GK_n. If the associated 1-form B becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by Ruse (1951) which is denoted by K_n.A Riemannian manifold (M^n, g) is called a semi-generalized recurrent manifold (Prasad, 2000) if its curvature tensor Rsatisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)\, g(Z, W)\, Y, \tag{1}$$

Where A and B are two 1-forms, Bis non-zero, P and Q are two vector fields such that

$$g(X, P) = A(X), g(X, Q) = B(X), \tag{2}$$

for any vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g.

The notion of almost α-cosymplectic manifolds is introduced by Kim and Pak (2005), where α is a scalar but it need not be constant. In Kumar *et al.*, 2000, the author studied some curvature conditions on α-cosymplectic manifolds. Recently Kumar *et.al.* (2010) studied semi generalized recurrent LP –Sasakian manifolds and obtained interesting results. In this

[1] Corresponding author

paper we study some properties of semi-generalized recurrent α-cosymplectic manifolds. The paper is organized as follows: In section 2,we give a brief account of an α–cosymplectic manifolds and Einstein manifold.

In section 3, we study a semi-generalized recurrent α-cosymplectic manifold. Section 4, deals with semi-generalized φ-recurrent α-cosymplectic manifolds. In the last section, it is proved that a semi-generalized M-projective φ-recurrent α-cosymplectic manifold is an Einstein manifold.

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold equipped with a triple (φ, ξ, η) where φ is a (1-1) tensor field, ξ is a vector field, η is a 1-form on M^n such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi, \tag{3}$$

which implies

$$\varphi\xi = 0, \eta \circ \varphi = 0, \qquad rank(\varphi) = n - 1. \tag{4}$$

If M^n admits a Riemannian metric g, such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5}$$

Then M^n is said to admit almost contact structure (φ, ξ, η, g).On such a manifold, the fundamental Φ on M^n is defined as

$$\Phi(X, Y) = g(\varphi X, Y), \qquad X, Y \in \Gamma(TM).$$

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be almost cosymplectic if $d\eta=0$ and $d\Phi=0$, where d is the exterior differential operator. The products of almost Kaehlerian manifolds and the real line $\mathbb{R}$ or the S^1 circle are the simplest examples of almost cosymplectic manifold. An almost contact manifold $(M, \varphi,)$ is said to be normal if the Nijenhuis torsion

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] + 2d\eta(X, Y)\xi,$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifold is called a cosymplectic manifold. As it is known that an almost contact metric structure is cosymplectic if and only if both $\nabla\eta$ and $\nabla\Phi$ vanish. An almost contact metric manifold M^n is said to be almost α-Kenmotsu if $d\eta=0$ and $d\Phi=2\alpha\, \eta \wedge \Phi$, α being a non-zero real constant. It is worthwhile to note that almost α-kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures.In order to treat these two classes in a unified way, we have a new notion of an almostα-cosymplectic manifold for any real number α that is defined as in the formula

$$d\eta = 0, \quad \text{and} \qquad d\Phi = 2\alpha\, \eta \wedge \Phi.$$

A normal almost α-cosymplectic manifold is called an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-kenmotsu under the conditions $\alpha \neq 0$ for $\alpha \in \mathbb{R}$.

On such an α-cosymplectic manifold, we have

$$(\nabla_X \varphi)(Y) = \alpha[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{6}$$

for $\alpha \epsilon \mathbb{R}$ on M^n.

Let M be a n-dimensional α-cosymplectic manifold. From equation (6), it is easy to see that

$$\nabla_X \xi = -\alpha\varphi^2 X, \tag{7}$$

where∇ denotes the Riemannian connection.

In an α-cosymplectic manifold M^n, the following relations hold:

$$R(\xi, X)Y = \alpha^2[\eta(Y)X - g(X, Y)], \tag{8}$$

$$R(X, Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X], \tag{9}$$

$$S(\xi, X) = -\alpha^2(n-1)\eta(X) \tag{10}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + \alpha^2(n-1)\eta(X)\eta(Y) \tag{11}$$

$$R(\xi, X)\xi = \alpha^2[X - \eta(X)\xi], \tag{12}$$

$$g(R(\xi, X)Y, \xi) = \alpha^2[\eta(X)\eta(Y) - g(X, Y)], \tag{13}$$

$$Q\xi = -\alpha^2(n-1)\xi, \tag{14}$$

$$S(\xi, \xi) = -\alpha^2(n-1) \tag{15}$$

$$(\nabla_X \eta)(Y) = -\alpha\, g(Y, \varphi^2 X), \tag{16}$$

for any vector fields X, Y and $\alpha \epsilon \mathbb{R}$. Kenmotsu manifold have been studied by Jun *et al.*, 2006 and he obtained the above results for $\alpha = 1$.

An α-cosymplectic manifold M^n is said to Einstein if its Ricci tensor S is of the form

$$S(X, Y) = \lambda\, g(X, Y) \tag{17}$$

where λ is constant.

3. Semi-Generalized Recurrent α-cosymplectic manifolds

Theorem 3.1. In a semi-generalized Ricci-recurrent α-cosymplectic manifold the 1-form A and B are related as

$$\alpha^2(n-1)A(X) - nB(X) = 0.$$

Proof: A Riemannian manifold (M^n, g) is semi-generalized Ricci-recurrent manifold (Kumar *et al.*, 2010) if

$$(\nabla_X S)(Y,Z) = A(X)\,S(Y,Z) + n\,B(X)g(Y,Z). \tag{18}$$

Taking Z = ξ in (18), we have

$$(\nabla_X S)(Y,\xi) = A(X)S(Y,\xi) + n\,B(X)g(Y,\xi). \tag{19}$$

The left hand side of (19), clearly can be written in the form

$$(\nabla_X S)(Y, \xi) = XS(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi),$$

in view of equation (7) and (10) above gives

$$\alpha^3\,(n-1)g(Y,\varphi^2 X) + \alpha\,S(Y,\varphi^2 X).$$

While the right hand side of (19) equals

$$A(X)S(Y,\xi) + nB(X)g(Y,\xi) = -(n-1)\alpha^2 A(X)\eta(Y) + nB(X)\eta(Y).$$

Hence

$$\alpha^3\,(n-1)g(Y,\varphi^2 X) + \alpha\,S(Y,\varphi^2 X) = -(n-1)\alpha^2 A(X)\eta(Y) + n\,B(X)\eta \tag{20}$$

Putting Y = ξ in (20) and then using (4), (5) and (11), we get

$$-\alpha^2(n-1)A(X) + nB(X) = 0. \tag{21}$$

which proves the theorem.

Theorem 3.2. If a semi-generalized Ricci-recurrent α-cosymplectic manifold is an Einstein manifold then 1-forms A and B are related as λ A(Y) + n B(Y).

Proof. For an Einstein manifold, we have S(Y, Z) = $\lambda\, g$(Y, Z) and (∇_US) = 0, where λ is constant.

Hence from (18), we have

$$[\lambda A(X) + nB\big((X)\big]g(Y,Z)] + [\lambda A(Y) + nB(Y)]g(Z,X)]$$
$$+[\lambda A(Z) + n\,B(Z)]\,g(X,Y) = 0. \tag{22}$$

Replacing Z by ξ in (22) and using (2), (4) and (5), we have

$$\lambda\eta(P) + n\,\eta(Q) = 0. \tag{23}$$

Using (4) in the above equation, it follows that

$$\lambda\,A(Y) + n\,B(Y). \tag{24}$$

4. Semi-Generalized φ-Recurrent α-cosymplectic manifolds

Definition4.1.A α-cosymplectic manifolds (M^n, g) is called semi-generalized φ-recurrent if its curvature tensor R satisfies the condition

$$\varphi^2((\nabla_W R(X,Y)Z) = A(W)R(X,Y)Z + B(W)g(Y,Z)X, \qquad (25)$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$\mathrm{A(W)}=g(W,P) \quad and \qquad B(W) = g(W,Q), \mathrm{(W, P)}, \qquad (26)$$

and P and Q are vector fields associated with 1-forms A and B respectively.

Theorem 4.1. A semi-generalized φ-recurrent α-cosymplectic manifold (M^n, g) is an Einstein manifold.

Proof. Let us consider a semi –generalized φ-recurrent α-cosymplectic manifold. Then by virtue of (3) and (25) we have

$$-(\nabla_W R)(X, Y) Z + \eta((\nabla_W R)(X, Y) Z)\xi = A(W) R(X, Y) Z + B(W) g(Y, Z)X, \qquad (27)$$

from which it follows that

$$-g((\nabla_W R)(X, Y) Z, U) + \eta((\nabla_W R)(X, Y) Z)(U)$$

$$= A(W)\, g(R(X, Y)Z, U) + B(W)\, g(Y, Z)\, g(X, U). \qquad (28)$$

Let $\{e_i\}$, i=1, 2, 3…..n be an orthonormal basis of the Tangent space at any point of the manifold. Then putting X = U = e_i in (28) and taking summation over i, 1≤ i ≤ n, we get

$$(\nabla_W S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_W R)(e_i, Y, Z)\eta(e_i)$$
$$= A(W)S(Y,Z) + n\,B(W)g(Y,Z). \qquad (29)$$

The second term of left hand side of (27) by putting Z = ξ takes the form $g((\nabla_W R)(e_i, Y)\xi, \xi)$, which is zero in this case. So, by replacing Z by ξ in (27) and using (10) we get

$$(\nabla_W S)(Y, \xi) = -\alpha^2(n-1) A(W)\eta(Y) - n B(W)\eta(Y). \qquad (30)$$

Now, we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (7) and (10) in the above relation, it follows

$$(\nabla_{WS}(Y,\xi) = (n-1)\alpha^3 g(\varphi^2 W, Y) + \alpha S(Y, \varphi^2 W), \qquad (31)$$

From (30) and (31) we obtain

$$(n-1)\alpha^3 g(\varphi^2 W, Y) + \alpha S(Y, \varphi^2 W) = -\alpha^2(n-1)A(W)\eta(Y) - n\,B(W)\eta(Y \qquad (32)$$

Replacing Y= ξ in (32) then using (3) we get

$$-\alpha^2(n-1)A(W) = n\,B(W). \qquad (33)$$

Using (33) in (32), we obtain

$$S(Y,\varphi^2 W) = -(n-1)\alpha^2\, g(Y,\varphi^2 W). \qquad (34)$$

In consequence of the equations (3) and (7) theabove equation reduces to

$$S(Y,W) = -(n-1)\alpha^2 g(Y,W),$$

which shows that the manifold is Einstein manifold.

Theorem 4.2. In a semi-generalized φ-recurrent α-cosymplectic manifold the characteristic vector field (P+Q) associated to the 1-form A + B are given by

$$A(W) = \left[\frac{2}{(n-2)\alpha^2}\eta(Q) - \frac{1}{(n-2)}\eta(P)\right]\eta(W).$$

Proof. From the relation (25), we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\,\xi - A(W)\,R(X,Y)Z - B(W)\,g(Y,Z)X. \qquad (35)$$

Then by use of second Bianchi's identity and (35), we get

$$A(W)\,\eta(R(X,Y)Z) + A(X)\,\eta(R(Y,W)Z) + A(Y)\,\eta(R(W,X)Z)$$

$$+B(W)\,g(Y,Z)\,\eta(X) + B(X)\,g(W,Z)\,\eta(Y) + B(Y)\,g(X,Z)\,\eta(W)=0. \qquad (36)$$

By virtue of (13) we obtain from (36)

$$[-\alpha^2 A(W) + B(W)]\,g(Y,Z)\,\eta(X) + [-\alpha^2 A(X) + B(X)]\,g(W,Z)\,\eta(Y)$$

$$+ [-\alpha^2 A(Y) + B(Y)]\,g(X,Z)\,\eta(W) + \alpha^2 A(W) g(X,Z)\,\eta(Y)$$

$$+\alpha^2 A(X)\,g(Y,Z)\,\eta(W) + \alpha^2 A(Y)\,g(W,Z)\,\eta(X) = 0. \qquad (37)$$

Putting Y = Z= e_i in (37), using (33) and taking summation over i, $1 \leq i \leq n$, we get

$$(n-2)\alpha^2 A(W)\eta(X) = [2B(X) - \alpha^2 A(X)]\eta(W), \qquad (38)$$

for all vector fields X, W. Replacing X by ξ in (38) it follows that

$$A(W) = \left[\frac{2}{(n-2)\alpha^2}\eta(Q) - \frac{1}{(n-2)}\eta(P)\right]\eta(W) \qquad (39)$$

for any vector field W, where A (ξ) = $g(\xi, P) = \eta(P)$and B (ξ) = $g(\xi, Q) = \eta(Q)$.

This completes the proof.

5. Semi-Generalized φ-Recurrent M-Projective α-cosymplectic manifolds

Definition 5.1. A α- cosymplectic manifold $(M^n, g)$is called semi-generalized M-Projective φ-recurrent if it's M-Projective curvature tensor (Pokhariyal and Mishra, 1971) given by

$$W^*(X,Y)Z = R(X, Y) Z - \frac{1}{2(n-1)} [S(Y, Z) X - S(X, Z) Y + g (Y, Z)QX - g(X, Z) QY], \quad (40)$$

satisfies the condition

$$\varphi^2((\nabla_V W^*) (X, Y) Z) = A (V) W^*(X, Y) Z + B (V) g(Y, Z) X (41)$$

where A and B are defined as (26).

Theorem 5.1. Let (M^n, g) be a semi-generalized M-projective φ- recurrent α-cosymplectic manifold then

$$[\frac{-n(n-1)\alpha^2 - r}{2(n-1)}] A (V) + n B (V) = 0.$$

Proof: Let us consider a semi-generalized φ-recurrent α-cosymplectic manifold. Then by virtue of (3) and (41) we have

$$-(\nabla_V W^*)(X, Y) Z + \eta ((\nabla_V W^*) (X, Y) Z) \xi = A (V) W^*(X, Y) Z + B (V) g(Y, Z) X. \quad (42)$$

From which it follows that

$$-g((\nabla_V W^*)(X, Y) Z, U) + \eta((\nabla_V W^*)(X, Y) Z) \eta (U)$$
$$= A (V) g(W^*(X, Y) Z, U) + B (V) g(Y, Z) g(X, U). \quad (43)$$

Let $\{e_i\}$, i=1, 2, 3…..n be an orthonormal basis of the Tangent space at any point of the manifold. Now putting X = U = e_i in (43) and taking summation over i, $1 \leq i \leq n$, we get

$$-\frac{n}{2(n-1)} (\nabla_V S)(X, U) + \frac{1}{2(n-1)} V(r) g(X, U)$$
$$+ \frac{n}{2(n-1)} (\nabla_V S) (X, \xi) \eta (U) - \frac{1}{2(n-1)} V(r) \eta(X) \eta(U)$$
$$= A (V) [\frac{n}{2(n-1)} S(X, U) - \frac{r}{2(n-1)} g(X, U)] + n B (V) g(X, U). \quad (44)$$

Replacing U by ξ in (44) and using (3) and (10) we get

$$[\frac{-n(n-1)\alpha^2 - r}{2(n-1)}] A (V) \eta(X) + n B (V) \eta(X) = 0. \quad (45)$$

Putting X = ξ in (45), we obtain

$$[\frac{-n(n-1)\alpha^2 - r}{2(n-1)}] A (V) + n B (V) = 0. \quad (46)$$

This proves the theorem.

Theorem5.2.A semi-generalized M-Projective φ-recurrent α-cosymplectic manifold is an Einstein manifold.

Proof. Putting X = U = e_i in (28) and taking summation over i, $1 \leq i \leq n$, we get

$$-\frac{n}{2(n-1)}(\nabla_V S)(Y, Z) = -\frac{1}{2(n-1)} V(r)\, g(Y, Z) - g((\nabla_V R)(e_i, Y)Z, \xi)\, g(e_i, \xi)$$

$$+\frac{1}{2(n-1)}[(\nabla_V S)(Y, Z)\, g(\xi, \xi) - (\nabla_V S)(\xi, Z)\, g(Y, \xi)$$

$$-(\nabla_V S)(\xi, \xi)\, g(Z, \xi)] + [\frac{n}{2(n-1)} S(Y, Z) - \frac{r}{2(n-1)} g(Y, Z)]A(V) + n\, B(V) g(Y, Z). \quad (47)$$

Replacing Z by ξ in (47) and using (3) and (10) we get

$$(\nabla_V S(Y, \xi) = \frac{1}{n} V(r)\eta(Y). \quad (48)$$

Now, we have

$(\nabla_V S)(Y, \xi) = \nabla_V S(Y, \xi) - S(\nabla_V Y, \xi) - S(Y, \nabla_V \xi)$.

Using (7) and (10) in above relation, it follows that

$$(\nabla_V S)(Y, \xi) = \alpha^3 (n-1)\, g(Y, \varphi^2 V) + \alpha\, S(Y, \varphi^2 V). \quad (49)$$

In view of (48) and (49) we have

$$\frac{1}{n} V(r)\eta(Y) = \alpha^3 (n-1)\, g(Y, \varphi^2 V) + \alpha\, S(Y, \varphi^2 V). \quad (50)$$

Replacing Y by φY in (50) then using (5), (9) and (11), we obtain

$$S(\varphi Y, V) = \alpha^2 (n-1)\, g(\varphi Y, V). \quad (51)$$

Again, Replacing V by φV in (51), we obtain

$$S(Y, V) = \alpha^2 (n-1)\, g(Y, V). \quad (52)$$

This prove the theorem

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