Characterization of almost W_1-curvature tensor on Riemannian manifold

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Abstract

In this paper we introduce a new tensor named almost W_1-curvature tensor $\mathbb{W}_1$, which generalizes the projective curvature tensor and W_1-curvature tensor. First we deduce some basic geometric properties of $\mathbb{W}_1$. Then, we study pseudo W_1-symmetric manifolds $(P\mathbb{W}_1S)_n$, $n > 2$. We provide several interesting results. Among others we prove that in a $(P\mathbb{W}_1S)_n$ if the associated vector field ρ is a unit parallel vector field, then the manifold reduces to a pseudo symmetric manifold. Finally, we construct a non-trivial Lorentzian metric of $(P\mathbb{W}_1S)_n$.

Keywords W_1-curvature tensor, pseudo W_1-symmetric manifold and pseudo symmetric manifold.

1. Introduction

It is well known that the symmetric spaces play an important role in differential geometry. The study of locally symmetric Riemannian spaces was initiated in the last twenties by Cartan (1926). The class of locally symmetric Riemannian manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been languished by many authors.

A non-flat pseudo-Riemannian manifold (M^n, g), $n > 2$, is said to be a pseudo symmetric (Chaki, 1987) if its curvature tensor $\mathcal{R}$ of type $(0, 4)$ satisfies the following condition:

$$(\mathcal{D}_X \mathcal{R})(Y, Z, U, V) = 2\mathcal{A}(X)\mathcal{R}(Y, Z, U, V) + \mathcal{A}(Y)\mathcal{R}(X, Z, U, V)$$
$$+ \mathcal{A}(Z)\mathcal{R}(Y, X, U, V) + \mathcal{A}(U)\mathcal{R}(Y, Z, X, V)$$
$$+ \mathcal{A}(V)\mathcal{R}(Y, Z, U, X),$$

where $\mathcal{A}$ is non-zero 1-form, ρ is a vector field defined by $g(X, \rho) = \mathcal{A}(X)$, for all X, $\mathcal{D}$ denotes the covariant differentiation with respect to the metric g and $\mathcal{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, where R is the curvature tensor of the type $(1, 3)$.

The 1-form $\mathcal{A}$ is called the associated 1-form of the manifold. If $\mathcal{A} = 0$, then the manifold reduces to a locally symmetric manifold in the sense of Cartan. An n-dimensional pseudo-symmetric manifold is denoted by $(PS)_n$.

Gray (1978) introduced the notion of cyclic parallel Ricci tensor and Codazzi type Ricci tensor. A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if its Ricci tensor Ric of the type $(0, 2)$ is non-zero and satisfies the condition:

$$(\mathcal{D}_X Ric)(Y,Z) + (\mathcal{D}_Y Ric)(Z,X) + (\mathcal{D}_Z Ric)(X,Y) = 0. \tag{1.1}$$

Again, Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor Ric of the type (0, 2) is non-zero satisfy the following condition:

$$(\mathcal{D}_X Ric)(Y,Z) = (\mathcal{D}_Y Ric)(X,Z). \tag{1.2}$$

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometry point of view.

Let M^n be an n-dimensional Riemannian manifold. If there exist a one to one correspondence between each co-ordinate neighbourhood of M^n and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, M^n is said to be locally projectively flat. For $n \geq 3$, M^n is locally projectively flat if and only if the projective curvature tensor vanishes.

Projective curvature tensor W in a Riemannian manifold (M^n, g), $n > 2$ is defined by Neil (1983)

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y], \tag{1.3}$$

where R is Riemannian curvature tensor of the type (1, 3) and Ric is the Ricci tensor of the type (0, 2).

Motivated by the projective curvature, Pokhariyal and Mishra (1971) defined and studied the relativistic significance of W_1-curvature tensor on an n-dimensional Remannian manifold, defined by

$$W_1(X,Y)U = R(X,Y)U + \frac{1}{n-1}[Ric(Y,U)X - Ric(X,U)Y], \tag{1.4}$$

for all vector fields X, Y and U on M^n, the properties of the same curvature tensor on the LP-Sasakian manifolds have been studied by Pokhariyal (1996).

Currently one of the author defined several novel curvature tensors on a Riemannian manifold such as pseudo projective curvature tensor (2002), Quasi conharmonic curvature tensor (2013), Quasi concircular curvature tensor (2007) and Generalized quasi conformal curvature (2011) and found numerous geometrical features.

Recently, in 2018, De and Majhi introduced a new curvature tensor of the type (1, 3) in an n-dimensional Riemannian manifold denoted by $\mathcal{P}$ and defined by

$$\mathcal{P}(X,Y)U = R(X,Y)U - \frac{\phi}{n-1}[Ric(Y,U)X - Ric(X,U)Y], \tag{1.5}$$

where ϕ is an arbitrary scalar function. Authors prefer the name "semi-projective curvature tensor", since it is clear from (1.3) that for $\phi = 1$, semi-projective curvature tensor reduces to projective curvature tensor.

With the preceding fact in mind, we define a new curvature tensor entitled "Almost W_1-curvature tensor" $\mathbb{W}_1$ $(AW_1CT)_n$, $n > 2$ of the type (1, 3) on Riemannian manifold as follows:

$$\mathbb{W}_1(X,Y)U = R(X,Y)U + \frac{a}{n-1}[Ric(Y,U)X - Ric(X,U)Y], \tag{1.6}$$

where a is an arbitrary real constant.

In particular,

    (i) If $a = 1$, then $\mathbb{W}_1(X,Y)U$ reduces in W_1-curvature (Pokhariyal and Mishra, 1972).

    (ii) If $a = -1$, then $\mathbb{W}_1(X,Y)U$ reduces in projective curvature tensor (Neill, 1983).

(iii) If we replace a by ϕ, where ϕ is an arbitrary scalar function, then $\mathbb{W}_1$-reduces in semi-projective curvature tensor (De and Majhi, 2018).

(iv) If $a = 0$, then W_1-curvature tensor and curvature tensor are equivalent.

Because of the existence of such specific situation of W_1-curvature tensor, we refer to it as a "Almost W_1-curvature tensor". This justifies their nomenclature.

We can express (1.4) as follows:

$$'\mathbb{W}_1(X,Y,U,V) = '\mathcal{R}(X,Y,U,V) + \frac{a}{n-1}[Ric(Y,U)g(X,V) - Ric(X,U)g(Y,V)], \qquad (1.7)$$

where $\quad '\mathbb{W}_1(X,Y,U,V) = g(\mathbb{W}_1(X,Y)U,V),$ (1.8)

and $\quad '\mathcal{R}(X,Y,U,V) = g(R(X,Y)U,V).$ (1.9)

A non-flat Riemannian manifold (M^n, g), $n > 2$, is said to be pseudo $\mathbb{W}_1$-symmetric manifold if the $'\mathbb{W}_1$-curvature tensor of type (0, 4) satisfies the condition:

$$(\mathcal{D}_X '\mathbb{W}_1)(Y,Z,U,V) = 2\mathcal{A}(X) \, '\mathbb{W}_1(Y,Z,U,V) + \mathcal{A}(Y) \, '\mathbb{W}_1(X,Z,U,V) +$$

$$\mathcal{A}(Z) \, '\mathbb{W}_1(Y,X,U,V) + \mathcal{A}(U) \, '\mathbb{W}_1(Y,Z,X,V) +$$

$$\mathcal{A}(V) \, '\mathbb{W}_1(Y,Z,U,X). \qquad (1.10)$$

An n-dimensional pseudo $\mathbb{W}_1$-symmetric manifold is denoted by $(P'\mathbb{W}_1 S)_n$, $n > 2$.

In particular, we have

(i) If $a = 0$, then $(P'\mathbb{W}_1 S)$ reduces in $(PS)_n$ Chaki etal. (1927).

(ii) If $a = 1$, then $(P'\mathbb{W}_1 S)$ reduces in $(PPS)_n$ Chaki et al. (1987).

(iii) If $a = \phi$, then $(P'\mathbb{W}_1 S)$ reduces in $(PSPS)_n$ De and Majhi (2018).

After introduction in section 2, we study some algebraic properties of $(AW_1CT)_n$, $n > 2$. Section 3 is devoted to study of curvature properties of $(P'\mathbb{W}_1 S)_n$, $n > 2$. In section 4, we study $(P'\mathbb{W}_1 S)_n$ admitting Codazzi type Ricci tensor. Section 5 and 6 deal with Einstein $(P'\mathbb{W}_1 S)_n$ and $(P'\mathbb{W}_1 S)_n$ with $div\mathbb{W}_1 = 0$ respectively. Section 7 is devoted to study of $(P'\mathbb{W}_1 S)_n$, $n > 2$ admitting parallel vector field ρ. Among others we prove that in a $(P'\mathbb{W}_1 S)_n$, $n > 2$, if the associated vector field ρ is a unit parallel vector field then the manifold reduces to pseudo symmetric manifold.

2. Preliminaries

Let Ric and r denote the Ricci tensor of type (0, 2) and scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric, i.e,

$$g(LX,Y) = Ric(X,Y). \qquad (2.1)$$

In this section, some formulas are derived, which will be useful to the study of $(P'\mathbb{W}_1 S)_n$. Let $\{E_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \le E_i \le n$.

In a Riemannian manifold the Ricci tensor Ric is defined by

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(X,E_i)E_i,Y).$$

From (1.6), it can be easily seen that

$$\mathbb{W}_1(X,Y)U + \mathbb{W}_1(Y,X)U = 0,$$

and

$$\mathbb{W}_1(X,Y)U + \mathbb{W}_1(Y,U)X + \mathbb{W}_1(U,X)Y = 0. \tag{2.2}$$

Also, from (1.7), we have

$$\sum_{i=1}^n {}'\mathbb{W}_1(X,Y,E_i,E_i) = \sum_{i=1}^n {}'\mathbb{W}_1(E_i,E_i,U,V) = 0, \tag{2.3}$$

$$\sum_{i=1}^n {}'\mathbb{W}_1(E_i,Y,U,E_i) = (1+a)Ric(Y,U) = \widetilde{W}_2(Y,U), \tag{2.4}$$

and

$$\sum_{i=1}^n {}'\mathbb{W}_1(X,E_i,E_i,V) = Ric(X,V)\left(1 - \frac{a}{n-1}\right) + \frac{ar}{n-1}\, g(X,V) = \widetilde{W}_3(X,V), \tag{2.5}$$

where $r = \sum_{i=1}^n Ric(E_i,E_i)$ is the scalar curvature.

From (1.7), it follows that

$$'\mathbb{W}_1(X,Y,U,V) + {}'\mathbb{W}_1(Y,X,U,V) = 0,$$

$$'\mathbb{W}_1(X,Y,U,V) + {}'\mathbb{W}_1(X,Y,V,U) \neq 0,$$

$$'\mathbb{W}_1(X,Y,U,V) + {}'\mathbb{W}_1(U,V,X,Y) \neq 0,$$

and

$$'\mathbb{W}_1(X,Y,U,V) + {}'\mathbb{W}_1(Y,U,X,V) + {}'\mathbb{W}_1(U,X,Y,V) = 0. \tag{2.6}$$

Theorem 2.1: If a Riemannian manifold is $'\mathbb{W}_1$-curvature flat then the scalar curvature vanishes.

Proof: The $'\mathbb{W}_1$-curvature tensor of type $(0,4)$ is given by

$$'\mathbb{W}_1(X,Y,U,V) = {}'\mathcal{R}(X,Y,U,V) + \frac{a}{n-1}[Ric(Y,U)g(X,V) - Ric(X,U)g(Y,V)], \tag{2.7}$$

where a is a real constant. If $'\mathbb{W}_1$-curvature tensor vanishes, then from (2.7),

$$'\mathcal{R}(X,Y,U,V) = -\frac{a}{n-1}[Ric(Y,U)g(X,V) - Ric(X,U)g(Y,V)]. \tag{2.8}$$

Contraction of (2.8) gives

$$Ric(X,V) = -\left(\frac{ra}{n-1-a}\right)g(X,V). \tag{2.9}$$

Again, contraction of (2.9) gives

$$r = 0. \tag{2.10}$$

This completes the proof.

Theorem 2.2: If the $'\mathbb{W}_1$-curvature tensor is symmetric in the sense of Cartan, then the manifold reduces in Ricci symmetric, provided $a \neq -1$.

Proof: $\mathbb{W}_1$-curvature tensor is given by

$$\mathbb{W}_1(Y,Z)U = R(Y,Z)U + \frac{a}{n-1}[Ric(Z,U)Y - Ric(Y,U)Z]. \tag{2.11}$$

Differentiating (2.11) covariantly, we get

$$(\mathcal{D}_X \mathbb{W}_1)(Y,Z)U = (\mathcal{D}_X R)(Y,Z)U + \frac{a}{n-1}[(\mathcal{D}_X Ric)(Z,U)Y - (\mathcal{D}_X Ric)(Y,U)Z]. \tag{2.12}$$

Here, we assume that $\mathbb{W}_1$-curvature is symmetric. Hence from (2.12), we get

$$(\mathcal{D}_X R)(Y,Z)U = -\frac{a}{n-1}[(\mathcal{D}_X Ric)(Z,U)Y - (\mathcal{D}_X Ric)(Y,U)Z]. \tag{2.13}$$

Contraction of (2.13), we get

$$(1 + a)(\mathcal{D}_X Ric)(Z, U) = 0,$$

$$\Rightarrow \quad (\mathcal{D}_X Ric)(Z, U) = 0, \text{ provided } 1 + a \neq 0.$$

This completes the proof.

Theorem 2.3 Almost W_1-curvature tensor $\mathbb{W}_1$ is divergence free if and only if it is of Codazzi type Ricci tensor.

Proof: $\mathbb{W}_1$-curvature tensor is given by

$$\mathbb{W}_1(Y, Z)U = R(Y, Z)U + \frac{a}{n-1}[Ric(Z, U)Y - Ric(Y, U)Z].$$

Differentiating covariantly above equation,

$$(\mathcal{D}_X \mathbb{W}_1)(Y, Z)U = (\mathcal{D}_X R)(Y, Z)U + \frac{a}{n-1}[(\mathcal{D}_X Ric)(Z, U)Y - (\mathcal{D}_X Ric)(Y, U)Z]. \tag{2.14}$$

Contracting (2.14) with respect to X, we get

$$(div\mathbb{W}_1)(Y, Z)U = (divR)(Y, Z)U + \frac{a}{n-1}[(\mathcal{D}_Y Ric)(Z, U) - (\mathcal{D}_Z Ric)(Y, U)]. \tag{2.15}$$

Making the use of $(divR)(Y, Z)U = (\mathcal{D}_Y Ric)(Z, U) - (\mathcal{D}_Z Ric)(Y, U)$, the equation (2.15) gives

$$(div\mathbb{W}_1)(Y, Z)U = \left(\frac{n-1+a}{n}\right)[(\mathcal{D}_Y Ric)(Z, U) - (\mathcal{D}_Z Ric)(Y, U)]. \tag{2.16}$$

Equation (2.16) proves the statement of theorem (2.3).

3. Curvature property of $(P'\mathbb{W}_1 S)_n$, $n > 2$

In this section, we prove that in a $(P'\mathbb{W}_1 S)_n$, $n > 2$, $'\mathbb{W}_1(Y, Z, U, V)$ satisfies Bianch's second identify, that is

$$(\mathcal{D}_X {}'\mathbb{W}_1)(Y, Z, U, V) + (\mathcal{D}_Y {}'\mathbb{W}_1)(Z, X, U, V) + (\mathcal{D}_Z {}'\mathbb{W}_1)(X, Y, U, V) = 0. \tag{3.1}$$

In view of (1.7) and (1.10), we get

$$(\mathcal{D}_X {}'\mathbb{W}_1)(Y, Z, U, V) + (\mathcal{D}_Y {}'\mathbb{W}_1)(Z, X, U, V) + (\mathcal{D}_Z {}'\mathbb{W}_1)(X, Y, U, V)$$

$$= A(U)[{}'\mathbb{W}_1(Y, Z, X, V) + {}'\mathbb{W}_1(Z, X, Y, V) + {}'\mathbb{W}_1(X, Y, Z, V) +$$

$$A(V)[{}'\mathbb{W}_1(Y, Z, U, X) + {}'\mathbb{W}_1(Z, X, U, Y) + {}'\mathbb{W}_1(X, Y, U, Z)]. \tag{3.2}$$

Using (2.6) in (3.2), we get

$$(\mathcal{D}_X {}'\mathbb{W}_1)(Y, Z, U, V) + (\mathcal{D}_Y {}'\mathbb{W}_1)(Z, X, U, V) + (\mathcal{D}_Z {}'\mathbb{W}_1)(X, Y, U, V)$$

$$= A(V)[{}'\mathbb{W}_1(Y, Z, U, X) + {}'\mathbb{W}_1(Z, X, U, Y) + {}'\mathbb{W}_1(X, Y, U, Z)]. \tag{3.3}$$

Again, in view of (1.7) and (3.3), we get

$$(\mathcal{D}_X {}'\mathbb{W}_1)(Y, Z, U, V) + (\mathcal{D}_Y {}'\mathbb{W}_1)(Z, X, U, V) + (\mathcal{D}_Z {}'\mathbb{W}_1)(X, Y, U, V)$$

$$= A(V)[-\{'\mathcal{R}(X, Y, Z, U) + '\mathcal{R}(Y, Z, X, U) + '\mathcal{R}(Z, X, Y, U)\} +$$

$$\frac{a}{n-1}\{Ric(Z, U)g(Y, X) - Ric(Y, U)g(Z, X) + Ric(X, U)g(Z, Y)$$

$$-Ric(Z, U)g(X, Y) + Ric(Y, U)g(X, Z) - Ric(X, U)g(Y, Z)\}]. \tag{3.4}$$

Making the use of Bianchi's first identity in (3.4), we get Bianchi's second identity.

Thus, we can state the following:

Theorem 3.1: Almost W_1-curvature tensor $'\mathbb{W}_1$ in $(P'\mathbb{W}_1 S)_n$, $n > 2$, satisfies Bianch's second identity.

4. A $(P'\mathbb{W}_1 S)_n$, $n > 2$, admitting Codazzi type of Ricci tensor.

$$(\mathcal{D}_X'\mathbb{W}_1)(Y,Z,U,V) = (\mathcal{D}_X\mathcal{R})(Y,Z,U,V) + \frac{a}{n-1}[(\mathcal{D}_X Ric)(Z,U)g(Y,V)$$
$$-(\mathcal{D}_X Ric)(Y,U)g(Z,V)]. \tag{4.1}$$

From (4.1), we get

$$(\mathcal{D}_X'\mathbb{W}_1)(Y,Z,U,V) + (\mathcal{D}_Y'\mathbb{W}_1)(Z,X,U,V) + (\mathcal{D}_Z'\mathbb{W}_1)(X,Y,U,V)$$

$$= (\mathcal{D}_X'\mathcal{R})(Y,Z,U,V) + (\mathcal{D}_Y'\mathcal{R})(Z,X,U,V) + (\mathcal{D}_Z'\mathcal{R})(X,Y,U,V) +$$

$$\frac{a}{n-1}[(\mathcal{D}_X Ric)(Z,U)g(Y,V) - (\mathcal{D}_X Ric)(Y,U)g(Z,V) + (\mathcal{D}_Y Ric)(X,U)g(Z,V) -$$

$$(\mathcal{D}_Y Ric)(Z,U)g(X,V) + (\mathcal{D}_Z Ric)(Y,U)g(X,V) - (\mathcal{D}_Z Ric)(X,U)g(Y,U)]. \tag{4.2}$$

Using (3.1) and Bianch's second identity in (4.2), we get

$$\frac{a}{n-1}[\{(\mathcal{D}_X Ric)(Z,U) - (\mathcal{D}_Z Ric)(X,U)\}g(Y,U) +$$

$$\{(\mathcal{D}_Y Ric)(X,U) - (\mathcal{D}_X Ric)(Y,U)\}g(Z,V) +$$

$$\{(\mathcal{D}_Z Ric)(Y,U) - (\mathcal{D}_Y Ric)(Z,U)\}g(X,V)] = 0. \tag{4.3}$$

Contracting (4.3) with respect to Y and V, we get

$$(\mathcal{D}_X Ric)(Z,U) = (\mathcal{D}_Z Ric)(X,U). \tag{4.4}$$

Therefore, we are in position to state the following theorem.

Theorem 4.1: In a $(P'\mathbb{W}_1 S)$, the Ricci tensor is of Codazzi type, provided $a \neq 0$.

Further from (2.4), we have

$$\widetilde{W}_2(Z,U) = (1+a)Ric(Z,U). \tag{4.5}$$

Contracting Z and U in (4.5), we get

$$\omega_2 = (1+a)r. \tag{4.6}$$

Again contracting Y and V in (1.10), we get

$$(\mathcal{D}_X\widetilde{W}_2)(Z,U) = 2\mathcal{A}(X)\widetilde{W}_2(Z,U) + {}'\mathbb{W}_1(X,Z,U,\rho) + \mathcal{A}(Z)\widetilde{W}_2(X,U) +$$

$$\mathcal{A}(U)\widetilde{W}_2(Z,U) + {}'\mathbb{W}_1(\rho,Z,U,X). \tag{4.7}$$

$$\Rightarrow \quad (\mathcal{D}_X\widetilde{W}_2)(Z,U) = 2\mathcal{A}(X)(1+a)Ric(Z,U) + \mathcal{A}(Z)(1+a)Ric(X,U) +$$

$$\mathcal{A}(U)(1+a)Ric(Z,X) + {}'\mathbb{W}_1(X,Z,U,\rho) + {}'\mathbb{W}_1(\rho,Z,U,X). \tag{4.8}$$

Further, contracting Z and U in (4.8), we get

$$\mathcal{D}_X\omega_2 = 2\mathcal{A}(X)(1+a)r + 2(1+a)Ric(X,\rho) + 2\widetilde{W}_3(X,\rho). \tag{4.9}$$

In view of (4.7) and (4.9), we get

$$(1+a)dr(X) = 2\mathcal{A}(X)(1+a)r + 2(1+a)Ric(X,\rho) +$$

$$2\left[Ric(X,\rho)\left\{1 - \frac{a}{n-1}\right\} + \frac{ar}{n-1}\mathcal{A}(X)\right]. \tag{4.10}$$

Some simplification of (4.10) gives

$$(1 + a)dr(X) = 2\mathcal{A}(X)\left[(1 + a) + \frac{a}{n-1}\right]r + 2\left[(2 + a) - \frac{a}{n-1}\right]Ric(X, \rho). \qquad (4.11)$$

From (4.11), we can state the following theorem:

Theorem 4.2: In a $(P'\mathbb{W}_1 S)_n$, $n > 2$, the following identity holds:

$$(1 + a)dr(X) = 2\mathcal{A}(X)\left[(1 + a) + \frac{a}{n-1}\right]r + 2\left[(2 + a) - \frac{a}{n-1}\right]Ric(X, \rho).$$

In particular, if we put $a = 0$ in theorem (4.2), then we have

$$dr(X) = 2\mathcal{A}(X)r + 4Ric(X, \rho).$$

Thus, we have recover the Chaki's (1987) result.

5. Einstein $(P'\mathbb{W}_1 S)_n$, $n > 2$

In this section we consider Einstein $(P'\mathbb{W}_1 S)_n$, $n > 2$. Since for every Einstein manifold the scalar curvature r is constant, hence for Einstein $(P'\mathbb{W}_1 S)_n$, $n > 2$, we have $dr(X) = 0$. Therefore, from the theorem (4.2), we get

$$2\mathcal{A}(X)\left[(1 + a) + \frac{a}{n-1}\right]r + 2\left[(2 + a) - \frac{a}{n-1}\right]Ric(X, \rho) = 0. \qquad (5.1)$$

Since in a Einstein manifold (M^n, g), we have

$$Ric(X, \rho) = \frac{r}{n}g(X, \rho),$$

hence (5.1) can be written as

$$\mathcal{A}(X)r[2a + 3] = 0$$

$$\Rightarrow \quad r = 0, \ \mathcal{A}(X) \neq 0, \ 2a + 3 \neq 0.$$

Thus, we have the following theorem:

Theorem 5.1: An Einstein $(P'\mathbb{W}_1 S)_n$, $n > 2$ is of zero scalar curvature, neither $A(X) \neq 0$ nor $(2a + 3) \neq 0$.

Let $(P'\mathbb{W}_1 S)_n$, $n > 2$, be a space of constant curvature. Then, we have

$$R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y], \qquad (5.2)$$

where k is a constant.

Contracting (5.2), we get

$$Ric(Y, Z) = k(n - 1)g(Y, Z). \qquad (5.3)$$

Again, contracting (5.3) with respect to Y and Z,

$$r = k(n - 1)n. \qquad (5.4)$$

In view of (5.2), (5.3) and (5.4), we get

$$R(X, Y)Z = \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \qquad (5.5)$$

Since, every space of constant curvature is an Einstein manifold, then from theorem (5.1), we have $r = 0$.

Hence, from (3.5), we get $R(X, Y)Z = 0$, which is contradiction to our assumption.

This leads to the following corollary:

Corollary (5.1): A $(P'\mathbb{W}_1 S)_n$, $n > 2$ can not be of constant curvature.

6. $(P'\mathbb{W}_1 S)_n$, $n > 2$ with $div\,'\mathbb{W} = 0$.

Putting E_i for X and V in (1.10), we get

$$(\mathcal{D}_X{}'\mathbb{W}_1)(Y,Z,U,E_i) = 2\mathcal{A}(E_i)'\mathbb{W}_1(Y,Z,U,E_i) + \mathcal{A}(Y)'\mathbb{W}_1(E_i,Z,U,E_i) +$$
$$\mathcal{A}(Z)'\mathbb{W}_1(Y,E_i,U,E_i) + \mathcal{A}(U)'\mathbb{W}_1(Y,Z,E_i,E_i) +$$
$$\mathcal{A}(E_i)'\mathbb{W}_1(Y,Z,U,E_i). \tag{6.1}$$

Simplified above equation, we obtain

$$(div'\mathbb{W}_1)(Y,Z)U = 3'\mathbb{W}_1(Y,Z,U,\rho) + \mathcal{A}(Y)\widetilde{W}_2(Z,U) - \mathcal{A}(Z)\widetilde{W}_2(Y,U). \tag{6.2}$$

According to our assumption,

$$div\,'\mathbb{W}_1 = 0. \tag{6.3}$$

Therefore, from (6.2) and (6.3), we get

$$3'\mathbb{W}_1(Y,Z,U,\rho) + \mathcal{A}(Y)\widetilde{W}_2(Z,U) - \mathcal{A}(Z)\widetilde{W}_2(Y,U) = 0. \tag{6.4}$$

Contracting Z and U in (6.4), we get

$$3\widetilde{W}_3(Y,\rho) + (1+a)\mathcal{A}(Y)r - (1+a)Ric(Y,\rho) = 0. \tag{6.5}$$

From (2.5) and (6.5), we get

$$Ric(Y,\rho) = \frac{3ar+(n-1)(1+a)r}{3a-3(n-1)+(1+a)(n-1)}\,\mathcal{A}(Y). \tag{6.6}$$

Let us put $\frac{3ar+(n-1)(1+a)r}{3a-3(n-1)+(1+a)(n-1)} = \lambda$ in (6.6), we get

$$Ric(Y,\rho) = \lambda.\,g(Y,\rho).$$

Hence, in view of (6.6), we are in position to state the following theorem:

Theorem 6.1: For $(P'\mathbb{W}_1 S)_n$, $n > 2$, with $div'\mathbb{W}_1 = 0$, λ is an eigen value of the Ricci tensor Ric corresponding to the eigen vector ρ.

7. $(P'\mathbb{W}_1 S)_n$, $n > 2$, admitting parallel vector field.

In this section, we obtain a sufficient condition for a $(P'\mathbb{W}_1 S)_n$ to be a $(PS)_n$ or $(P\mathcal{P}S)_n$. For this we require a notion of parallel vector field defined as follows:

A vector field V' is said to be parallel if

$$\mathcal{D}_X V' = 0. \tag{7.1}$$

Here, we assume that a $(P'\mathbb{W}_1 S)_n$, admitting a unit parallel vector field ρ, that is

$$\mathcal{D}_X \rho = 0. \tag{7.2}$$

Applying Ricci identity to (7.2), we get

$$R(X,Y,\rho) = 0. \tag{7.3}$$

Contracting Y in (7.3), we get

$$Ric(Y,\rho) = 0. \tag{7.4}$$

Therefore, (2.4) and (7.4) gives

$$\widetilde{W}_2(X,\rho) = (1+a)Ric(X,\rho) = 0. \tag{7.5}$$

Again, from the definition of $(P'\mathbb{W}_1 S)_n$,

$$(\mathcal{D}_X{}'\mathbb{W}_1)(Y,Z,U,V) = 2\mathcal{A}(X){}'\mathbb{W}_1(Y,Z,U,V) + \mathcal{A}(Y){}'\mathbb{W}_1(X,Z,U,V) +$$
$$\mathcal{A}(Z){}'\mathbb{W}_1(Y,X,U,V) + \mathcal{A}(U){}'\mathbb{W}_1(Y,Z,X,V) +$$
$$\mathcal{A}(V){}'\mathbb{W}_1(Y,Z,U,X), \tag{7.6}$$

where $\mathcal{A}$ is defined by

$$g(X,\rho) = \mathcal{A}(X), \ \rho \text{ is a vector field.}$$

Put $Y = V = E_i$ in (7.6), we get

$$\left(\mathcal{D}_X\widetilde{W}_2\right)(Z,U) = \sum_{i=1}^n\{2\mathcal{A}(X){}'\mathbb{W}_1(E_i,Z,U,E_i) + \mathcal{A}(E_i){}'\mathbb{W}_1(X,Z,U,E_i) +$$
$$\mathcal{A}(Z){}'\mathbb{W}_1(E_i,X,U,E_i) + \mathcal{A}(U){}'\mathbb{W}_1(E_i,Z,X,E_i) +$$
$$\mathcal{A}(E_i){}'\mathbb{W}_1(E_i,Z,U,X)\},$$

$$\Rightarrow \quad \left(\mathcal{D}_X\widetilde{W}_2\right)(Z,U) = 2\mathcal{A}(X)\widetilde{W}_2(Z,U) + {}'\mathbb{W}_1(X,Z,U,\rho) + \mathcal{A}(Z)\widetilde{W}_2(X,U) +$$
$$\mathcal{A}(U)\widetilde{W}_2(Z,X) + {}'\mathbb{W}_1(\rho,Z,U,X).$$

Substituting $U = \rho$ in above equation, we get

$$\left(\mathcal{D}_X\widetilde{W}_2\right)(Z,\rho) = 2\mathcal{A}(X)\widetilde{W}_2(Z,\rho) + \mathcal{A}(Z)\widetilde{W}_2(X,\rho) + \mathcal{A}(\rho)\widetilde{W}_2(Z,X) +$$
$$'\mathbb{W}_1(X,Z,\rho,\rho) + {}'\mathbb{W}_1(\rho,Z,\rho,X). \tag{7.7}$$

Now, using ${}'\mathbb{W}_1(X,Z,\rho,\rho) = {}'\mathbb{W}_1(\rho,Z,\rho,X) = 0$, in (7.7), we have

$$\left(\mathcal{D}_X\widetilde{W}_2\right)(Z,\rho) = (1+a)\mathcal{A}(\rho)Ric(Z,X). \tag{7.8}$$

Using (7.5) in (7.8), we get

$$(1+a)Ric(Z,X) = 0. \tag{7.9}$$

Therefore, either $1+a = 0$ or $Ric(Z,X) = 0$.

Thus, we have

For $a = -1$, $(P'\mathbb{W}_1 S)_n$ reduces in pseudo projective symmetric manifold, i.e, $(P\mathcal{P}S)_n$. Also if $Ric(Z,X) = 0$, then $(P'\mathbb{W}_1 S)_n$ reduces in pseudo symmetric manifolds, that is, $(PS)_n$.

Therefore, we can state the following theorem:

Theorem 7.1: In a $(P'\mathbb{W}_1 S)_n$ if the associated vector field, then either the manifold reduces to a pseudo-symmetric manifold or pseudo projective symmetric manifold.

8. Example

Let us consider a Lorentzian manifold (M^4, g) endowed with the metric

$$ds^2 = g_{ij}\, dx^i\, dx^j = \sin(x^2)(dx^1)^2 + e^{(x^3)}(dx^2)^2 - (dx^3)^2 + (dx^4)^2, \tag{8.1}$$

where $i,j = 1,2,3,4$ and $\sin(x^2)$, $e^{(x^3)}$ are two non-zero.

Then, the only non-vanishing components of the Christoffel symbols are

$$\Gamma^2_{11} = -\frac{\cos(x^2)}{2e^{(x^3)}}, \ \Gamma^3_{22} = \frac{e^{(x^3)}}{2}, \ \Gamma^1_{12} = \frac{\cos(x^2)}{2}, \ \Gamma^2_{23} = \frac{1}{2}. \tag{8.2}$$

The curvature tensor of type (1,3) is defined by

$$R^a_{ijk} = -\frac{\partial}{\partial x^k}\Gamma^a_{ij} + \frac{\partial}{\partial x^j}\Gamma^a_{ik} + \Gamma^a_{bj}\Gamma^b_{ik} - \Gamma^a_{bk}\Gamma^b_{ij}. \tag{8.3}$$

Using (8.2) and (8.3), the non-zero components of the curvature tensor are

$$R^3_{112} = \frac{\cos(x^2)}{2}, \ R^1_{221} = \left[\frac{-2\csc^2(x^2)+\cot(x^2)}{4}\right]. \tag{8.4}$$

The curvature tensor of type (0,4) is defined by

$$R_{hijk} = g_{ha}R^a_{ijk}. \tag{8.5}$$

From (8.4) and (8.5), we obtain

$$R_{3112} = -\frac{\cos(x^2)}{2}, \ R_{1221} = \left[\frac{-2\csc(x^2)+\cos(x^2)}{4}\right]. \tag{8.6}$$

The non-vanishing components of the Ricci tensor are

$$R_{ij} = g^{hk}R_{hijk},$$

which gives

$$R_{11} = \left[\frac{-2\csc(x^2)+\cos(x^2)}{4\sin(x^2)}\right], \ R_{23} = \frac{\cot(x^2)}{2}. \tag{8.7}$$

The scalar curvature is given by

$$r = \left[\frac{-2\csc(x^2)+\cos(x^2)}{4\sin^2(x^2)}\right] = a \text{ (say)}.$$

We now show that M^4 is a $\mathbb{W}_1$-symmetric spacetime, that is, it satisfies the defining relation (1.10).

The only non-vanishing component of $\mathbb{W}_1$ and its covariant derivatives are

$$(\mathbb{W}_1)_{3112} = -\frac{\cos(x^2)}{2}, (\mathbb{W}_1)_{3112,2} = \left[\frac{\sin(x^2)+\sin(x^2)\cos(x^2)+1}{4\sin(x^2)}\right], (\mathbb{W}_1)_{3112,3} = \frac{\cos(x^2)}{4}. \tag{8.8}$$

Let us choose the associated 1-form as

$$A_i = \begin{cases} 0, & \text{for } i = 1,4 \\ -\frac{1}{6}, & \text{for } i = 2 \\ \frac{\sin(x^2)+\sin(x^2)\cos(x^2)+1}{6\sin(x^2)\cos(x^2)}, & \text{for } i = 3. \end{cases} \tag{8.9}$$

at any point $x \in M^4$. In our, (M^4, g), (1.10) reduces with these 1-forms to the following equations:

$$(\mathbb{W}_1)_{3112.2} = 2A_2(\mathbb{W}_1)_{3112} + A_3(\mathbb{W}_1)_{2112} + A_1(\mathbb{W}_1)_{3212}$$
$$+A_1(\mathbb{W}_1)_{3122} + A_2(\mathbb{W}_1)_{3112}. \tag{8.10}$$

Similarly,

$$(\mathbb{W}_1)_{3112.3} = 2A_2(\mathbb{W}_1)_{3112} + A_3(\mathbb{W}_1)_{3112} + A_1(\mathbb{W}_1)_{3312}$$
$$+ A_1(\mathbb{W}_1)_{3132} + A_2(\mathbb{W}_1)_{3113}. \tag{8.11}$$

It can be easily verified that equations (8.10) and (8.11) are satisfied. Hence, the manifold under consideration is a pseudo $\mathbb{W}_1$-symmetric spacetime, that is, a $(P\mathbb{W}_1S)_4$.

Theorem 8.1: Let (M^4, g) be be Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = \sin(x^2)(dx^1)^2 + e^{(x^3)}(dx^2)^2 - (dx^3)^2 + (dx^4)^2,$$

where $\sin(x^2)$ and $e^{(x^3)}$ are non-zero. Then (M^4, g) is a $(P\mathbb{W}_1S)_4$ spacetime.

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