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Riemannian manifolds admitting pseudo W_2-curvature tensor

Shobhit Srivastava and Rajeev Kumar Singh
Department of Mathematics, P.B.G. College, Pratapgharh, India
Email: shobhit2830@gmail.com & rajeevpbh5959@gmail.com

Abstract

In this paper, we obtaine some basic geometrical properties of pseudo W_2-curvature tensor. Then, we study pseudo $'\widetilde{W}_2$ symmetric $(P'\widetilde{W}_2S)_n$ manifold which recover some known results of De and Ghosh (1994). We also provide several interesting results of $(P'\widetilde{W}_2S)_n$. Moreover, we deal with pseudo W_2-flat perfect fluid and dust fluid spacetimes respectively. As a consequence, we obtain some important theorems. Finally, we construct a non-trivial Lorentzian metric of $(P'\widetilde{W}_2S)_4$.

Keywords Pseudo W_2-curvature tensor, pseudo $'\widetilde{W}_2$ symmetric, pseudo W_2-flat perfect fluid, pseudo W_2-flat dust fluid spacetimes.

1. Introduction

In differential geometry, the investigation of curvature characteristics is the prime problem among others. In this context, S.S. Cherm had uttered in 1990. "A fundamental notion is curvature, in its different forms". Hence, the discovery of the Riemann curvature tensor creates an extremely significant subject matter.

According to Chaki (1987), for a non-vanishing 1-form $\mathcal{A}$, a non-flat Riemannian or semi-Riemannian manifold (M^n, g), $n > 2$, is named pseudo symmetric if its curvature tensor obeys

$$(\mathcal{D}_X\mathcal{R})(Y,Z,U,V) = 2\mathcal{A}(X)\mathcal{R}(Y,Z,U,V) + \mathcal{A}(Y)\mathcal{R}(X,Z,U,V) + \mathcal{A}(Z)\mathcal{R}(Y,X,U,V) + \mathcal{A}(U)\mathcal{R}(Y,Z,X,V) + \mathcal{A}(V)\mathcal{R}(Y,Z,U,X), \tag{1.1}$$

where $\mathcal{D}$ is the Levi-Civita connection and $\mathcal{R}(Y,Z,U,V) = g(R(Y,Z)U,V)$, R and R being the curvature of the type (1, 3) and (0, 4) respectively. Let ρ be the associated vector field corresponding to the one 1-form $\mathcal{A}$ i.e. $g(X,\rho) = \mathcal{A}(X)$, for all X. Pseudo symmetric manifolds have been investigated by several authors.

In a Riemannian or a semi-Riemannian manifold the Ricci tensor B is said to be of Codazzi type Gray (1978) if the covariant derivative of the Ricci tensor satisfies

$$(\mathcal{D}_XB)(Y,Z) = (\mathcal{D}_YB)(X,Z). \tag{1.2}$$

A non-flat Riemannian or semi-Riemannian manifold following the condition:

$$(\mathcal{D}_XB)(Y,Z) = \alpha(X)B(Y,Z) + \beta(X)g(Y,Z), \tag{1.3}$$

where α and β are two non-zero 1-forms, is named a generalized Ricci-recurrent manifold De, Guha and Kamily (1995). The manifold becomes a Ricci recurrent when $\beta = 0$.

Lorentzian manifold is a special category of Riemannian manifold with a Lorentzian manifold g. The spacetime of general relativity (GR) is nothing but a time-oriented connected Lorentzian manifold (M^n, g) with the signature $(-,+,+,+)$. The study of the casual character of vectors of the Lorentzian manifold is started with the geometry of the Lorentz metric.

A perfect fluid spacetime, the energy-momentum tensor (EMT) of the type (0, 2) is of the form Neill (1983)

$$\mathcal{T}(X,Y) = (\sigma + p)C(X)C(Y), \tag{1.4}$$

where σ and p stands for the energy density and the isotropic pressure, respectively and ρ is a unit time vector field $(g(\rho,\rho) = -1)$, metrically to the one-form C.

The Einstein's field equation (EFE) without cosmological constant is written by

$$B(X,Y) - \frac{r}{2}g(X,Y) = k\,\mathcal{T}(X,Y), \tag{1.5}$$

where r and k are scalar curvature and the gravitational constant.

In 2016, a spacetime admitting a pseudo-projective curvature tensor have been investigated by Mallick, Suh and De. Moreover, Mantica and Suh (2014) have studied pseudo Z-symmetric spacetimes and Zengin (2012) have studied M-projectively flat spacetimes. Recently, Zhao etal. (2021) investigated a pseudo symmetric spacetime and obtained a condition for which it would be a perfect fluid spacetime (PFS). Also, numerous authors in many ways have investigated spacetimes of GR.

Pokhariya and Mishra (1970) introduced a new curvature tensor in a Riemannian or semi-Riemannian manifold (M^n, g) denoted by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{1.6}$$

where Q is a Ricci operator defined by $g(QX,Y) = B(X,Y)$.

The authors (1970) investigated the relativistic importance of this kind of tensor. Some researches investigated $W_2(X,Y)Z$ in Riemannian manifold and contact manifolds, such as Prasad (1997), Zengin (2011) and Venkatesha et al. (2018) and many others. The W_2-curvature tensor was introduced on the line of projective curvature tensor. It is shown that Pokhariyal and Mishra (1970) except the vanishing of complexion vector and properly of being identical in two spaces which are in geodesic correspondence, the W_2-curvature possesses the properties, almost similar to the projective curvature tensor. Thus, we can use W_2-curvature tensor in various physical and geometrical sphere in place of the projective curvature tensor.

A few years ago, Prasad and Maurya in 2005 defined pseudo W_2-curvature tensor $\widetilde{W}_2$ as follows

$$\widetilde{W}_2(X,Y)Z = \tilde{a}\,R(X,Y)Z + \tilde{b}[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{\tilde{a}}{n-1} + \tilde{b}\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.7}$$

where $\tilde{a}$ and $\tilde{b}$ are constant such that $\tilde{a}, \tilde{b} \neq 0$ and r is the scalar curvature.

It can also be put as

$$\widetilde{W}_2(X,Y)Z = -(n-1)W_2(X,Y)Z + \left[\tilde{a} + (n-1)\tilde{b}\right]L(X,Y)Z,$$

where L is the concircular curvature tensor.

It can be easily verified that

$$\widetilde{W}_2(X,Y)Z + \widetilde{W}_2(Y,X)Z = 0,$$

and

$$\widetilde{W}_2(X,Y)Z + \widetilde{W}_2(Y,Z)X + \widetilde{W}_2(X,Y)Z + \widetilde{W}_2(Z,X)Y = 0. \quad (1.8)$$

If $\tilde{a} = 1$ and $\tilde{b} = -\frac{1}{n-1}$, then (1.7) takes the form

$$\widetilde{W}_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX] = W_2(X,Y)Z,$$

where W_2-curvature tensor Pokhariyal and Mishra (1970). Hence cuvature tensor W_2 is a particular case of the tensor $\widetilde{W}_2$. For this reason $\widetilde{W}_2$ is called pseudo W_2-curvature tensor $\widetilde{W}_2$.

We can express (1.7) as follows:

$$'\widetilde{W}_2(X,Y,U,V) = \tilde{a}\mathcal{R}(X,Y,U,V) + \tilde{b}[g(Y,U)B(X,V) - g(X,U)B(Y,V)] - \frac{r}{n}\left(\frac{\tilde{a}}{n-1} + \tilde{b}\right)[g(Y,U)g(X,V) - g(X,U)g(Y,V)], \quad (1.9)$$

where

$$'\widetilde{W}_2(X,Y,U,V) = g(W_2(X,Y)U,V) \text{ and } \mathcal{R}(X,Y,U,V) = g(R(X,Y)U,V).$$

A non-flat Riemannian manifold (M_n, g), $n > 2$ is said to be a pseudo $'\widetilde{W}_2$ of type (0, 4) satisfies the condition:

$$(\mathcal{D}_X{}'\widetilde{W}_2)(Y,Z,U,V) = 2\mathcal{A}(X){}'\widetilde{W}_2(Y,Z,U,V) + \mathcal{A}(Y){}'\widetilde{W}_2(X,Z,U,V) + \mathcal{A}(Z){}'\widetilde{W}_2(Y,X,U,V) + \mathcal{A}(U){}'\widetilde{W}_2(Y,Z,X,V) + \mathcal{A}(V){}'\widetilde{W}_2(Y,Z,U,X), \quad (1.10)$$

where $\mathcal{A}$ is a non-zero 1-form ρ is a vector field defined by

$$g(X,\rho) = \mathcal{A}(X).$$

An n-dimensional pseudo $'\widetilde{W}_2$ symmetric manifold is denoted by $(P'\widetilde{W}_2S)_n$, $n > 2$, where ρ stands for pseudo, $'\widetilde{W}_2$ stands for pseudo W_2-curvature tensor, and S denotes for symmetric.

In particular, if $\tilde{a} = 1$, $\tilde{b} = -\frac{1}{n-1}$, then pseudo $'\widetilde{W}_2$ symmetric manifold reduces to pseudo W_2 symmetric manifolds studies by De and Ghosh in 1994.

The present paper is organized as follows:

After introduction in section 2, we study some basic geometric properties of pseudo W_2-curature tensor. Section 3 is devoted to the study of curvature properly of $(P'\widetilde{W}_2S)_n$. In section 4, we study Einstein $(P'\widetilde{W}_2S)_n$ manifold is of zero scalar curvature tensor. Section 5 deals with pseudo W_2-flat spacetime. Moreover in section 6 and 7 we consider pseudo W_2-flat perfect fluid and dust fluid spacetimes respectively. Finally we construct a non-trivial Lorentzian metric of $(P'\widetilde{W}_2S)_4$.

2. Preliminaries

In this section, some basic formulas are derived, which will be useful to study of $(P'\widetilde{W}_2S)_n$. Let $\{E_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq E_i \leq n$. In a Riemannian manifold the Ricci tensor B is defined by

$$B(X,Y) = \sum_{i=1}^{n} {}'R(X,E_i,E_i,Y), \text{ where } g(E_i,E_i) = \pm 1.$$

From (1.9), we get

$$\sum_{i=1}^{n} {'\widetilde{W}_2}(X,Y,E_i,E_i) = \sum_{i=1}^{n} {'\widetilde{W}_2}(E_i,E_i,U,V) = 0, \tag{2.1}$$

$$\sum_{i=1}^{n} {'\widetilde{W}_2}(E_i,Y,U,E_i) = \left(\tilde{a} - \tilde{b}\right)\left[B(Y,U) - \frac{r}{n} g(Y,U)\right] = W_3(Y,U), \tag{2.2}$$

$$\sum_{i=1}^{n} {'\widetilde{W}_2}(X,E_i,E_i,V) = [\tilde{a} + \tilde{b}(n-1)]\left[B(X,V) - \frac{r}{n} g(X,V)\right] = W_4(X,V), \tag{2.3}$$

$$'\widetilde{W}_2(X,Y,U,V) + {'\widetilde{W}_2}(Y,X,U,V) = 0, \tag{2.4}$$

$$'\widetilde{W}_2(X,Y,U,V) + {'\widetilde{W}_2}(X,Y,V,U) \neq 0, \tag{2.5}$$

$$'\widetilde{W}_2(X,Y,U,V) - {'\widetilde{W}_2}(U,V,X,Y) \neq 0,$$

and

$$'\widetilde{W}_2(X,Y,U,V) + {'\widetilde{W}_2}(Y,U,X,V) + {'\widetilde{W}_2}(U,X,Y,V) = 0. \tag{2.6}$$

Proposition 2.1: If pseudo $\widetilde{W}_2$ on a Riemannian manifold is flat, then it is Einstein manifold, provided $[\tilde{a} + \tilde{b}(n-1)] \neq 0$.

Proof: For $'\widetilde{W}_2$ flat manifold, we have

$$'\widetilde{W}_2(X,Y,U,V) = 0. \tag{2.7}$$

Hence, in view of (1.9), we get

$$\tilde{a}'\mathcal{R}(X,Y,U,V) = -\tilde{b}[g(Y,U)B(X,U) - g(X,U)B(Y,V)] + \frac{r}{n}\left(\frac{\tilde{a}}{n-1} + \tilde{b}\right)[g(Y,U)g(X,V) - g(X,U)g(Y,V)]. \tag{2.8}$$

Put E_i for Y and U in (2,8), we get

$$B(X,V) = \frac{r}{n} g(X,V), \text{ provided } \tilde{a} + \tilde{b}(n-1) \neq 0. \tag{2.9}$$

This proves proposition (2.1).

Proposition 2.2: If the pseudo W_2-curvature tensor is symmetric in the sence of Carten, then the manifold becomes Ricci symmetric if and only if $dr(X) = 0$, provided $(\tilde{a} - \tilde{b}) \neq 0$.

Proof: Differentiating covariantly (1.7), we get

$$\left(\mathcal{D}_X{'\widetilde{W}_2}\right)(Y,Z)U = \tilde{a}(\mathcal{D}_X R)(Y,Z)U + \tilde{b}[(\mathcal{D}_X Q)(Y)g(Z,U) - (\mathcal{D}_X Q)(Z)g(Y,U)] - \frac{dr(X)}{n}\left(\frac{\tilde{a}}{n-1} + \tilde{b}\right)[g(Z,U)Y - g(Y,U)Z]. \tag{2.10}$$

According to our assumption, we get from (2.10).

$$\tilde{a}(\mathcal{D}_X R)(Y,Z)U + \tilde{b}[(\mathcal{D}_X Q)(Y)g(Z,U) - (\mathcal{D}_X Q)(Z)g(Y,U)] - \frac{dr(X)}{n}\left(\frac{\tilde{a}}{n-1} + \tilde{b}\right)[g(Z,U)Y - g(Y,U)Z] = 0. \tag{2.11}$$

Contracting (2.11) with respect to Y, we get

$$(\mathcal{D}_X B)(Z,U) = \frac{dr(X)}{n} g(Z,U), \quad \tilde{a} - \tilde{b} \neq 0. \tag{2.12}$$

Now, we assume that

$$dr(X) = 0. \tag{2.13}$$

Hence, from (2.12), we get

$$(\mathcal{D}_X B)(Z,U) = 0. \tag{2.14}$$

Conversely, from (2.14) and (2.12), we get $dr(X) = 0$.

This proves the theorem.

Preposition 2.3: On a pseudo W_2-curvature tensor, divergence of $\widetilde{W}_2$ is equivalent to divergence of W_2 if and only if scalar curvature is constant.

Proof: Contracting (2.10) with respect to X, we get

$$\left(div\,'\widetilde{W}_2\right)(Y,Z)U = \tilde{a}(divR)(Y,Z)U + \left[\frac{\tilde{b}}{2} - \frac{\tilde{a}+\tilde{b}(n-1)}{n(n-1)}\right][g(Z,U)dr(Y)$$
$$-g(Y,U)dr(Z)]. \tag{2.15}$$

From (1.6), we have

$$(divW_2)(Y,Z)U = (divR)(Y,Z)U + \frac{1}{2(n-1)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \tag{2.16}$$

From (2.15) and (2.16), we get

$$\left(div\widetilde{W}_2\right)(Y,Z)U = (divW_2)(Y,Z)U + \left[\frac{\tilde{b}(n-2)}{2n} - \frac{\tilde{a}(n+2)}{2n(n-1)}\right][g(Z,U)dr(Y)$$
$$-g(Y,U)dr(Z)]. \tag{2.17}$$

Now, if scalar curvature is constant i.e. $dr(Y) = 0$, then from (2.17), we get

$$\left(div\widetilde{W}_2\right)(Y,Z)U = (divW_2)(Y,Z)U. \tag{2.18}$$

Conversely, if $\left(div\widetilde{W}_2\right)(Y,Z)U = (divW_2)(Y,Z)U$, then from (2.17), we get

$$dr(Y) = 0.$$

This proves the statement.

3. Algebraic properties of $(P'\widetilde{W}_2S)_n, n > 2.$

In this section, we prove that in $(P'\widetilde{W}_2S)_n$, $n > 2$, pseudo W_2-curvature tensor satisfies Bianch's second identity.

In view of (1.9) and (1.10), we get

$$\left(\mathcal{D}_X{}'\widetilde{W}_2\right)(Y,Z,U,V) + \left(\mathcal{D}_Y{}'\widetilde{W}_2\right)(Z,X,U,V) + \left(\mathcal{D}_Z{}'\widetilde{W}_2\right)(X,Y,U,V)$$
$$= \mathcal{A}(U)[{}'\widetilde{W}_2(X,Y,U,V) + {}'\widetilde{W}_2(Z,X,Y,V) + {}'\widetilde{W}_2(X,Y,Z,V) +$$
$$\mathcal{A}(V)[{}'\widetilde{W}_2(Y,Z,U,X) + {}'\widetilde{W}_2(Z,X,U,Y) + {}'\widetilde{W}_2(X,Y,U,Z). \tag{3.1}$$

Again, using (1.9) and (2.6) in (3.1), we get

$$\left(\mathcal{D}_X{}'\widetilde{W}_2\right)(Y,Z,U,V) + \left(\mathcal{D}_Y{}'\widetilde{W}_2\right)(Z,X,U,V) + \left(\mathcal{D}_Z{}'\widetilde{W}_2\right)(X,Y,U,V) = 0. \tag{3.2}$$

Thus, we can state the following theorem:

Theorem 3.1: Pseudo W_2-curvature tensor in $(P'\widetilde{W}_2S)_n, n > 2$, satisfies the Bianch's second identity.

Moreover, in view of (1.9), we get

$$\left(\mathcal{D}_X{}'\widetilde{W}_2\right)(Y,Z,U,V) + \left(\mathcal{D}_Y{}'\widetilde{W}_2\right)(Z,X,U,V) + \left(\mathcal{D}_Z{}'\widetilde{W}_2\right)(X,Y,U,V)$$
$$b[\{(\mathcal{D}_X B)(Y,V) - (\mathcal{D}_Y B)(X,V)\}g(Z,U) + \{(\mathcal{D}_Z B)(X,V) - (\mathcal{D}_X B)(Z,V)\}g(Y,U)$$

$$+\{(\mathcal{D}_Y B)(Z,V)-(\mathcal{D}_Z B)(Y,V)\}g(X,U)]-\frac{dr(X)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]-\frac{dr(Y)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(X,U)g(Z,V)-g(Z,U)g(X,V)]-\frac{dr(Z)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Y,U)g(X,V)-g(X,U)g(Y,V)]. \quad (3.3)$$

We assume that $(P'\widetilde{W}_2S)_n, n>2$ admits Codazzi type Ricci tensor, then from (3.3), we get

$$\left(\mathcal{D}_X{}'\widetilde{W}_2\right)(Y,Z,U,V)+\left(\mathcal{D}_Y{}'\widetilde{W}_2\right)(Z,X,U,V)+\left(\mathcal{D}_Z{}'\widetilde{W}_2\right)(X,Y,U,V)$$
$$=-\frac{dr(X)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]$$
$$-\frac{dr(Y)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(X,U)g(Z,V)-g(Z,U)g(X,V)]$$
$$-\frac{dr(Z)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Y,U)g(X,V)-g(X,U)g(Y,V)]. \quad (3.4)$$

Using (3.2) and (3.4), we get

$$\frac{dr(X)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]+$$
$$\frac{dr(Y)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(X,U)g(Z,V)-g(Z,U)g(X,V)]+$$
$$\frac{dr(Z)}{n}\left(\frac{\tilde{a}}{n-1}+\tilde{b}\right)[g(Y,U)g(X,V)-g(X,U)g(Y,V)]=0. \quad (3.5)$$

Contracting of (3.5) gives

$$\left[\tilde{a}+(n-1)\tilde{b}\right]dr(X)=0\Rightarrow dr(X)=0, \text{ provided } \tilde{a}+(n-1)\tilde{b}\neq 0. \quad (3.6)$$

In view of (3.6), we have the following theorem:

Theorem 3.2: If $(P'\widetilde{W}_2S)$ admits Codazzi type Ricci tensor and Bianch's second identity, then scalar curvature is constant, provided $\tilde{a}+(n-1)\tilde{b}\neq 0$.

4. Einstein $(P'\widetilde{W}_2S)_n, n>2$

In this section, we consider an Einstein $(P'\widetilde{W}_2S)_n, n>2$.

An Einstein manifold is defined by

$$B(X,Y)=\frac{r}{n}g(X,Y), \quad (4.1)$$

for which the scalar curvature is constant. From (4.1), we get

$$(D_X B)(X,Y)=0 \ \Rightarrow dr=0. \quad (4.2)$$

From (1.9), (4.1) and (4.2), we get

$$\left(\mathcal{D}_X{}'\widetilde{W}_2\right)(Y,Z,U,V)=\tilde{a}(\mathcal{D}_X R)(Y,Z,U,V). \quad (4.3)$$

From (1.10) and (4.3), we get

$$2\mathcal{A}(X)'\widetilde{W}_2(Y,Z,U,V)+\mathcal{A}(Y)'\widetilde{W}_2(X,Z,U,V)+\mathcal{A}(Z)'\widetilde{W}_2(Y,X,U,V)+$$
$$\mathcal{A}(U)'\widetilde{W}_2(Y,Z,X,V)+\mathcal{A}(V)'\widetilde{W}_2(Y,Z,U,X)=\tilde{a}[2\mathcal{A}(X)\mathcal{R}(Y,Z,U,V)+$$
$$\mathcal{A}(Y)\mathcal{R}(X,Z,U,V)+\mathcal{A}(Z)\mathcal{R}(Y,X,U,V)+\mathcal{A}(U)\mathcal{R}(Y,Z,X,V)$$
$$+\mathcal{A}(V)\mathcal{R}(Y,Z,U,X)]. \quad (4.4)$$

Now using (4.1) in (1.9), we get

$$'\widetilde{W}_2(Y,Z,U,V) = \tilde{a}[\mathcal{R}(Y,Z,U,V) - \frac{r}{n(n-1)}\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}]. \quad (4.5)$$

From (4.4) and (4.5), we get

$$\frac{r\tilde{a}}{n(n-1)}[2\mathcal{A}(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} + \mathcal{A}(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\} + \mathcal{A}(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\} + \mathcal{A}(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\} + \mathcal{A}(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}] = 0.$$

That is,

$$r\tilde{a}[2\mathcal{A}(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} + \mathcal{A}(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\} + \mathcal{A}(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\} + \mathcal{A}(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\} + \mathcal{A}(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}] = 0. \quad (4.6)$$

Putting E_i for Y and V in (4.6) and summing over $1 \leq E_i \leq n$, we get

$$r\tilde{a}[(n-1)2\mathcal{A}(X)g(Z,U) + g(Z,U)\mathcal{A}(X) - g(X,U)\mathcal{A}(Z) + (n-1)\mathcal{A}(Z)g(X,U) + (n-1)2\mathcal{A}(U)g(X,Z) + \mathcal{A}(X)g(Z,U) - \mathcal{A}(U)g(Z,X)] = 0. \quad (4.7)$$

Again, putting E_i for Z and U in equation (4.7), we get

$$r.\,\tilde{a}\mathcal{A}(X)(n^2 + 2n - 3) = 0,$$

$$\Rightarrow \quad r = 0,\ \mathcal{A}(X) \neq 0,\ \tilde{a} \neq 0. \quad (4.8)$$

Hence, we can state the following theorem:

Theorem 4.1: An Einstein $(P'\widetilde{W}_2S)_n$, $n > 2$, manifold is of zero scalar curvature.

5. Spacetime with vanishing pseudo W_2-curvature tensor $\widetilde{W}_2$.

Let (M^n, g) be the spacetime of GR, then from (1.9)

$$'\widetilde{W}_2(X,Y,Z,W) = \tilde{a}\mathcal{R}(X,Y,Z,W) + \tilde{b}[g(X,Z)B(Y,W) - g(Y,Z)B(X,W)] - \frac{r}{4}\left(\frac{\tilde{a}}{3} + \tilde{b}\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (5.1)$$

If $'\widetilde{W}_2(X,Y,Z,W) = 0$, then (5.1) gives

$$\tilde{a}\mathcal{R}(X,Y,Z,W) + \tilde{b}[g(X,Z)B(Y,W) - g(Y,Z)B(X,W)] + \frac{r}{4}\left(\frac{\tilde{a}}{3} + \tilde{b}\right)[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] = 0. \quad (5.2)$$

Taking a frame field over X and W, we get from (5.2)

$$\left(\tilde{a} - \tilde{b}\right)\left[B(Y,Z) - \frac{r}{4}g(Y,Z)\right] = 0. \quad (5.3)$$

Thus, we can state the following theorem:

Theorem 5.1: A pseudo W_2-flat spacetime is an Einstein spacetime, provided $\tilde{a} - \tilde{b} \neq 0$.

From (5.2) and (5.3), we get

$$\mathcal{R}(X,Y,Z,W) = \frac{r}{12}[g(Y,Z)g(X,W) - g(Y,Z)g(X,W)]. \tag{5.4}$$

Hence, we can state the following theorem:

Theorem 5.2: A pseudo W_2-flat spacetime is a spacetime of constant curvature, provided $\tilde{a} - \tilde{b} \neq 0$.

Let us consider a spacetime satisfying the Einstein field equation (EFE) with cosmological constant λ as

$$B(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k\mathcal{T}(X,Y). \tag{5.5}$$

Using (5.3) and (5.5), we get

$$\mathcal{T}(X,Y) = \frac{1}{k}\left[\lambda - \frac{r}{4}\right]g(X,Y). \tag{5.6}$$

Taking the covariant derivative of (5.6), we get

$$(D_Z\mathcal{T})(X,Y) = -\frac{1}{4k}dr(Z)g(X,Y). \tag{5.7}$$

Since pseudo W_2-flat spacetime is Einstein, therefore scalar curvature is constant. That is,

$$dr(X) = 0, \tag{5.8}$$

for all X.

Hence (5.7) and (5.8) together gives

$$(D_Z\mathcal{T})(X,Y) = 0. \tag{5.9}$$

Thus, we can state the following theorem:

Theorem 5.3: In a pseudo W_2-flat spacetime with non-zero scalar curvature satisfying EFE with cosmological constant, then EMT is covariant constant.

In 1996, Chaki et al. proved that in a general relativistic spacetime $\mathcal{D}\mathcal{T} = 0 \Leftrightarrow \mathcal{D}B$. Thus we can state the following:

Corollary (5.1). In a pseudo W_2-flat spacetime with non-zero scalar curvature $\mathcal{D}\mathcal{T} = 0$ and $\mathcal{D}B = 0$ are equivalent.

6. Pseudo W_2-flat perfect fluid spacetimes

Now, we consider the matter distribution is perfect fluid whose velocity vector field is the vector field ρ corresponding to the 1-form $\mathcal{A}$ of the spacetime. Therefore the EMT of type (0, 2) is of the form O'Neill (1983):

$$\mathcal{T}(X,Y) = pg(X,Y) + (\sigma + p)\mathcal{A}(X)\mathcal{A}(Y), \tag{6.1}$$

where σ and p are the energy density and the isotropic pressure respectively. Hence from the EFE, we get

$$B(X,Y) - \frac{r}{2}g(X,Y) = k[pg(X,Y) + (\sigma + p)\mathcal{A}(X)\mathcal{A}(Y)]. \tag{6.2}$$

Contracting (6.2) X and Y, we get

$$r = k(\sigma - 3p). \tag{6.3}$$

Using (2.9) in (6.3), we get

$$-\frac{r}{6}g(X,Y) = k[pg(X,Y) + (\sigma + p)\mathcal{A}(X)\mathcal{A}(Y)]. \tag{6.4}$$

Setting $Y = \rho$ in (6.4) and solving them, we get

$$r = 6k\sigma. \tag{6.5}$$

From (6.3) and (6.5), we get

$$5\sigma + 3\rho = 0, \;\; k \neq 0. \tag{6.6}$$

Thus in view of the above equation we can state the following theorem:

Theorem 6.1: In a pseudo W_2-flat perfect fluid spacetime with non-zero scalar curvature following EFE without cosmological constant the energy density and the isotropic pressure are given by $5\sigma + 3p = 0$, $k \neq 0$, $a + (n-1)b \neq 0$.

Remark: In a pseudo W_2-flat spacetime with non-zero scalar curvature tensor, we have $\sigma = -\frac{3}{5}p$. It follows that $p = -\frac{5}{3}\sigma$, that is, of the form $p = p(\sigma)$.

Hence, we conclude that fluid is isentropic Hawking and Ellis (1973).

7. Dust fluid spacetime with vanishing pseudo W_2-curvature tensor $\widetilde{W}_2$

In a dust or pressureless fluid spacetime, the EMT is in the form (2008)

$$\mathcal{T}(X,Y) = \sigma\mathcal{A}(X)\mathcal{A}(Y), \tag{7.1}$$

where σ is the energy density of the dust-like matter and $\mathcal{A}$ is a non-zero 1-form such that $g(X,U) = \mathcal{A}(X)$, for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$.

Using (5.6) and (7.1), we get

$$\left(\lambda - \frac{r}{4}\right) g(X,Y) = k\sigma\mathcal{A}(X)\mathcal{A}(Y). \tag{7.2}$$

A frame field after contraction over X and Y gives to

$$\lambda = \frac{r}{4} - \frac{k\sigma}{4}. \tag{7.3}$$

Again, if we put $X = Y = U$ in (7.2), we get

$$\lambda = \frac{r}{4} - k\sigma. \tag{7.4}$$

Hence, from (7.3) and (7.4), we obtain

$$\sigma = 0. \tag{7.5}$$

Thus, from (7.1) and (7.5), we conclude that

$$\mathcal{T}(X,Y) = 0.$$

This means that the spacetime is devoid of the matter.

Thus, we can state the following theorem:

Theorem 7.1: A pseudo W_2-flat dust fluid spacetime satisfying Einstein's field equation with cosmological constant is vacuum.

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