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On the existence of R-$\ominus$ symmetric finsler spaces

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Abstract

Cartan (1926, 1927) and Helgasan (1962) discussed the existence of symmetric spaces in a Riemannian space. Certain symmetric Finsler spaces and it's properties have been discussed by Tiwari and Srivastava (2001). The object of present paper is to extend the concept of symmetric space for $R \ominus -$ symmetric Finsler space admitting non-symmetric connections. Some properties of such a space have been investigated.

1. Introduction

Consider an n-dimensional Finsler space F_n (Rund,1959), having 2n-line elements $(x^i, \dot{x}^i)$ equipped with a non-symmetric connection $\Gamma^i_{jk} \neq \Gamma^i_{jk}$ based on non-symmetric metre tensor $g_{ij} \neq g_{ij}$. Here we assume that $\Gamma^i_{jk}(x, \dot{x})$ is homogeneous of degree zero in it's directional arguements $\dot{x}^i$'s.

Nitescu [3] defined non-symmetric connection Γ^i_{jk} as follows:

$$\Gamma^i_{jk} = M^i_{jk}(x,\dot{x}) + \frac{1}{2} N^i_{jk}(x,\dot{x}) \quad , \tag{1.1}$$

where M^i_{jk} and N^i_{jk} denote symmetric and skew symmetric parts of Γ^i_{jk} respectively.

Let us introduce another connection $\hat{\Gamma}^i_{jk}(x, \dot{x})$ defined as below:

$$\hat{\Gamma}^i_{jk}(x,\dot{x}) = M^i_{jk} - \frac{1}{2} N^i_{jk}, \quad [6] \tag{1.2}$$

From (1.1) and (1.2) it is easily seen that

$$\hat{\Gamma}^i_{jk}(x,\dot{x}) = \Gamma^i_{kj}(x,\dot{x}) \tag{1.3}$$

Here, we define covariant derivative of any contravariant vector X^i $(x, \dot{x})$ in two distinct ways as follows : (Catalina,1983-86), (Pandey and Gupta,1979)

$$X^i_{|j} = \partial_j X^i - (\dot{\partial}_m X^i)(\hat{\Gamma}^m_{kj}\dot{x}^k) + X^k \hat{\Gamma}^i_{kj}, \tag{1.4}$$

$$X^i_{\underset{|j}{+}} = \partial_j X^i - \left(\dot\partial_m X^i\right)\Gamma^m_{kj}\dot x^k + X^k\Gamma^i_{kj} \qquad (1.5)$$

where

$$\partial_j = \frac{\partial}{\partial x^j} \quad and \quad \dot\partial_j = \frac{\partial}{\partial \dot x^j}.$$

But, on account of (1.3), the equation (1.4) and (1.5) may be re-written as :

$$X^i_{\underset{|j}{-}} = \partial_j X^i - \left(\dot\partial_m X^i\right)\Gamma^m_{jk}\dot x^m + X^k\Gamma^i_{jk}$$

$$X^i_{\underset{|j}{+}} = \partial_j X^i - \left(\dot\partial_m X^i\right)\hat\Gamma^m_{jk}\dot x^k + X^k\hat\Gamma^i_{jk}.$$

The covariant differentiations defined in (1.4) and (1.5) may be called "$\ominus$ − covariant derivative" of tensor $X^i(x,\dot x)$ with respect to x^j and "$\oplus$ -covariant derivative of $X^i(x,\dot x)$ with respect to x^j respectively.

The commutation formulae involving $\ominus$ -covariant derivative and $\oplus$ -covariant derivative for a vector X^i are given by (Pandey and Gupta, 1979)

$$X^i_{\underset{|hk}{-}} - X^i_{\underset{|kh}{-}} = -\left(\dot\partial_m X^i\right)\hat R^m_{hk} + X^m\hat R^i_{mhk} + X^i_{\underset{|m}{-}} N^m_{kh}, \qquad (1.6)$$

$$X^i_{\underset{|hk}{+}} - X^i_{\underset{|kh}{+}} = -\left(\dot\partial_m X^i\right)R^m_{hk} + X^m R^i_{mhk} + X^i_{\underset{|m}{-}} N^m_{kh}, \qquad (1.7)$$

where $\hat R^h_{ijk}$ and R^h_{ijk} are curvature tensors defined as : (Pandey and Gupta, 1979)

$$\hat R^h_{ijk} = \partial_k\hat\Gamma^h_{ij} - \partial_j\hat\Gamma^h_{ik} + \left(\dot\partial_m\hat\Gamma^h_{ik}\right)\hat\Gamma^m_{sj}\dot x^s - \left(\dot\partial_m\hat\Gamma^h_{ij}\right)\hat\Gamma^m_{sk}\dot x^s + \hat\Gamma^p_{ij}\hat\Gamma^h_{pk} - \hat\Gamma^p_{ik}\hat\Gamma^h_{pj}. \qquad (1.8)$$

$$R^h_{ijk} = \partial_k\Gamma^h_{ij} - \partial_j\Gamma^h_{ik} + \left(\dot\partial_m\Gamma^h_{ik}\right)\Gamma^m_{sj}\dot x^s - \left(\dot\partial_m\Gamma^h_{ij}\right)\Gamma^m_{sk}\dot x^s + \Gamma^p_{ij}\Gamma^h_{pk} - \Gamma^p_{ik}\Gamma^h_{pj}. \qquad (1.9)$$

2. R-$\ominus$ Symmetric Finsler Space

Definition (2.1): A Finsler space in which the curvature tensor satisfies the relation

$$R^i_{hjk\underset{|s}{-}} = 0, \qquad (2.1)$$

is called R-$\ominus$ symmetric Finsler space.

Definition (2.2): An n-dimensional Finsler space F_n is said to be R-$\ominus$ recurrent Finsler space if it's curvature tensor R^i_{hjk} satisfies the relation

$$R^i_{hjk\underset{|s}{-}} = \beta_s R^i_{hjk}, \qquad (2.2)$$

where β_s(x) is a non-zero covariant recurrence vector field. We shall denote such a space by F_n^* throughout the paper.

We shall extensively use the following identities and notations in the sequal:

(a) $\hat R^i_{hjk} = -\hat R^i_{hkj}$ (b) $R^i_{hjk} = -R^i_{hkj}$ (2.3)

(a) $\hat{R}^i_{hji} \underset{=\!=}{def} \hat{R}_{hj}$ (b) $R^i_{hji} \underset{=\!=}{def} R_{hj}$ (2.4)

(a) $\hat{R}^i_{jk} \underset{=\!=}{def} \dot{x}^h \hat{R}^i_{hjk}$ (b) $R^i_{jk} \underset{=\!=}{def} \dot{x}^h R^i_{hjk}$ (2.5)

(a) $\hat{R}^i_{j} \underset{=\!=}{def} \dot{x}^h \hat{R}^i_{hj}$ (b) $R^i_{j} \underset{=\!=}{def} \dot{x}^h R^i_{hj}$ (2.6)

$$N^i_{jk} = -N^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj}. \quad (2.7)$$

$$\dot{x}^i_{\overset{+}{|}k} = \dot{x}^i_{\overset{-}{|}k} = 0. \quad (2.8)$$

From (1.1) and (1.2), we have

$$\hat{\Gamma}^i_{jk} - \Gamma^i_{jk} = -N^i_{jk}. \quad (2.9)$$

Now contracting (2.1), we get

$$R^i_{hji} = 0 \quad (2.10)$$

In view of (2.4), the above equation takes the form

$$R_{hj\overset{-}{|}s} = 0. \quad (2.11)$$

From above, we may establish the following theorem

Theorem (2.1): Every R-⊖ symmetric Finsler space is Ricci-symmetric Finsler space.

Multiplying (2.1) successively by $\dot{x}^h$ and $\dot{x}^j$, we shall have

$$\left(R^i_{hjk}\dot{x}^h\right)_{\overset{-}{|}s} = 0 \text{ and } \left(R^i_{hjk}\dot{x}^h\dot{x}^j\right)_{\overset{-}{|}s} = 0 \quad (2.12)$$

where, we have used the relation (2.8).

Contracting equation (2.12) for the indices i and k and using (2.4), we get

$$\left(R_{hj}\dot{x}^h\right)_{\overset{-}{|}s} = 0 \text{ and } \left(R_{hj}\dot{x}^h\dot{x}^j\right)_{\overset{-}{|}s} = 0 \quad (2.13)$$

Thus, we have following theorem:

Theorem (2.2): In a R-⊖ symmetric Finsler space, the relation (2.13) holds.

Theorem (2.3): In a R-⊖ symmetric Finsler space with metrical connection

$$R^i_{k\overset{-}{|}l} = 0 \text{ holds,} \quad (2.14)$$

where

$$R^i_k \underset{=\!=}{def} g^{hj} R^i_{hjk} \quad (2.15)$$

Proof : Multiplying (2.1) by g^{hj}, we get

$$g^{hj}\left(R^i_{hjk\overset{-}{|}l}\right) = 0 \quad (2.16)$$

Since the non-symmetric connection Γ^i_{jk} is metrical i.e.,

$$g^{hj}_{\bar{|}k} = 0 \tag{2.17}$$

From (2.16) and (2.17), the result follows:

3. Decomposition in a R-⊖ Symmetric Finsler Space

We consider the decomposition of curvature tensor field R^i_{jkh} of recurrent Finsler space F^*_n in the following form:

$$R^i_{jkh} = X^i_j \phi_{kh}, \quad \text{(Pandey and Gupta, 1979)} \tag{3.1}$$

where $X^i_j(x,\dot{x})$ and $\phi_{kh}(x,\dot{x})$ are two non-zero tensor fields such that

$$X^i_j \beta_i = a_j \tag{3.2}$$

a_j is a non-zero vector field and is called a decomposed vector field.

Here X^i_j and ϕ_{kh} are homogeneous of degree zero in it's directional arguments $\dot{x}^i$.

⊖-covariant derivative of (3.1) with respect to x^m gives

$$X^i_{j\bar{|}l}\phi_{kh} + X^i_j \phi_{kh\bar{|}l} = R^i_{jkh\bar{|}l} \tag{3.3}$$

From (2.1) and (3.3), we get

$$X^i_{j\bar{|}l}\phi_{kh} + X^i_j \phi_{kh\bar{|}l} = 0 \tag{3.4}$$

Now if decomposed tensor field X^i_j is R-⊖ recurrent, then

$$X^i_{j\bar{|}l} = \beta_l X^i_j \tag{3.5}$$

In view of (3.5), (3.4) may be expressed as

$$\beta_l X^i_j \phi_{kh} + X^i_j \phi_{kh\bar{|}l} = 0 . \tag{3.6}$$

Transvecting (3.6) by β_i and using (3.2), we get

$$a_j\left(\phi_{kh\bar{|}l} + \beta_l \phi_{kh}\right) = 0 \tag{3.7}$$

Interchanging k and h in (3.7), we get

$$a_j\left(\phi_{hk\bar{|}l} + \beta_l \phi_{kh}\right) = 0 . \tag{3.8}$$

Adding (3.7) and (3.8) and using the relation $a_j\phi_{(kh)} = 0$, we get

$$a_j \phi_{(kh)\bar{|}l} = 0 , \tag{3.9}$$

where

$$\phi_{(kh)} = \phi_{kh} + \phi_{hk} .$$

Since a_j is non-zero vector field, therefore (3.9) reduces to

$$\phi_{kh|[lm]} = 0 \tag{3.10}$$

Thus, we have the following

Theorem (3.1): In a R-⊖ symmetric Finsler space, the tensor field ϕ_{kh} satisfies the identity (3.10) if the tensor field X_j^i is R-⊖ recurrent.

⊖-covariant derivative of (3.4) is given by

$$X^i_{j|\bar{l}\bar{m}}\varphi_{kh} + X^i_{j|\bar{l}}\varphi_{kh|\bar{m}} + X^i_{j|\bar{m}}\varphi_{kh|\bar{l}} + X^i_j\varphi_{kh|\bar{l}\bar{m}} = 0 \tag{3.11}$$

If $X_j^i(x - \dot{x})$ be ⊖-covariantly invariant, then

$$X^i_{j|\bar{l}} = 0 \tag{3.12}$$

Hence, from (3.11) and (3.12), we get

$$X^i_j\varphi_{kh|\bar{l}\bar{m}} = 0 \tag{3.13}$$

Commuting the indices l and m in the equation (3.12), we obtain

$$X^i_j\left(\varphi_{kh|\bar{l}\bar{m}} - \varphi_{kh|\bar{m}\bar{l}}\right) = 0 \tag{3.14}$$

Since X_j^i be a non-zero tensor field, therefore, we must have

$$\varphi_{kh|[\bar{l}\bar{m}]} = 0 \tag{3.15}$$

Thus, we have

Theorem (3.2): In a R-⊖ symmetric Finsler space, the tensor field satisfies the identity (3.15) if the tensor field X_j^i is ⊖-covariantly invariant.

Refereces

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