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On a certain type of semi-symmetric non-metric connection on a Riemannian manifold

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Abstract

Recently, Melohtra (2012) studied a semi-symmetric non-metric connection ($\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\tilde{C}$) on a Riemannian manifold. In this paper, we study a $\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\tilde{C}$ with recurrent torsion.

Keywords Semi-symmetric non-metric connection, recurrent torsion, Einstein manifold, Schouten tensor, Projective curvature tensor, Concircular curvature tensor.

1. Introduction

Let $\overline{D}$ be a linear connection in an n-dimensional differentiable manifold M^n. The torsion tensor $\overline{T}$ and the curvature tensor $\overline{R}$ of $\overline{D}$ are given by $\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y]$, and $\overline{R}(X,Y)Z = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z$, respectively. The connection $\overline{D}$ is symmetric if its torsion tensor $\overline{T}(X,Y) = 0$, otherwise, it is non symmetric. The connection $\overline{D}$ is is metric if $(\overline{D}_X\, g)(X,Y) = 0$, otherwise, it is non-metric. It is well known that linear connection is symmetric and metric if and only if it is the Riemannian connection. In 1994, Friedmann and Schouten introduced the concept of semi-symmetric linear connection in a differentiable manifold. Hayden (1932) introduced a metric connection $\overline{D}$ with a non-zero torsion on a Riemannian manifold. A linear connection is said to be a semi-symmetric connection ($\tilde{S}\tilde{S}\tilde{C}$), if the torsion tensor ($\tilde{T}\tilde{T}$) is of the form $\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y$, where η is a 1-form defined by $\eta(X) = g(X,\xi)$. A connection with torsion tenor of the above form is a semi-symmetric metric connection ($\tilde{S}\tilde{S}\widetilde{M}\tilde{C}$) which appeared in a study of Pok (1969). A systematic study of the $\tilde{S}\tilde{S}\widetilde{M}\tilde{C}$ $\overline{D}$ on a Riemannian manifold was initiated by Yano (1970). He proved that a Riemannian manifold is conformally flat if and only if it admits a $\tilde{S}\tilde{S}\widetilde{M}\tilde{C}$ where curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a $\tilde{S}\tilde{S}\widetilde{M}\tilde{C}$ for which the manifold is group manifold. Some different kind of semi-symmetric non-metric connection ($\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\tilde{C}$) was studied by De and Kamilya (1994), Agashe and Chafle (1992), Sengupta et al. (2000), Prasad and Verma (2004), Prasad and Singh (2006) and many others. In this paper we studied a type of $\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\tilde{C}$ $\overline{D}$ where $\tilde{T}\tilde{T}$ satisfies certain condition.

Let (M^n, g) be an n-dimensional Riemannian manifold class C^∞ with a metric tensor g. A linear connection $\overline{D}$ is said to be a $\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\tilde{C}$ Melhotra (2012) if its torsion tensor $\overline{T}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

and

$$(\bar{D}_X\, g)(X,Y) = -2\eta(X)g(Y,Z), \tag{1.2}$$

for arbitrary vector fields X, Y and Z in M^n, where η is a 1-form defined by $g(X,\xi) = \eta(X)$.

We now, suppose that the Riemannian manifold (M^n, g) admits a $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$ given by

$$\bar{D}_X Y = D_X + \eta(X)Y, \tag{1.3}$$

for arbitrary vector fields X, Y and Z in M^n, where D denotes the Levi-Civita connection ($\tilde{L}\tilde{C}\tilde{C}$).

Further, if $\bar{R}$ and R denote the curvature tensor of $\bar{D}$ and D, respectively, then it is known Melhotra (2012) that

$$\bar{R}(X,Y)Z = R(X,Y)Z + [(D_X\eta)(Y) - (D_Y\eta)(X)]Z\ . \tag{1.4}$$

In the present paper, we consider a Riemannian manifold (M^n, g) admitting a type of $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$ whose torsion tensor is recurrent. Here we have deduced a necessary and sufficient condition for symmetry and skew-symmetry of the Ricci tensor of the $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor is recurrent. In section 3 and 4, the Einstein tensor and Schouten tensor are studied with respect to $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor satisfies the prescribed condition. In section 5 and 6, the necessary and sufficient conditions are obtained under which the projective curvature tensor and the concircular curvature tensor corresponds to the connection $\bar{D}$ and D coincide.

2. Curvature tensor, Ricci tensor and scalar curvature of a special type of $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$

In this section, we consider a type of $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}\ \tilde{T}\tilde{T}\ \bar{T}$ satisfies the following condition De and Kamilya (1994)

$$(\bar{D}_X\, \bar{T}\,)(X,Y) = B(X)\bar{T}(Y,Z), \tag{2.1}$$

where B is a non-zero 1-form such that $B(X) = g(X,\rho)$.

From (1.1), we have

$$(C_1^1\bar{T})(Y) = -(n-1)\eta(Y). \tag{2.2}$$

From (2.2), we get

$$\left(\bar{D}_X\, C_1^1\bar{T}\,\right)(Y) = -(n-1)\left(\bar{D}_X\,\eta\right)(Y). \tag{2.3}$$

From (2.3) and (2.4), we get

$$\left(\bar{D}_X\, C_1^1\bar{T}\,\right)(Y) = B(X)(C_1^1\bar{T}\,)(Y). \tag{2.4}$$

From (2.3) and (2.4), we get

$$\left(\bar{D}_X\,\eta\right)(Y) = B(X)\eta(Y). \tag{2.5}$$

From (1.3) and (2.5), we get

$$(D_X\eta)(Y) = B(X)\eta(Y) + \eta(X)\eta(Y). \tag{2.6}$$

From (1.4) and (2.6), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z + [B(X)\eta(Y) - B(Y)\eta(X)]Z\ . \tag{2.7}$$

Equation (2.7) can be put as

$$'\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W). \tag{2.8}$$

Contraction of (2.8) gives

$$\bar{R}ic(Y,Z) = Ric(Y,Z) + [B(Z)\eta(Y) - B(Y)\eta(Z)], \tag{2.9}$$

where $\bar{R}ic$ and Ric denote the Ricci tensor corresponding to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ and $\tilde{L}\tilde{C}\tilde{C}\ \ D$, respectively.

Again, contracting (2.9), we get

$$\bar{r} = r + \eta(\rho) - B(\xi), \tag{2.10}$$

where $\bar{r}$ and r denote the scalar curvature with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ and $\tilde{L}\tilde{C}\tilde{C}\ \ D$, respectively.

From (2.8), we have

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = 0, \tag{2.11}$$

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) = 2[B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W), \tag{2.12}$$

$$'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) = [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) - [B(Z)\eta(W) - B(W)C(Z)]g(X,Y), \tag{2.13}$$

and

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,Z,X,W) + {}'\bar{R}(Z,X,Y,W) = [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) + [B(Y)\eta(Z) - B(Z)\eta(Y)]g(X,W) \; [B(Z)\eta(X) - B(X)\eta(Z)]g(Y,W)]. \tag{2.14}$$

In consequences of (2.8), (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14), we get

Theorem (2.1): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}\ \ \bar{T}$ satisfies the condition (2.1), then

(i) the curvature tensor $\bar{R}$ is given by (2.8),

(ii) the Ricci tenor $\bar{R}ic$ is given by (2.9) ,

(iii) the scalar curvature of $\bar{r}$ is given by (2.10),

(iv) $'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = 0$,

(v) $'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) = 0$, if and only if $B(X)\eta(Y) = B(Y)\eta(X)$,

(vi) $'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) = 0$, if and only if $B(X)\eta(Y) = B(Y)\eta(X)$,

(vii) $'\bar{R}(X,Y,Z,W) + '\bar{R}(Y,Z,X,W) + '\bar{R}(Z,X,Y,W) = 0$, if and only if $B(X)\eta(Y) = B(Y)\eta(X)$,

(viii) A necessary and sufficient condition for the Ricci tensor Ric of the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ to be symmetric is that $B(X)\eta(Y) = B(Y)\eta(X)$,

(ix) A necessary and sufficient condition for the Ricci tensor Ric of the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ to be skew-symmetric is that Ricci tensor of the $\tilde{L}\tilde{C}\tilde{C}\ \ D$ flat,

(x) the scalar curvature of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ is equal to the scalar curvature of $\tilde{L}\tilde{C}\tilde{C}\ \ D$ if and only if $\eta(\rho) = B(\xi)$.

3. Einstein manifold with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition

A Riemannian manifold (M^n, g) is called an Einstein manifold with respect to $\tilde{L}\tilde{C}\tilde{C}\ \ D$ if

$$Ric(X,Y) = \frac{r}{n} g(X,Y). \tag{3.1}$$

Analogous to the definition (3.1), we define a Riemannian manifold with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition (2.1) is called an Einstein manifold if

$$\bar{R}ic(X,Y) = \frac{\bar{r}}{n} g(X,Y). \tag{3.2}$$

From (2.9), (2.10) and (3.2), we get

$$\bar{R}ic(X,Y) - \frac{\bar{r}}{n} g(X,Y) = Ric(X,Y) - \frac{r}{n} g(X,Y) + [B(Y)\eta(X) - B(X)\eta(Y)]$$
$$+\frac{1}{n}[\eta(\rho) - B(\xi)]g(X,Y). \tag{3.3}$$

In view of (3.3), we can state the following theorem:

Theorem (3.1): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor satisfies the condition (2.1). Then, the notion of Einstein manifold with respect to the Riemannian connection D and the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ coincide if and only if

$$n[B(Y)\eta(X) - B(X)\eta(Y)] + [\eta(\rho) - B(\xi)]g(X,Y) = 0.$$

4. Schouten tensor S with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition

Schouten tensor on a Riemannian manifold with respect to $\tilde{L}\tilde{C}\tilde{C}\ \ D$ is given by Guan et al. (2003)

$$S(X,Y) = \frac{1}{n-2} Ric(X,Y) - \frac{r}{2(n-1)(n-2)} g(X,Y). \tag{4.1}$$

From (4.1), we get

$$S(X,Y) - S(Y,X) = 0, \tag{4.2}$$

and

$$S(X,Y) + S(Y,X) = \frac{2}{n-2} Ric(X,Y) - \frac{1}{(n-1)(n-2)} g(X,Y). \tag{4.3}$$

It follows from (4.2) that Schouten tensor is symmetric while (4.3) shows that it is skew-symmetric if and only if $Ric(X,Y) = \frac{1}{2(n-1)} g(X,Y)$ with respect to $\tilde{L}\tilde{C}\tilde{C}\ \ D$.

Analogous to the definition (4.1), we define Schouten tensor on a Riemannian manifold with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition (2.1 as

$$\bar{S}(X,Y) = \frac{1}{n-2}\bar{R}ic(X,Y) - \frac{\bar{r}}{2(n-1)(n-2)}g(X,Y). \tag{4.4}$$

From (2.9), (2.10) and (4.1), we get

$$\bar{S}(X,Y) = S(X,Y) + \frac{1}{n-2}[B(X)\eta(Y) - B(Y)\eta(X)] - \frac{[\eta(\rho)-B(\xi)]}{2(n-1)(n-2)}g(X,Y). \tag{4.5}$$

From (4.2), (4.3) and (4.5), we can state the following theorem:

Theorem (4.1): Let a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then, the Schouten tensor with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ coincides with the Schouten tensor of the $\tilde{L}\tilde{C}\tilde{C}\ D$ if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)] = \frac{[\eta(\rho)-B(\xi)]}{2(n-1)}g(X,Y).$$

Theorem (4.2): If a Riemannian manifold (M^n, g) admits a $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then, the Schouten tensor with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ is symmetric if and only if

$$B(Y)\eta(X) - B(X)\eta(Y) = 0.$$

Theorem (4.3): If a Riemannian manifold (M^n, g) admits a $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then, the Schouten tensor with respect to the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ is skew-symmetric if and only if

$$Ric(X,Y) = \frac{[r-\{\eta(\rho)-B(\xi)\}]}{2(n-1)}g(X,Y).$$

5. Projective curvature tensor with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition

Let $'\bar{P}(X,Y,Z,W)$ and $'P(X,Y,Z,W)$ be the projective curvature tensor of the connection $\bar{D}$ and D, respectively. Then

$$'\bar{P}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{1}{n-1}[\bar{R}ic(Y,Z)g(X,W) - \bar{R}(X,Z)g(Y,W)], \tag{5.1}$$

where $'\bar{P}(X,Y,Z,W) = g(\bar{P}(X,Y)Z,W)$.

In view of (2.8), (2.9), and (5.1), we get

$$\begin{aligned} '\bar{P}(X,Y,Z,W) = {}&'P(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) \\ &-\frac{1}{n-1}[\{B(Z)\eta(Y) - B(Y)\eta(Z)+\}g(X,W) \\ &-\{B(Z)\eta(X) - B(X)\eta(Z)\}g(Y,W)], \end{aligned} \tag{5.2}$$

where

$$'P(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{n-1}[Ric(Y,Z)g(X,W) - R(X,Z)g(Y,W)],$$

and

$$'P(X,Y,Z,W) = g(P(X,Y)Z,W).$$

In view of (5.3), we can state the following theorem:

Theorem (5.1): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then, the projective curvature tensor of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ is equal to projective curvature tensor of the $\tilde{L}\tilde{C}\tilde{C}\ D$ if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)] = 0.$$

Further, suppose that Ricci tensor of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ vanishes, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1),

$$\bar{R}ic(X,Y) = 0. \tag{5.3}$$

Applying the condition (5.3) to (5.1), we get

$$'\bar{P}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W). \tag{5.4}$$

Using (5.4) and (5.2), we get

$$'\bar{R}(X,Y,Z,W) = {}'P(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W)$$
$$-\frac{1}{n-1}[\{B(Z)\eta(Y) - B(Y)\eta(Z)+\}g(X,W)$$
$$-\{B(Z)\eta(X) - B(X)\eta(Z)\}g(Y,W)], \tag{5.5}$$

In view of (5.5), we have the following theorem:

Theorem (5.2): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1) and Ricci tensor $\bar{R}ic$ of $\bar{D}$ vanishes. Then, the curvature tensor $'\bar{R}$ of the connection $\bar{D}$ is equal to projective curvature tensor $'P$ of the manifold if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)] = 0.$$

Further, we assume that if the curvature tensor of the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ vanishes, then its Ricci tensor $\bar{R}ic$ is also vanishes, hence from (5.5), we get

$$'P(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) -$$
$$\frac{1}{n-1}[\{B(Z)\eta(Y) - B(Y)\eta(Z)+\}g(X,W) - \{B(Z)\eta(X) - B(X)\eta(Z)\}g(Y,W)] = 0. \tag{5.6}$$

Hence, from (5.6), we can state the following theorem:

Theorem (5.3): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1) and curvature tensor $'\bar{R}$ of $\bar{D}$ vanishes. Then, the manifold is projectively flat if and only if $[B(X)\eta(Y) - B(Y)\eta(X)] = 0$.

From (5.2), we have the following theorem:

Theorem (5.4): If a Riemannian manifold (M^n, g) admits a $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then the algebraic properties of projective curvature tensor $'\bar{P}$ of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ are given by

$$'\bar{P}(X,Y,Z,W) + {'\bar{P}}(Y,X,Z,W) = 0.$$

and

$'\bar{P}(X,Y,Z,W) + {'\bar{P}}(Y,Z,X,W) + {'\bar{P}}(Z,X,Y,W) = 0$, if and only if $[B(X)\eta(Y) - B(Y)\eta(X)] = 0$.

6. Concircular curvature tensor with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose $\tilde{T}\tilde{T}$ satisfies the certain condition

Let $'\bar{L}(X,Y,Z,W)$ and $'L(X,Y,Z,W)$ be the concircular curvature tensor of the connection $\bar{D}$ and D, respectively. Then

$$'\bar{L}(X,Y,Z,W) = {'\bar{R}}(X,Y,Z,W) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{6.1}$$

where $'\bar{L}(X,Y,Z,W) = g(\bar{L}(X,Y)Z,W)$.

In view of (2.8), (2.10) and (6.1), we get

$$'\bar{L}(X,Y,Z,W) = {'L}(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) - \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{6.2}$$

where $'L(X,Y,Z,W) = {'R}(X,Y,Z,W) - \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$,

and $'L(X,Y,Z,W) = g(L(X,Y)Z,W)$.

Therefore, from (6.2), we can state the following theorem:

Theorem (6.1): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then, the cocircular curvature tensor $'\bar{L}$ of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ is equal to concircular curvature tensor $'L$ of D if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) = \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

Further, if we assume that the scalar curvature of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \bar{D}$ vanishes, that is

$$\bar{r} = 0. \tag{6.3}$$

Then, from (6.1) and (6.3), we get

$$'\bar{L}(X,Y,Z,W) = {'\bar{R}}(X,Y,Z,W). \tag{6.4}$$

Hence, from (6.4) and (6.2), we get

$$'\bar{R}(X,Y,Z,W) = {'L}(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) - \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{6.5}$$

Therefore, from (6.5), we can state the following theorem:

Theorem (6.2): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1) and scalar curvature of $\bar{D}$ vanishes. Then, $'\bar{R}(X,Y,Z,W) = {'L}(X,Y,Z,W)$ if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) = \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

Again, if we assume that

$$'\bar{R}(X,Y,Z,W) = 0. \tag{6.6}$$

Then, from (6.6) and (6.5), we get

$$'L(X,Y,Z,W) + [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) - \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \tag{6.7}$$

In view of (6.7), we can have the following theorem:

Theorem (6.3): If a Riemannian manifold (M^n, g) admits $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1) and curvature tensor $'\bar{R}$ of $\bar{D}$ vanishes. Then, the manifold is concircularly flat if and only if

$$[B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) = \frac{[\eta(\rho) - B(\xi)]}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

Theorem (6.4): If a Riemannian manifold (M^n, g) admits a $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$, whose torsion tensor $\tilde{T}\tilde{T}$ satisfies the condition (2.1). Then the algebraic properties of cocircular curvature tensor $'\bar{L}$ of $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\tilde{C}\ \bar{D}$ are given by

$$'\bar{L}(X,Y,Z,W) + {'\bar{L}}(Y,X,Z,W) = 0,$$

$$'\bar{L}(X,Y,Z,W) + {'\bar{L}}(X,Y,W,Z) = 2[B(X)\eta(Y) - B(Y)\eta(X)],$$

$$'\bar{L}(X,Y,Z,W) - {'\bar{L}}(Z,W,X,Y) = [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) - [B(Z)\eta(W) - B(W)\eta(Z)]g(X,Y).$$

and

$$'\bar{L}(X,Y,Z,W) + {'\bar{L}}(Y,Z,X,W) + {'\bar{L}}(Z,X,Y,W) = [B(X)\eta(Y) - B(Y)\eta(X)]g(Z,W) + [B(Y)\eta(Z) - B(Z)\eta(Y)]g(X,W) + [B(Z)\eta(X) - B(X)\eta(Z)]g(Y,W).$$

In particular, above results can also be rewritten as

$'\bar{L}(X,Y,Z,W) + {'\bar{L}}(X,Y,W,Z) = 0$, if and only if $[B(X)\eta(Y) - B(Y)\eta(X)] = 0$,

$'\bar{L}(X,Y,Z,W) - {'\bar{L}}(Z,W,X,Y) = 0$, if and only if $[B(X)\eta(Y) - B(Y)\eta(X)] = 0$,

and

$'\bar{L}(X,Y,Z,W) + {'\bar{L}}(Y,Z,X,W) + {'\bar{L}}(Z,X,Y,W) = 0$, if and only if $[B(X)\eta(Y) - B(Y)\eta(X)] = 0$.

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