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On Concircular nearly recurrent manifolds

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Abstract

The object of the present paper is to study concircular nearly recurrent manifolds.

Keywords Nearly recurrent manifold, concircular curvature tensor and scalar curvature.

1. Introduction

Recurrent spaces have been of significant importance and have been the subject of research by numerous authors including (Ruse, 1946; Patterson, 1952; Walker, 1951; Singh and Khan, 2000) etc. In 1991, De and Guha introduced and investigated a generalized recurrent manifold whose curvature tensor of the form (1, 3) satisfies the following condition:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where A and B are two non-zero one-forms and D denotes the operator of covariant differentiation with respect to metric tensor g. Such a space has been denoted by GK_n. In a recent work (Bandyopadhaya, 2011; Prakasha and Yildiz, 2010; Khan, 2017) etc investigated some geometrical characteristics by applying generalized recurrent manifold on different structure.

Furthermore, one of the author, (Prasad, 2000), investigated a non-flat Riemannian manifold (M^n, g), $n > 3$, whose curvature tensor R meets the following criterion:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)g(Y,Z)X, \tag{1.2}$$

where A, B and D have the meaning already mentioned. Such a manifold called by the author as semi-generalized recurrent manifold and denoted by $(SGK)_n$. Singh, Singh and Kumar (2016), (2015) and Chaudhary, Kumar and Singh (2016) extended this notation to Lorentzian α-Sasakian manifold, P-Sasakian manifold and Trans-Sasakian manifold.

Recently in 2023, Prasad and Yadav investigated and studied another type of non-flat recurrent Riemannian manifold (M^n, g), $n > 2$, whose curvature tensor $R(X,Y)Z$ of the type (1, 3) fulfill the condition:

$$(D_U R)(X,Y)Z = [A(U) + B(U)]R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

where A and B are two non-zero 1-forms and ρ_1 and ρ_2 are two vector fields such that

$$g(U,\rho_1) = A(U) \text{ and } g(U,\rho_2) = B(U). \tag{1.4}$$

Such a manifold called by them as a nearly recurrent manifold and 1-form A and B are called its associated 1-forms an n-dimensional recurrent manifold of this kind denoted by $(NR)_n$. If in particular $B = 0$, then (1.3) reduces to a recurrent space according to (Ruse, 1947) and (Walker, 1951) which was denoted by K_n.

Moreover, in particular if $A = B = 0$, then (1.3) becomes $(D_U R)(X,Y)Z = 0$. That is, Riemannian manifold is symmetric accordingly Kobayashi and Nomizu (1963) and Desai and Amur (1975). The name nearly recurrent Riemannian manifold was chosen because if $B = 0$ in (1.3), then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name "Nearly recurrent Riemannian manifold" for the manifold defined by (1.3) and the use of the symbol $(NR)_n$ for it.

A transformation of a Riemannian manifold which transforms every geodesic circle of manifold into a geodesic circle is called a concircular transformation and the geometry which deals with such transformation is called the concircular geometry (Yano, 1940). A concircular transformation is always a conformal transformation (Yano, 1940). Here geodesic circle means a curve in a manifold whose first curvature is constant and second curvature is zero. Keeping this fact in mind Yano in 1940 defined the concircular curvature tensor C of the type (1, 3) tensor that invariant under concircular transformation for an n-dimensional Riemannian manifold:

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.5}$$

where r is the scalar curvature of (M^n, g).

In 1994, De and Guha studied a type of non-flat Riemannian manifold (M^n, g), $n \geq 2$, whose concircular curvature tensor satisfies the condition:

$$(D_X C)(Y,Z)W = A(X)C(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z].$$

Such a manifold denoted by authors as $G\{CK_n\}$.

Motivation of the above concept, in the present paper investigator defined a new type of non-flat Riemannian manifold whose concircular curvature tensor C satisfies the condition:

$$(D_X C)(Y,Z)W = [A(X) + B(X)]C(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z], \tag{1.6}$$

where A, B and D have the meaning already mentioned. Such a manifold shall be called concircular nearly recurrent manifold, B shall be called its associated 1-form an n-dimensional manifold of the kind shall be denoted by $(CNR)_n$.

In this paper, after introduction, a necessary and sufficient for a $(CNR)_n$ to be $(NR)_n$ has been obtained. Also a necessary and sufficient condition for constant scalar curvature has been obtained. It is further shown that if a $(CNR)_n$ with constant scalar curvature is a $(NR)_n$, then $\frac{2r}{n}$ is a eigen value of the Ricci tensor Ric and ρ_1 is an eigen vector corresponding to this eigen value. In the last section the necessary and sufficient condition for zero scalar curvature has been obtained.

2. Necessary and sufficient condition for a $(CNR)_n$ to be $(NR)_n$.

From (1.6), we get

$$(D_X C)(Y,Z)W - [A(X) + B(X)]C(Y,Z)W - B(X)[g(Z,W)Y - g(Y,W)Z] = 0. \tag{2.1}$$

From (1.5) and (2.1), we get

$$(D_X R)(Y,Z)W - \frac{dr(X)}{n(n-1)}[g(Z,W)Y - g(Y,W)Z] - [A(X) + B(X)]R(Y,Z)W -$$

$$\frac{[A(X)+B(X)]r}{n(n-1)}[g(Z,W)Y - g(Y,W)Z] - B(X)[g(Z,W)Y - g(Y,W)Z] = 0.$$

$\Rightarrow \quad (D_X R)(Y,Z)W - [A(X)+B(X)]R(Y,Z)W - B(X)[g(Z,W)Y - g(Y,W)Z]$

$$= \frac{1}{n(n-1)}[g(Z,W)Y - g(Y,W)Z][dr(X) - \{A(X)+B(X)\}r]. \tag{2.2}$$

Hence, from (2.2), we can state the following theorem:

Theorem 2.1: A necessary and sufficient condition for a $(CNR)_n$ to be a $(NR)_n$ is that

$$dr(X) = \{A(X)+B(X)\}r.$$

3. Nature of scalar curvature

Equation (1.5) can be put as

$$'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{r}{n(n-1)}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)], \tag{3.1}$$

where $\quad 'C(X,Y,Z,W) = g(C(X,Y)Z,W),$ (3.2)

and $\quad 'R(X,Y,Z,W) = g(R(X,Y)Z,W).$ (3.3)

Let $\quad 'C(X,E_i,E_i,W) = \bar{P}(X,W),$ (3.4)

where $\{E_i\}$, $i = 1,2,\dots n$ is an orthogonal basis of the tangent space at a point and i is summed for $1 \le i \le n$. Hence from (3.1), we get

$$\bar{P}(X,W) = Ric(X,W) - \frac{r}{n}g(X,W). \tag{3.5}$$

Let l and L be respectively the symmetric endomorphisms of the tangent space at a point corresponding to the tensors $\bar{P}$ and Ric. That is,

$$g(lX,Y) = \bar{P}(X,Y), \tag{3.6}$$

and

$$g(LX,Y) = Ric(X,Y). \tag{3.7}$$

Differentiating (1.5) covariantly, we get

$$(D_U C)(X,Y)Z = (D_U R)(X,Y)Z - \frac{dr(U)}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.8}$$

Contraction of (3.8) gives

$$(divC)(X,Y)Z = [(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z)] - \frac{1}{n(n-1)}[g(Y,Z)dr(X) - dr(Y)g(X,Z)], \tag{3.9}$$

where div denotes divergence.

Contracting (1.6), we get

$$(divC)(Y,Z)W = A(C(Y,Z)W) + B(C(Y,Z)W) + B(Y)g(Z,W) - B(Z)g(Y,W). \tag{3.10}$$

Writing X, Y, Z for Y, Z, W in (3.10), we get

$$(divC)(X,Y)Z = A(C(X,Y)Z) + B(C(X,Y)Z) + B(X)g(Y,Z) - B(Y)g(X,Z). \tag{3.11}$$

From (3.9) and (3.11), we get

$$(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z) - \frac{1}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] = A(C(X,Y)Z) + B(C(X,Y)Z) + B(X)g(Y,Z) - B(Y)g(X,Z). \tag{3.12}$$

Contracting (3.12), we get

$$(n-2)dr(X) = 2n[A(lX) + B(lX)] + 2n(n-1)B(X). \tag{3.13}$$

Let us assume that r is constant then from (3.13) it follow that

$$A(lX) + B(lX) + (n-1)B(X) = 0. \tag{3.14}$$

Conversely, if (3.14) exist then from (3.13), we get

$$r = \text{Constant}. \tag{3.15}$$

Thus, we can state the following theorem:

Theorem 3.1: In a $(CNR)_n$, the scalar curvature is constant if and only if (3.14) holds.

Next, we assume that $(CNR)_n$ with constant scalar curvature is a $(NR)_n$. Since the scalar curvature is constant, hence we get

$$A(lX) + B(lX) + (n-1)B(X) = 0. \tag{3.16}$$

From (3.5) and (3.6), we get

$$A(lX) = Ric(X, \rho_1) - \frac{r}{n} g(X, \rho_1), \tag{3.17}$$

where ρ_1 is defined by (1.4).

In view of (3.16) and (3.17), we get

$$Ric(X, \rho_1 + \rho_2) - \frac{r}{n} Ric(X, \rho_1 + \rho_2) = -(n-1)B(X). \tag{3.18}$$

Contraction of (1.3) gives

$$dr(X) = [A(X) + B(X)]r + n(n-1)B(X). \tag{3.19}$$

Since r is constant, hence from (3.19), we get

$$[A(X) + B(X)]r + (n-1)nB(X) = 0,$$

$$\Rightarrow \quad r[g(X, \rho_1) + g(X, \rho_2)] + n(n-1)B(X) = 0,$$

$$\Rightarrow \quad \frac{r}{n} g(X, \rho_1) + \frac{r}{n} g(X, \rho_2) = -(n-1)B(X),$$

$$\Rightarrow \quad \frac{r}{n} g(X, \rho_1 + \rho_2) = -(n-1)B(X). \tag{3.20}$$

Hence, from (3.18) and (3.20), we get

$$Ric(X, \rho_3) = \frac{2r}{n} g(X, \rho_3), \tag{3.21}$$

where $\rho_1 + \rho_2 = \rho_3$.

Thus, we have the following theorem:

Theorem 3.1: If a $(CNR)_n$ with constant scalar curvature is a $(NR)_n$, then $\frac{2r}{n}$ is an eigen value of the Ricci tensor Ric and ρ_3 is an eigen vector corresponding to this eigen value.

4. Necessary and sufficient condition for zero scalar curvature of $(CNR)_n$.

From (1.5), we get

$$C(X,Y)\rho_2 = R(X,Y)\rho_2 - \frac{r}{n(n-1)}[g(Y, \rho_2)X - g(X, \rho_2)Y], \tag{4.1}$$

where ρ_2 is defined by $g(X, \rho_2) = B(X)$.

First, we assume that $r = 0$ in $(NCR)_n$.

Then, from (4.1), we get

$$C(X, Y)\rho_2 = R(X, Y)\rho_2. \quad (4.2)$$

Next, we assume that in a $(CNR)_n$ the relation (4.2) holds.

Hence, from (4.1), we get

$r = 0$, provided, $[g(Y, \rho_2)X - g(X, \rho_2)Y] \neq 0$, i.e, $B(X) \neq 0$.

This leads to the following theorem:

Theorem 4.1: A $(CNR)_n$ is of the zero scalar curvature if and only if

$$C(X, Y)\rho_2 = R(X, Y)\rho_2.$$

References

1. De, U.C., Guha, N. and Kamilya, D. (1991). On generalized recurrent manifolds, National Academy of Math, Indian, 9(1):1-4.
2. Desai, P. and Amur, K. (1975). On symmetric spaces, Tensor N.S., 29:119-124.
3. Khan, Q. (2017). On generalized recurrent Sasakian manifold with special curvature tensor $\mathcal{T}(X, Y)Z$, Ganita, 67:189-194.
4. Kobayshi, S. and Nomizu, K. (1963). Foundation of differential geometry, Vol1, New York.
5. Mahuya, B. (2011). On generalized ϕ-recurrent Sasakian manifolds, Mathematica Pannonica, 22(1):19-23.
6. Prakasha, D.G. and Yildiz, A. (2010). On generalized ϕ-recurrent Lorentzian α-Sasakian manifolds, Commun Fac. Sci. Uni. Ank. Series A1, 59(1):53-62.
7. Prasad, B. (2000). On semi-generalized recurrent manifold, Mathematica Balkanica N.S., 14:1-4.
8. Prasad, B. and Yadav, R.P.S. (2023). On nearly recurrent Riemannian manifold, Malaya J. Math., 11(02):200-209.
9. Paatterson, E.M. (1952). Some theorem on Ricci-recurrent space, J. London Math. Soc., 52:287-295.
10. Ruse, H.S. (1946). On simply harmonic space, J. London Math. Soc., 21.
11. Ruse, H.S. (1947). Three dimensional spaces of recurrent curvature, Proc. London. Math. Soc., 50(2):438-446.
12. Singh, H. and Khan, Q. (2000). On generalized recurrent Riemannian manifolds, Publ. Math. Debrecen, 56(1-2):87-95.
13. Singh, J.P., Singh, A. and Kumar, R. (2016). On a type of semi-generalized recurrent P-Sasakian manifold, Facta Univ. (NIS) Ser. Math. Inform, 31(1):213-225.
14. Singh, J.P., Singh, A. and Kumar, R. (2015). On a type of semi-generalized recurrent Lorentzian α-Sasakian manifold, Bull. Cal. Math. Soc., 107(5):357-370.
15. Yano, K. (1940). Concircular geometry. Imp. Acad. Sci. of Japan, 16:165.
16. Walker, A.G. (1951). On Ruse spaces of recurrent curvature, Proc. London Math. Soc., 52.

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