ISSN:0976-4933
Journal of Progressive Science
A Peer-reviewed Research Journal
Vol.16, No.01, pp 27--32 (2025)
https://doi.org/10.21590/jps.16.01.04

Semi-symmetric non-metric connection on Sasakian manifold

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Abstract

The prime goal of this paper is to investigate Ricci solitons on quasi-concircularly flat, pseudo W_2-flat and pseudo W_8-flat Sasakian manifold using a type of semi-symmetric non-metric connection $\bar{D}$.

Keywords Sasakian manifold, Semi-symmetric non-metric connection, various different curvature tensors, Ricci solitons.

1. Introduction

Let M^n be a n-dimensional Riemannian manifold and let D denote the Levi-Civita connection corresponding to the metric g on the manifold. A linear connection $\bar{D}$ on M^n is said to be symmetric if torsion tensor $\bar{T}$ of $\bar{D}$ defined by

$$\bar{T}(X,Y) = \bar{D}_X Y - \bar{D}_Y X - [X,Y],$$

is zero for all X and Y on M^n; otherwise, it is semi-symmetric. In 1924, Friedmann and Schouten introduced the idea of a semi-symmetric linear connection on a differentiable manifold. A linear connection $\bar{D}$ on M^n is said to be semi-symmetric if

$$\bar{T}(X,Y) = (\alpha - \beta)[\eta(X)Y - \eta(Y)X], \tag{1.1}$$

holds for all vector fields $X$and Y on M^n, where η is a 1-form associated with the vector field ξ and satisfies

$$g(X,\xi) = \eta(X). \tag{1.2}$$

In 1932, Hayden gave the idea of metric connection $\bar{D}$ on a Riemannian manifold and later named such connection a Hayden connection. After a long gap, Pak (1969) considered the Hayden connection $\bar{D}$ equipped with the torsion tensor $\bar{T}$ defined as (1.1) and proved that it is a semi-symmetric metric connection. A linear connection $\bar{D}$ is said to be a metric on M^n if $\bar{D}_g = 0$; otherwise, it is non-metric. A systematic study of the semi-symmetric metric connection $\bar{D}$ on a Riemannian manifold was initiated by Yano (1970). In 1992, Agashe and Chafle introduced a new class of semi-symmetric connection, called the semi-symmetric non-metric connection on a Riemannian manifold and studied some of its geometric properties. Riemannian manifold with a special type of semi-symmetric non-metric connection was defined by Chaturvedi and Pandey in 2009.

Prasad and Maurya (2004 and 2007) explored the quasi concircular curvature tensor, pseudo W_2-curvature on an n-dimensional Riemannian manifold as follows:

$$V(X,Y)Z = a\,R(X,Y)Z + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

$$W_2(X,Y)Z = a\,R(X,Y)Z + b[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.4}$$

where a, b are non-zero constant, R, Ric, Q and r denote the Riemannian curvature tensor, Ricci tensor Q is the Ricci operator and r is the scalar curvature defined by

$$g(QX,Y) = Ric(X,Y).$$

In a similar fashion, Prasad, Yadav and Pandey defined an another new type of curvature tensor in 2018 as follows:

$$W_8(X,Y)Z = aR(X,Y)Z + b[Ric(X,Y)Z - Ric(Y,Z)X] - \frac{r}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)Z - g(Y,Z)X]. \tag{1.5}$$

If $V(X,Y)Z = 0$, $W_2(X,Y)Z = 0$, $W_4(X,Y)Z = 0$ and $W_8(X,Y)Z = 0$, then all are flat manifold.

The definition of a Ricci solitons on the Riemannian manifold M^n is expressed by Cho and Kimura (2009) as:

$$(\mathcal{L}_\xi\, g)(X,Y) + 2Ric(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.6}$$

for all X, Y belongs to the manifold, $(\mathcal{L}_\xi\, g)$ is the Lie derivative and λ is a real constant.

If $\lambda < 0$, then the Ricci solitons is shrinking.

If $\lambda > 0$, then the Ricci solitons is expanding.

If $\lambda = 0$, then the Ricci solitons is steady.

2. Sasakian manifolds

An $n = (2m+1)$-dimensional smooth manifold M^n is announced as an almost contact metric manifold if it admits a 1-form η, a (1,1) tensor field ϕ, a Riemannian metric g and a contravariant vector field ξ which fulfill the following condition:

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\quad \phi\xi = 0,\quad \eta(\phi\xi) = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),\quad g(\phi X,Y) = -g(X,\phi Y),$$

$g(X,\xi) = \eta(X)$, for all $X, Y \in M^n$.

An almost contact metric manifold $(M^n, g, \eta, \xi, \phi)$ is named a contact manifold if

$$g(X,\phi Y) = d\eta(X,Y). \tag{2.2}$$

Also, an almost contact metric manifold $(M^n, g, \eta, \xi, \phi)$ is named normal if

$$2d\eta(X,Y)\xi + [\phi,\phi](X,Y) = 0, \tag{2.3}$$

where $[\phi,\phi]$ denotes the Nijenhuis tensor.

A normal contact metric manifold is called a Sasakian manifold. For a Sasakian manifold, the following relations are satisfied: Sasaki (1965), Yano and Kon (1984),

$$D_X\xi = -\phi X, \tag{2.4)a}$$

$$(D_X\eta)(Y) = -g(\phi X, Y), \tag{2.4)b}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.4)c}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.4)d}$$

$$Ric(X,\xi) = (n-1)\eta(X), \tag{2.5)d}$$

$$Ric(\phi X, \phi Y) = Ric(X,Y) - (n-1)\eta(X)\eta(Y),$$

and

$$Q\xi = (n-1)\xi.$$

3. Semi-symmetric non-metric connection $\overline{D}$ on Sasakian manifold

Let (M^n, g) be a Riemannian manifold of dimension endowed with a Levi-Civita connection D corresponding to the Riemannian metric g. A linear connection $\overline{D}$ defined by Chaturvedi and Pandey (2009) as

$$\overline{D}_X Y = D_X Y + \alpha\eta(X)Y + \beta\eta(Y)X, \tag{3.1}$$

for arbitrary vector fields X and Y on M^n and α, β are real constants is said to be semi-symmetric non-metric connection, where torsion tensor satisfies the equations (1.1) and the metric g holds the relation:

$$(\overline{D}_X g)(Y,Z) = -2\alpha\eta(X)g(Y,Z) - \beta\eta(Y)g(X,Z) - \beta\eta(Z)g(X,Y), \tag{3.2}$$

for all vector fields X, Y, Z on M^n.

The curvature tensor with respect to semi-symmetric non-metric connection $\overline{D}$ is given by

$$\overline{R}\,(X,Y)Z = R(X,Y)Z + \alpha[(D_X\eta)(Y) - (D_Y\eta)(X)]Z + \beta[(D_X\eta)(Z)Y - (D_Y\eta)(Z)X] + \beta^2[\eta(Y)X - \eta(X)Y]\eta(Z), \tag{3.3}$$

where $\overline{R}\,(X,Y)Z$ and $R(X,Y)Z$ are the curvature tensor with respect to $\overline{D}$ and D.

In view of (2.2), (2.4)b and (3.3), we get

$$\overline{R}\,(X,Y)Z = R(X,Y)Z + 2\alpha d\eta(X,Y)Z - \beta[d\eta(Y,Z)X - d\eta(X,Z)Y] + \beta^2[\eta(Y)X - \eta(X)Y]\eta(Z). \tag{3.4}$$

Contracting (3.4) with respect to X, we get

$$\overline{Ric}(Y,Z) = Ric(Y,Z) - [2\alpha + (n-1)\beta]d\eta(Y,Z) + (n-1)\beta^2\eta(Y)\eta(Z). \tag{3.5}$$

Further, contraction of (3.5) gives

$$\bar{r} = r + (n-1)\beta^2, \tag{3.6}$$

where $\overline{Ric}(Y,Z)$, $Ric(Y,Z)$, $\bar{r}$ and r are Ricci tensor and scalar curvature with respect to $\overline{D}$ and D.

4. Ricci solitons on quasi-concircularly flat Sasakian manifold admitting semi-symmetric non-metric connection $\overline{D}$.

Quasi-concircular curvature tensor V with respect to $\overline{D}$ from (1.5) is given by

$$\overline{V}(X,Y)Z = a\overline{R}(X,Y)Z + \frac{\bar{r}}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \tag{4.1}$$

Here, we assume that $\bar{V}(X,Y)Z = 0$, then from (4.1), we get

$$a\bar{R}(X,Y)Z + \frac{\bar{r}}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \quad (4.2)$$

Contracting (4.2), we get

$$\bar{R}ic(Y,Z) = -\frac{\bar{r}}{an}[a + 2(n-1)b]g(Y,Z). \quad (4.3)$$

In view of (3.5), (3.6) and (4.3), we get

$$Ric(Y,Z) = -\left[\frac{r+(n-1)\beta^2}{an}\right][a + 2(n-1)b]g(Y,Z) -$$
$$(n-1)\beta^2\eta(Y)\eta(Z) + [2\alpha + n - 1]d\eta(Y,Z). \quad (4.4)$$

Hence from (4.4), we have the following theorem:

Theorem 4.1: The quasi-concircularly flat Sasakian manifold with respect to $\bar{D}$ is η-Einstein manifold with respect to D if and only if one form η is closed, provided $a \neq 0$.

Putting ξ for Z in (4.4), and using (2.1) and (2.4)b, we get

$$Ric(Y,\xi) = -\left[\left\{\frac{r+(n-1)\beta^2}{an}\right\}\{a + 2(n-1)b\} + (n-1)\beta^2\right]\eta(Y). \quad (4.5)$$

From (1.7) and (2.1), we get

$$Ric(Y,Z) = -\lambda g(Y,Z). \quad (4.6)$$

Putting ξ for Z in (4.6), we have

$$Ric(Y,\xi) = -\lambda\eta(Y). \quad (4.7)$$

Using (4.7) in (4.5), we obtain

$$\lambda = \left[\frac{a+2(n-1)b}{an}\right]\left[r - \left\{\frac{a(n+1)+2(n-1)b}{a+2(n-1)b}\right\}(n-1)\beta^2\right]. \quad (4.8)$$

Hence from (4.8), we have the following theorem:

Theorem 4.2: For a given quasi-concircularly flat Sasakian manifold admitting Ricci solitons (λ,ξ,g) with respect to $\bar{D}$, the Ricci solitons is steady, shrinking or expanding if

$$r = \left[\frac{a(n+1)+2(n-1)b}{a+2(n-1)b}\right](n-1)\beta^2, \quad r < \left[\frac{a(n+1)+2(n-1)b}{a+2(n-1)b}\right](n-1)\beta^2$$

or $\quad r > \left[\frac{a(n+1)+2(n-1)b}{a+2(n-1)b}\right](n-1)\beta^2,$

provided $a + 2(n-1)b \neq 0$.

5. Ricci solitons on pseudo W_2-flat Sasakian manifold admitting semi-symmetric non-metric connection $\bar{D}$

Pseudo W_2-curvature tensor $\mathcal{W}_2$ with respect to $\bar{D}$ from (1.4) is given by

$$\bar{\mathcal{W}}_2(X,Y)Z = a\bar{R}(X,Y)Z + b[g(Y,Z)\overline{QX} - g(X,Z)\overline{QY}] -$$
$$\frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y]. \quad (5.1)$$

Here we assume that $\bar{\mathcal{W}}_2(X,Y)Z = 0$, then from (5.1), we get

$$a\bar{R}(X,Y)Z + b[g(Y,Z)\overline{QX} - g(X,Z)\overline{QY}] - \frac{\bar{r}}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] = 0. \quad (5.2)$$

Contracting (5.2), we get

$$a\overline{Ric}(Y,Z) + b[g(Y,Z)\bar{r} - \overline{Ric}(Y,Z)] - -\frac{\bar{r}}{n}\left(\frac{a}{n-1} + b\right)(n-1)g(Y,Z) = 0.$$

Solving above equation, we get

$$(a-b)\left[\overline{Ric}(Y,Z) - \frac{\bar{r}}{n}g(Y,Z)\right] = 0. \tag{5.3}$$

Using (3.5), (3.6) in (5.3), we get

$$Ric(Y,Z) = \left[\frac{r+(n-1)\beta^2}{n}\right]g(Y,Z) - (n-1)\beta^2\eta(Y)\eta(Z)$$
$$+[2\alpha + (n-1)\beta]d\eta(Y,Z). \tag{5.4}$$

Hence from (5.4), we have the following theorem:

Theorem 5.1: A Pseudo W_2-flat Sasakian manifold with respect to $\bar{D}$ is η-Einstein with respect to D if and only if one-form η is closed, $a - b \neq 0$.

Putting ξ for Z in (5.4) and using (2.1) and (2.5), we get

$$Ric(Y,\xi) = \left[\frac{r-(n-1)^2\beta^2}{n}\right]\eta(Y). \tag{5.5}$$

In view of (4.7) and (5.5) d, we have the following theorem:

Theorem 5.2: If pseudo W_2-flat Sasakian manifold appreciating a Ricci solitons (λ, ξ, g) with respect to $\bar{D}$. Then, the Ricci solitons is steady, shrinking or expanding if

$$r = (n-1)^2\beta^2,\ r > (n-1)^2\beta^2 \text{ or } r < (n-1)^2\beta^2,\ a - b \neq 0.$$

6. Ricci solitons on pseudo W_8-flat Sasakian manifold with respect to semi-symmetric non-metric connection $\bar{D}$.

Pseudo W_8-curvature tensor $\mathcal{W}_8$ with respect to $\bar{D}$ from (1.5) is given by

$$\bar{\mathcal{W}}_8(X,Y)Z = a\bar{R}(X,Y)Z + b[\overline{Ric}(X,Y)Z - \overline{Ric}(Y,Z)X]$$
$$-\frac{\bar{r}}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)Z - g(Y,Z)X]. \tag{6.1}$$

For flat manifold, we have $\bar{\mathcal{W}}_8(X,Y)Z = 0$. Hence from (6.1), we get

$$a\bar{R}(X,Y)Z + b[\overline{Ric}(X,Y)Z - \overline{Ric}(Y,Z)X] - \frac{\bar{r}}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)Z - g(Y,Z)X] = 0. \tag{6.2}$$

Contracting (6.2), we get

$$\left[\overline{Ric}(Y,Z) + \frac{\bar{r}}{n}g(Y,Z)\right][a - (n-1)b] = 0. \tag{6.3}$$

Using (3.5) and (3.6) in (6.3), we get

$$Ric(Y,Z) = -\left[\frac{r+(n-1)\beta^2}{n}\right]g(Y,Z) - (n-1)\beta^2\eta(Y)\eta(Z)$$
$$+[2\alpha + (n-1)\beta]d\eta(Y,Z). \tag{6.4}$$

Hence, from above equation we can state the following theorem:

Theorem 6.1: A pseudo $\mathcal{W}_8$-flat Sasakian manifold with respect to $\bar{D}$ is η-Einstein with respect to D if and only if one-form η is closed, provided $[a - (n-1)b] \neq 0$.

Further, putting ξ for Z in (6.4), we get

$$Ric(Y,\xi) = -\left[\frac{r+(n^2-1)\beta^2}{n}\right]\eta(Y). \quad (6.5)$$

In view of (2.1) and (2.5)d, equation (6.5) gives

$$\lambda = \frac{r+(n^2-1)\beta^2}{n}. \quad (6.6)$$

From (6.6), we have the following theorem:

Theorem 6.2: If a pseudo W_8-flat Sasakian manifold admits a Ricci soliton (λ, ξ, g) with respect to $\bar{D}$. In this case Ricci solitons is shrinking, expanding or steady if and only if

$r < -(n^2-1)\beta^2$, $r > -(n^2-1)\beta^2$ or $r = (n^2-1)\beta^2$, $[a-(n-1)b] \neq 0$.

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Received on 28.11.2024 and accepted on 06.04.2025