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On pseudo W_8-flat Lorentzian α-para Kenmotsu manifold

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Abstract

The notion of Lorentzian α-para Kenmotsu manifold has been introduced by Prasad, Verma and Yadav (2023). In this paper, we investigate some properties of pseudo W_8-flat Lorentzian α-para Kenmotsu manifold and derived conditions under which such manifolds are η-Einstein and of constant curvature. As a special case, corresponding results for W_8-flat Lorentzian α-para Kenmotsu manifold are also obtained.

Keywords: Lorentzian α-para Kenmotsu manifold, pseudo W_8-curvature tensor, manifold of constant curvature.

1. Introduction

The Lorentzian α-para Kenmotsu manifold is playing in important role in mathematical physics specially in the development of the theory of relativity and cosmology. It is one of the most important subclass of pseufo-Riemannian manifold. In 1989, Matsumoto gave the idea of Lorentzian para-Sasakain (LP-Sasakian) manifold. Later such a manifolds have been studied by several authors. De, Shaikh and Sengupta (2002) introduced the notion of LP-Sasakian manifold with a coefficient α which generalized the notion of LP-Sasakian manifold. Many researchers have investigate these manifolds extensively and a variety of geometric properties have been established in the literature. In recent work, Haseeb et al. (2021) introduced of the notion of Lorentzian para-Kenmotsu manifolds and investigated their fundamental geometric properties as a distinguished subclass of LP-contact manifolds,

In view of the above developments, Prasad, Verma and Yadav (2023) defined Lorentzian α-para Kenmotsu manifolds as a new class of Lorentzian para contact manifolds and examined their geometric properties.

In 1982, Pokhariyal introduced $'W_8$-curvature tensor if the type (0,4) on the line of projective curvature tensor and studied its relativistic significance and the expression is

$$'W_8(X,Y,Z,W) = 'R(X,Y,Z,W) + \frac{1}{n-1}[Ric(X,Y)g(Z,W) - Ric(Y,Z)g(X,W)], \qquad (1.1)$$

where $'W_8(X,Y,Z,W) = g(W_8(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

In 2018, Prasad, Yadav and Pandey proposed a generalization of the $'W_8$-curvature tensor, known as the pseudo $'W_8$-curvature tensor $'\widetilde{W}_8$ on the Riemannian manifolds with the following expression:

$$'\widetilde{W}_8(X,Y,Z,W) = a\,'R(X,Y,Z,W) + b[Ric(X,Y)g(Z,W) - Ric(Y,Z)g(X,W)]$$
$$-\frac{r}{n}\left[\frac{a}{n-1} - b\right][g(X,Y)g(Z,W) - g(Y,Z)g(X,W)], \qquad (1.2)$$

where a and b are real constant , not equal to zero.

In particular, if $a = 1$ and $b = \frac{1}{n-1}$, then equation (1.2) reduces in $'W_8$-curvature tensor Pokhariyal and Mishra (1982). This fact justifies calling it the pseudo $'W_8$-curvature tensor.

If $'\widetilde{W}_8(X,Y,Z,W) = 0$, then Prasad, Yadav and Pandey (2018) obtained

$$Ric(Y,Z) = -\frac{r}{n}g(Y,Z), \qquad (1.3)$$

provided $[a - b(n-1)] \neq 0$.

In this paper, we study some properties of pseudo $'W_8$-flat Lorentzian α-para Kenmotsu manifold. We proved that a pseudo $'W_8$-flat Lorentzian α-para Kenmotsu manifold is always a η-Einstein manifold, provided α and σ are constants. Further, we prove that if $a - (n-1)b \neq 0$ and scalar curvature r is constant, then a pseudo $'W_8$-flat Lorentzian α-para Kenmotsu manifold is is of constant curvature.

2. Preliminaries

An n-dimensional smooth manifold M^n is said to be Lorentzian almost para-contact manifold, provided M^n is equipped with a (1,1)-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η, and a (0,2) type Lorentzian metric g . Let $g_m: T_mM^n \times T_mM^n \to \tilde{R}$ be an inner product of signature $(-,+,+,\ldots,+)$, here m is a point in M^n, T_mM^n represents tangent space of smooth manifold M^n at m and $\tilde{R}$ is real number space. Some basic results, given below hold:

$$\phi^2X = X + \eta(X)\xi, \ \eta(\xi) = -1, \qquad (2.1)$$

$$g(X,\xi) = \eta(X), \ \eta(\phi X) = 0, \ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \qquad (2.2)$$

$\forall\, X, Y$ on M^n, and structure (ϕ,ξ,η,g) is said to be Lorentzian almost paracontact structure. An n-dimensional smooth manifold M^n endowed with structure (ϕ,ξ,η,g) is said to be Lorentzian almost paracontact manifold De and Shaikh (2009). Results given below hold for Lorentzian almost paracontact manifold,

$$\phi\xi = 0, \ \eta(\phi X) = 0, \ \Omega(X,Y) = \Omega(Y,X), \qquad (2.3)$$

here $\Omega(X,Y) = g(X,\phi Y)$.

Definition 2.1: A Lorentzian almost paracontact manifold M^n is said to be Lorentzian para-Kenmotsu manifold if Haseeb et al. (2021)

$$(D_X\phi)(Y) = -g(\phi X, Y)\xi - \eta(Y)\phi X,$$

$\forall\, X, Y$ on M^n.

Definition 2.2: A Lorentzian para-Kenmotsu manifold is said to be Lorentzian α-para Kenmotsu manifold if Prasad et al. (2023)

$$(D_Z\Omega)(X,Y) + \alpha\eta(X)\Omega(Y,Z) + \alpha\eta(Y)\Omega(X,Z) = 0, \tag{2.4}$$

$\forall\, X, Y$ on M^n, where α is a non-zero smooth function and

$$\Omega(\phi X, Y) = -\frac{1}{\alpha}(D_X\eta)(Y).$$

We define

$$\bar{\Omega}(X,Y) = \Omega(\phi X, Y),$$

then we have

$$\bar{\Omega}(X,Y) = -\frac{1}{\alpha}(D_X\eta)(Y), \tag{2.5}$$

where D is covariant differentiation operator.

From equation (2.4), we get

$$(D_X\phi)(Y) = -\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X. \tag{2.6}$$

Putting ξ for Y in above equation, we get

$$(D_X\phi)(\xi) = -\alpha g(\phi X, \xi)\xi - \alpha\eta(\xi)\phi X.$$

Using (2.1) and (2.3), we get

$$-\phi(D_X\xi) = \alpha\phi X.$$

Operating ϕ on both sides of the above relation and using relation (2.1), it yields

$$D_X\xi + \eta(D_X\xi)\xi = -\alpha(X + \eta(X)\xi).$$

Relation (2.1) implies $\eta(D_X\xi) = 0$. Using this relation in the above equation, we get

$$D_X\xi = -\alpha X - \alpha\eta(X)\xi. \tag{2.7}$$

Also

$$(D_X\eta)(Y) = D_X\eta(Y) - \eta(D_XY) = g(Y, D_X\xi). \tag{2.8}$$

Relation (2.7) and (2.8) together yield

$$(D_X\eta)(Y) = -\alpha[g(X,Y) + \eta(X)\eta(Y)]. \tag{2.9}$$

In particular, if α satisfies (2.9) together with the following relation

$$D_X\alpha = d\alpha(X) = \sigma\eta(X), \tag{2.10}$$

Then, ξ is said to be concircular vector field. Here, σ is smooth function and η is 1-form.

For Lorentzian α-para Kenmotsu manifold $M(\phi, \xi, \eta, g)$, following results hold Prasad et al. (2023)

$$\eta(R(X,Y)Z) = (\alpha^2 + \sigma)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$Ric(X,\xi) = (n-1)(\alpha^2 + \sigma)\eta(X), \tag{2.12}$$

$$R(X,Y)\xi = (\alpha^2 + \sigma)[\eta(Y)X - \eta(X)Y], \tag{2.13}$$

$$R(\xi, Y)X = (\alpha^2 + \sigma)[g(X, Y)\xi - \eta(X)Y], \tag{2.14}$$

$$Ric(\phi X, \phi Y) = Ric(X, Y) + (n-1)(\alpha^2 + \sigma)\eta(X)\eta(Y), \tag{2.15}$$

$\forall\ X, Y, Z$ on M^n.

3. Pseudo W_8-flat Lorentzian α-para Kenmotsu manifold

Let us consider a pseudo W_8-flat Lorentzian α-para Kenmotsu manifold. Then, $\widetilde{W}_8 = 0$. Then from (1.2), we get

$$'R(X, Y, Z, W) = \frac{b}{a}[Ric(Y, Z)g(X, W) - Ric(X, Y)g(Z, W)] + \frac{r}{an}\left[\frac{a}{n-1} - b\right][g(X, Y)g(Z, W) - g(Y, Z)g(X, W)]. \tag{3.1}$$

Putting ξ for W in (3.1) and using (2.2) and (2.11), we get

$$(\alpha^2 + \sigma)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] = \frac{b}{a}[Ric(Y, Z)\eta(X) - Ric(X, Y)\eta(Z)] + \frac{r}{an}\left[\frac{a}{n-1} - b\right][g(X, Y)\eta(Z) - g(Y, Z)\eta(X)]. \tag{3.2}$$

Again, putting ξ for X in (3.2) and using (2.1), (2.2) and (2.12), we get

$$Ric(Y, Z) = \left[\frac{a}{b}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right] g(Y, Z) + \left[\left\{\frac{a - b(n-1)}{b}\right\}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right]\eta(Y)\eta(Z). \tag{3.3}$$

From (3.3), we can state the following theorem:

Theorem (3.1): A pseudo W_8-flat Lorentzian α-para Kenmotsu manifold is always an η-Einstein manifold, provided α and σ are constants.

Differentiating (3.3) along $X$and using (2.2) and (2.3), we get

$$(D_X Ric)(Y, Z) = \frac{dr(X)}{bn}\left[\frac{a}{n-1} - b\right][g(Y, Z) + \eta(Y)\eta(Z)] - \alpha\left[\left\{\frac{a - b(n-1)}{b}\right\}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)], \tag{3.4}$$

provided α and σ are constants.

Using (3.4), we get

$$(D_X Ric)(Y, Z) - (D_Y Ric)(X, Z) = \frac{dr(X)}{bn}\left[\frac{a}{n-1} - b\right][g(Y, Z) + \eta(Y)\eta(Z)] - \frac{dr(Y)}{bn}\left[\frac{a}{n-1} - b\right][g(X, Z) + \eta(X)\eta(Z)] - \alpha\left[\left\{\frac{a - b(n-1)}{b}\right\}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right] \times [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \tag{3.5}$$

Differentiating (1.3) along X, we get

$$(D_X Ric)(Y,Z) = -\frac{dr(X)}{n} g(Y,Z), \tag{3.6}$$

provided $[a - b(n-1)] \neq 0$.

From (3.6), we get

$$(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z) = \frac{dr(Y)}{n} g(X,Z) - \frac{dr(X)}{n} g(Y,Z). \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{1}{n}[dr(Y)g(X,Z) - dr(X)g(Y,Z)] = \frac{dr(X)}{bn}\left[\frac{a}{n-1} - b\right][g(Y,Z) + \eta(Y)\eta(Z)] - \frac{dr(Y)}{bn}\left[\frac{a}{n-1} - b\right][g(X,Z) + \eta(X)\eta(Z)] - \alpha\left[\left\{\frac{a-b(n-1)}{b}\right\}(\alpha^2+\sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right] \times [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{3.8}$$

If r is constant, then from (3.8), we get

$$r = -n(n-1)(\alpha^2+\sigma), \tag{3.9}$$

provided $[a - b(n-1)] \neq 0, \alpha, \sigma$ and r are all constants.

From (3.3) and (3.9), we get

$$Ric(Y,Z) = (n-1)(\alpha^2+\sigma)g(Y,Z). \tag{3.10}$$

Using (3.9) and (3.10) in (3.1). we get

$$'R(X,Y,Z,W) = \frac{b}{a}[(n-1)(\alpha^2+\sigma)g(Y,Z)g(X,W) - (n-1)(\alpha^2+\sigma)g(X,Y)g(Z,W)] - \frac{n(n-1)(\alpha^2+\sigma)}{an}\left[\frac{a}{n-1} - b\right][g(X,Y)g(Z,W) - g(Y,Z)g(X,W)].$$

That is

$$'R(X,Y,Z,W) = (\alpha^2+\sigma)[g(Y,Z)g(X,W) - g(X,Y)g(Z,W)]. \tag{3.11}$$

From (3.11), we can state the following theorem:

Theorem (3.2): In a pseudo W_8-flat Lorentzian α-para Kenmotsu manifold, if the scalar curvature r is constant, then the manifold is of constant curvature, provided $[a - b(n-1)] \neq 0$, α and σ are constants.

From (3.11), we get

$$R(X,Y)Z = (\alpha^2+\sigma)[g(Y,Z)X - g(X,Y)Z]. \tag{3.12}$$

Using (2.1), (2.2), (2.3), (3.10) and (3.12), we can state the following theorem:

Theorem (3.3): In a pseudo W_8-flat Lorentzian α-para Kenmotsu manifold, if the scalar curvature r is constant, then

(i) $R(X,\xi)Y = (\alpha^2 + \sigma)[\eta(Y)X - \eta(X)Y]$,
(ii) $R(\xi,X)Y = (\alpha^2 + \sigma)[g(X,Y)\xi - \eta(X)Y]$,
(iii) $R(X,Y)\xi = (\alpha^2 + \sigma)[\eta(Y)X - g(X,Y)\xi]$,
(iv) $Ric(X,\xi) = (n-1)(\alpha^2 + \sigma)\eta(X)$,
(v) $Ric(\xi,X) = (n-1)(\alpha^2 + \sigma)\eta(X)$,

provided $[a - b(n-1)] \neq 0$, α and σ are constants.

From (3.3), we get

$$QY = \left[\frac{a}{b}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right]Y + \left[\left\{\frac{a-b(n-1)}{b}\right\}(\alpha^2 + \sigma) + \frac{r}{bn}\left(\frac{a}{n-1} - b\right)\right]\eta(Y)\xi.$$

Contracting above with respect to Y and using (2.1), we get

$$r = \left[\frac{n(n-1)(a+b)(\alpha^2+\sigma)}{(2n-1)b-a}\right], \tag{3.13}$$

provided $[(2n-1)b - a] \neq 0$.

Using (3.13) in (3.3), we get

$$Ric(Y,Z) = \left[\frac{(\alpha^2+\sigma)\{(n+1)a-(n-1)b\}}{(2n-1)b-a}\right]g(Y,Z) + 2n[a - b(n-1)]\eta(Y)\eta(Z), \tag{3.14}$$

provided $[(2n-1)b - a] \neq 0$.

Using (3.13) and (3.14) in (3.1), we get

$$R(X,Y)Z = (\alpha^2 + \sigma)[g(Y,Z)X - g(X,Y)Z] + \left[\frac{2n(\alpha^2+\sigma)\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right][\eta(Z)X - \eta(X)Z]\eta(Y). \tag{3.15}$$

provided $[(2n-1)b - a] \neq 0$.

Since $\alpha \neq 0$, from (3.15), we can state the following theorem:

Theorem (3.4): In a pseudo W_8-flat Lorentzian α-para Kenmotsu manifold, if $[(2n-1)b - a] \neq 0$, then the manifold cannot be of constant curvature, provided α and σ are constants.

In particular

If $a = 1$ and $b = \frac{1}{n-1}$, then from (1.1) and (1.2), we get $'\widetilde{W}_8 = {}'W_8$. Also from (3.15), we get

$$R(X,Y)Z = (\alpha^2 + \sigma)[g(Y,Z)X - g(X,Y)Z]. \tag{3.16}$$

Therefore, from (3.16), we can state the following corollary:

Corollary (3.5): In a W_8-flat Lorentzian α-para Kenmotsu manifold, the manifold is of constant curvature, provided α and σ are constants.

Using (2.1), (2.2), (2.3), (3.14) and (3.15), we can state the following theorem:

Theorem (3.6): In a pseudo W_8-flat Lorentzian α-para Kenmotsu manifold, if $[(2n-1)b-a] \neq 0$, the following relation hold:

(i) $R(X,\xi)Y = (\alpha^2+\sigma)\left[1 - \frac{2n\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right][\eta(Y)X - \eta(X)Y]$,

(ii) $R(\xi,X)Y = (\alpha^2+\sigma)g(X,Y)\xi - (\alpha^2+\sigma)\left[1 - \frac{2n\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right]\eta(X)Y + \left[\frac{2n(\alpha^2+\sigma)\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right]\eta(X)\eta(Y)\xi$,

(iii) $R(X,Y)\xi = (\alpha^2+\sigma)\left[1 - \frac{2n\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right]\eta(Y)X - (\alpha^2+\sigma)g(X,Y)\xi + \left[\frac{2n(\alpha^2+\sigma)\{a-b(n-1)\}}{a\{(2n-1)b-a\}}\right]\eta(X)\eta(Y)\xi$,

(iv) $Ric(X,\xi) = (n-1)(\alpha^2+\sigma)\eta(X)$,

(v) $Ric(\phi X,\phi Y) = \left[\frac{(\alpha^2+\sigma)\{(n+1)a-(n-1)b\}}{(2n-1)b-a}\right][g(X,Y)+\eta(X)\eta(Y)]$,

provided $[a\{(2n-1)b-a\}] \neq 0$, α and σ are constants.

In particular

If $a = 1$ and $b = \frac{1}{n-1}$, then from (3.14), we get

$$Ric(Y,Z) = (n-1)(\alpha^2+\sigma)g(Y,Z). \tag{3.17}$$

Using (2.1), (2.2), (2.3), (3.16) and (3.17), we can state the following theorem:

Corollary (3.7): In a W_8-flat Lorentzian α-para Kenmotsu manifold, the following relation hold:

(i) $R(X,\xi)Y = (\alpha^2+\sigma)[\eta(Y)X - \eta(X)Y]$,

(ii) $R(\xi,X)Y = (\alpha^2+\sigma)[g(X,Y)\xi - \eta(X)Y]$,

(iii) $R(X,Y)\xi = (\alpha^2+\sigma)[\eta(Y)X - g(X,Y)\xi]$,

(iv) $Ric(X,\xi) = (n-1)(\alpha^2+\sigma)\eta(X)$,

(v) $Ric(\phi X,\phi Y) = Ric(X,Y) + (n-1)(\alpha^2+\sigma)\eta(X)\eta(Y)$,

provided α and σ are constants.

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