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Generalized semi-pseudo Ricci-symmetric manifold admitting W_2-Ricci tensor

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Abstract

The present paper deals with the study of generalized semi-pseudo Ricci-symmetric on W_2-Ricci tensor and obtained some interesting results.

Keywords and phrases: Pseudo Ricci-symmetric manifolds, semi-pseudo Ricci–symmetric manifolds, Extended pseudo Ricci-symmetric manifolds, generalized semi-pseudo Ricci symmetric manifolds, W_2-curvature tensor.

1. Introduction

The study of Riemann symmetric manifolds began with the work of Cartan (1926). According to Cartan, a Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies the condition $DR = 0$, where D is the covariant differentiation operator with respect to the metric tensor g. During the last five decades the notion of locally symmetric manifolds has been weakened by many investigators in several ways to a different extent such as recurrent manifold (Walker, 1951), semi symmetric manifold (Szabo, 1982), Pseudo symmetric manifold (Chaki, 1987; Deszcz, 1992) etc. If the Ricci tensor Ric of the type (0, 2) in a Riemannian manifold M satisfies the relation $D.Ric = 0$, then Ric is said to be Ricci symmetric. The notion of Ricci symmetric has been studied extensively by many authors in several ways to a different extent viz., Ricci recurrent manifold (Patterson, 1952), Pseudo Ricci symmetric manifold (Chaki, 1988; Deszcz, 1989) and Weakly Ricci symmetric manifold (Tamassy and Binh, 1993).

Let Q be the symmetric endomorphism corresponding to the Ricci tensor defined by $Ric(X,Y) = g(QX,Y)$ for all vector fields X and Y.

A non-flat Riemannian manifold is called pseudo Ricci-symmetric and denoted by $(PRS)_n$ if the Ricci tensor Ric of type (0, 2) of the manifold is non-zero satisfies the condition (Chaki, 1988):

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(X,Y), \tag{1.1}$$

for any vector fields X, Y and Z, where A is no where vanishing 1-form on M.

The pseudo Ricci symmetric manifolds have been studied by many authors.

In 1993 Tarafdar and Jawarneh introduced another type of non-flat Riemannian manifold whose Ricci tensor of type (0, 2) satisfies the condition:

$$(D_X Ric)(Y,Z) = A(Y)Ric(X,Z) + A(Z)Ric(X,Y). \quad (1.2)$$

Such a manifold was called by them a semi-pseudo Ricci–symmetric manifold and n-dimensional manifold of this kind was denoted by $(SPRS)_n$. Some contributions in this direction was due to (Prasad, 1998), (Tarafdar et.al, 2011) and others discussed some additional properties on different structures.

Extended pseudo Ricci-symmetric manifolds investigated by Prasad and Doulo in 2016. In this paper, authors find the nature of the scalar curvature r and also obtained condition on $E(PRS)_n$ such that Ricci tensor as a cyclic tensor. Finally conformally flat $E(PRS)_n$ is being considered, as follows:

$$(D_X S)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(Y,X).$$

Pseudo Ricci symmetric manifolds $(PRS)_n$, semi-pseudo Ricci-symmetric manifold $(SPRS)_n$ and Extended pseudo Ricci-symmetric manifolds $E(PRS)_n$ have great impotance in the general theory of Relativity. Considering this aspect Jawarneh and Tarhtoush in 2012 defined generalized semi-pseudo Ricci symmetric manifolds by the expression:

$$(D_X Ric)(Y,Z) = A(Y)Ric(X,Z) + B(Z)Ric(X,Y), \quad (1.3)$$

where A and B are two non-zero 1-forms defined by

$$A(X) = g(X,\rho_1), \quad (1.4)$$

and

$$B(X) = g(X,\rho_2). \quad (1.5)$$

Such a manifold were called by them a generalized semi-pseudo Ricci symmetric manifold and denoted by $G(SPRS)_n$. In Particular, if $A = B$ then equation (1.3) redues to semi-pseudo Ricci-symmetric manifold (Tarafdar and Jawarneh, 1993). This justified the name generalized semi-pseudo Ricci-symmetric manifold defined by the equation (1.3) and the symbol for it. Recently Halder and Bhattacharyya, (2017) studied this structure under the title “Some properties of generalized semi-pseudo Ricci-symmetric manifold”.

In a Riemannian manifold Ricci tensor is Ricci recurrent (Patterson, 1952) if

$$(D_Z Ric)(X,Y) = A(Z)Ric(X,Y). \quad (1.6)$$

In a Riemannian manifold the Ricci tensor are Codazzi type and Cyclic Ricci tensor if the (Hicks, 1965) following condition holds

$$(D_Z Ric)(X,Y) = (D_X Ric)(Y,Z), \quad (1.7)$$

and

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0. \quad (1.8)$$

Pokhariyal and Mishra in 1970 introduced W_2-curvature tensor on a manifold (M^n, g), $(n > 2)$by

$$W_2(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \quad (1.9)$$

where $Ric(X,Y) = g(QX,Y)$, Q be the symmetric endomorphism corresponding to the Ricci tensor. The W_2-curvature tensor on different manifolds have been examined by many authors as (Prasad, 1997; Prakasha, 2010; Mallik and De, 2014; Hui, 2012; Ozen et.al, 2019). In this paper, it is known that scalar curvature of $G(SPRS)_n$ with $\widetilde{W}_2$-flat Ricci tensor is not necessarily a constant. Further a necessary and sufficient condition is obtained for $\widetilde{W}_2$-Ricci tensor to be conservative. Finally $G(SPRS)_n$ with its $\widetilde{W}_2$-Ricci tensor as a Codazzi type and Cyclic Ricci tensor are examined.

2. $G(SPRS)_n$ with $\widetilde{W}_2$-flat Ricci tensor

In this section, we denote the contracted W_2-curvature tensor which is type of (0, 2) as $\widetilde{W}_2$ and call it $\widetilde{W}_2$-Ricci tensor. Now contracting equation (1.9), we get

$$\widetilde{W}_2(X,Y) = \frac{n}{n-1}\left[Ric(X,Y) - \frac{r}{n}g(X,Y)\right]. \tag{2.1}$$

Here we assume that manifold is $\widetilde{W}_2$-flat . Hence from (2.1), we get

$$Ric(X,Y) = \frac{r}{n}g(X,Y). \tag{2.2}$$

From (2.2), we get

$$(D_Z Ric)(X,Y) = \frac{1}{n}g(X,Y)(D_z r). \tag{2.3}$$

Here, we assume that manifold is $G(SPRS)_n$. Hence we have from (1.3),

$$(D_Z Ric)(X,Y) = A(X)Ric(Z,Y) + B(Y)Ric(Z,X). \tag{2.4}$$

In view of (2.2), (2.3) and (2.4), we obtain

$$(D_Z r)g(X,Y) = r[A(X)g(Z,Y) + B(Y)g(Z,X)]. \tag{2.5}$$

Contracting equation (2.5) with respect to X and Y, we get

$$(D_Z r)n = r[A(Z) + B(Z)]. \tag{2.6}$$

Again contracting (2.4) with respect to X and Z, we get

$$(D_Z r) = A(QZ) + B(Z)r. \tag{2.7}$$

From (2.6) and (2.7), we get

$$r = \frac{nA(QZ)}{A(Z)-(n-1)B(Z)}\ ,\ A(Z) - (n-1)B(Z) \neq 0. \tag{2.8}$$

Thus, we have the following theorem:

Theorem 2.1: For $\widetilde{W}_2$-flat $G(SPRS)_n$, relation (2.8) shows that r is not necessarily a constant, provided $A(Z) - (n-1)B(Z) \neq 0$.

3. $G(SPRS)_n$ with $\widetilde{W}_2$-flat Ricci tensor as conservative

In this section, we assume that $\widetilde{W}_2$ is conservative (Hicks, 1965), then

$$div\widetilde{W}_2 = 0. \tag{3.1}$$

Now, differentiating covariantly (2.1), we get

$$(D_Z\widetilde{W}_2)(X,Y) = \frac{n}{n-1}\left[(D_Z Ric)(X,Y) - \frac{1}{n}(D_Z r)g(X,Y)\right]. \tag{3.2}$$

Contracting (2.4) over $X$and Y, we get

$$(D_Z r) = A(QZ) + B(QZ). \tag{3.3}$$

From (2.4), (3.2) and (3.3), we get

$$(D_Z\widetilde{W}_2)(X,Y) = \frac{n}{n-1}\Big[A(X)Ric(Z,Y) + B(Y)Ric(Z,X) - \frac{1}{n}A(QZ)g(X,Y) - \frac{1}{n}B(QZ)g(X,Y)\Big]. \tag{3.4}$$

This gives on contraction

$$(div\widetilde{W}_2)(Y) = \frac{n}{n-1}\Big[A(QY)\frac{n-1}{n} + B(Y)r - \frac{1}{n}B(QY)\Big]. \tag{3.5}$$

Hence in view of (3.5), we can state the following theorem:

Theorem 3.1: For an $G(SPRS)_n$, contracted W_2-curvature tensor $\widetilde{W}_2$ be divergence free if and only if

$$\frac{n-1}{n}A(QY) + B(Y)r - \frac{1}{n}B(QY) = 0.$$

4. $G(SPRS)_n$ with $\widetilde{W}_2$-flat Ricci tensor as a Ricci recurrent tensor

In this section, we assume that $\widetilde{W}_2$ be Ricci recurrent then from (1.6), we get

$$(D_Z\widetilde{W}_2)(X,Y) = \alpha(Z)\widetilde{W}_2(X,Y), \tag{4.1}$$

where α is 1-form.

Thus, in view of (2.1), (3.4) and (4.1), we get

$$\alpha(Z)\Big[Ric(X,Y) - \frac{r}{n}g(X,Y)\Big] = \Big[A(X)Ric(Z,Y) + B(Y)Ric(Z,X) - \frac{1}{n}A(QZ)g(X,Y) - \frac{1}{n}B(QZ)g(X,Y)\Big]. \tag{4.2}$$

Contracting (4.2) with respect to X and Z, we get

$$\alpha(QZ) - \frac{r}{n}\alpha(Z) = \frac{n-1}{n}A(QZ) + B(Z)r - \frac{1}{n}B(QZ). \tag{4.3}$$

Let us take $\alpha(Z) = A(Z)$ in (4.3) and nothing, we get

$$A(QZ) - rA(Z) = B(Z)\frac{r}{n} - B(QZ). \tag{4.4}$$

Hence, from (4.4), we have the following theorem:

Theorem 4.1: In an $G(SPRS)_n$, if $\widetilde{W}_2$-Ricci tensor is recurrent with recurrence vector generated by the 1-form A. A necessary and sufficient condition for $-\frac{r}{n}$ be an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ_2 where $g(X,\rho_2) = B(X)$ is that r be an eigen value of the Ricci tensor Ric corresponding to the eigenvector ρ_1 where $g(X,\rho_1) = A(X)$.

5. $G(SPRS)_n$ with its $\widetilde{W}_2$-flat Ricci tensor as Codazzi type and cyclic type

Now, we assume that r is a non-zero constant then from (3.2), we get

$$(D_Z\widetilde{W})(X,Y) = \frac{n}{n-1}(D_Z Ric)(X,Y). \tag{5.1}$$

Using (1.3) in (5.1), we get

$$(D_Z\widetilde{W})(X,Y) = \frac{n}{n-1}[A(X)Ric(Z,Y) + B(Y)Ric(Z,X). \tag{5.2}$$

If $\widetilde{W}_2$ be Codazzi type, then from (1.7) and (5.2), we get

$$A(X)Ric(Z,Y) + B(Y)Ric(Z,X) = A(Y)Ric(X,Z) + B(Z)Ric(X,Y). \tag{5.3}$$

Contracting (5.3) over X and Y, we get

$$B(QZ) = B(Z)r. \tag{5.4}$$

In view of (5.4), we have the following theorem:

Theorem 5.1: For an $G(SPRS)_n$ with constant scalar curvature, if $\widetilde{W}_2$-Ricci tensor is Codazzi type then r is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector $\rho_2$where $g(X,\rho_2) = B(X)$.

Next, we assume that $\widetilde{W}_2$-Ricci tensor is a cyclic tensor. Hence in view of (1.8), we have

$$(D_X\widetilde{W}_2)(Y,Z) + (D_Y\widetilde{W}_2)(Z,X) + (D_Z\widetilde{W}_2)(X,Y) = 0. \tag{5.5}$$

Thus from (5.2) and (5.5), we get

$$\begin{aligned} A(X)Ric(Z,Y) + A(Y)Ric(X,Z) + A(Z)Ric(Y,X) = B(X)Ric(Y,Z) + B(Y)Ric(Z,X) \\ +B(Z)Ric(X,Y). \end{aligned} \tag{5.6}$$

Contraction of (5.6) over X and Y, we get

$$A(QZ) + \frac{r}{2}A(Z) = -B(QZ) - \frac{r}{2}B(Z). \tag{5.7}$$

Equation (5.7) can be put as

$$Ric(Z,\rho_1) + Ric(Z,\rho_2) = -\frac{r}{2}g(Z,\rho_1) - \frac{r}{2}g(Z,\rho_2),$$

or $$Ric(Z,\rho_1+\rho_2) = -\frac{r}{2}g(Z,\rho_1+\rho_2),$$

or $$Ric(Z,\rho) = -\frac{r}{2}g(Z,\rho), \quad \text{where } \rho_1+\rho_2 = \rho. \tag{5.8}$$

Hence in view of (5.8), we can state the following theorem:

Theorem 5.2: For an $G(SPRS)_n$ with constant scalar curvature, if $\widetilde{W}_2$-Ricci tensor is cyclic Ricci tensor then $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ.

Let r is non-zero constant i.e $dr(Z) = 0$ and Let $\widetilde{W}_2$-Ricci is covariantly constant i.e $\left(D_z\widetilde{W}_2\right) = 0$.

Hence we have from (2.1)

$$(D_ZRic)(X,Y) = 0. \tag{5.9}$$

Hence we have from (1.3)

$$(D_ZRic)(X,Y) = A(X)Ric(Z,Y) + B(Y)Ric(Z,X). \tag{5.10}$$

Thus in view of (5.9) and (5.10), we get

$$A(X)Ric(Z,Y) + B(Y)Ric(Z,X) = 0. \tag{5.11}$$

Put $Y = Z = e_i$ in (5.11), we get

$$A(X)r + Ric(X, \rho_2) = 0. \tag{5.12}$$

That is, from (5.12), we get

$$B(QX) = -A(X)r. \tag{5.13}$$

Theorem 5.3: For an $G(SPRS)_n$ with non-zero constant scalar curvature, if $\widetilde{W}_2$-Ricci tensor is cyclic Ricci tensor then $-r$ is none eigenvalue of the Ricci tensor Ric corresponding to the any eigenvector.

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