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Semi-symmetric non-metric π-recurrent connection on a Riemannian manifold

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Abstract

The aim of the present paper is to define a linear connection on a Riemannian manifold which is semi-symmetric non-metric and study some properties of the curvature tensor, Ricci tensor, scalar curvature and conformal curvature tensor with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$.

Keywords Riemannian manifold, semi-symmetric metric and non-metric connection, π-recurrent connection and conformal curvature tensor.

1. Introduction

The theory of linear connections placed an important place in Differential Geometry. In 1924, Friedmann and Schouten introduced the concept of semi-symmetric connection ($\tilde{S}\tilde{S}\tilde{C}$), while in 1932, Hayden defined metric connections with torsion. Pak (1969) studied semi-symmetric connection on the pseudo Riemannian manifold and established the relationship between semi-symmetric connection $\bar{D}$ and Levi-Civita connection D. In the paper 1970,Yano studied semi-symmetric metric connection ($\tilde{S}\tilde{S}\tilde{M}\tilde{C}$) on the Riemannin manifold in detail. Afterward many authors studied the application of this $\tilde{S}\tilde{S}\tilde{M}\tilde{C}$ to the other manifolds. Agashe and Chafle (1992) defined semi-symmetric non-metric connection ($\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}$) on a Riemannian manifold and observed properties of the curvature tenor and Weyl projective curvature tensor. This was furthure studied by Agashe and Chafle (1994), Sengupta et al. (2000), Prasad and verma (2004), Prasad and Singh (2006) and Many others. In continuation of this concept, Chaubey and Yildiz (2019) and Ram and Gupta (2022) established another new type of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}$ on a Riemannian manifold and obtained various geometrical properties.

The present paper deals with a Riemannian manifold admitting a semi-symmetric non-metric π-recurrent connetion ($\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$). After introduction, in section 2 of the present paper, an expression for the curvature tensor $\bar{R}(X,Y)Z$, the Ricci tensor $\bar{R}ic(Y,Z)$ and the scalar curvature $\bar{r}$ of the connection $\bar{D}$ have been deduced.

2. Curvature tensor, Ricci tensor and scalar curvature tensor of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$

A special type of linear connection on a Riemannian manifold given by Ram and Gupta (2022)

$$\bar{D}_X Y = D_X Y + a.\pi(X)Y + b.\pi(Y)X + c.g(X,Y)\xi \tag{2.1}$$

where a, b and c are non-zero real numbers and ξ is a vector field defined by $\pi(X) = g(X, \xi)$ for all X and $Y \epsilon M^n$. Using (2.1), the torsion $\bar{T}$ of M^n with respect to the connection $\bar{D}$ is given by

$$\bar{T}(X,Y) = \bar{D}_X Y - \bar{D}_Y X - [X,Y] = (a-b)[\pi(X)Y - \pi(Y)X]. \quad (2.2)$$

A linear connection satisfying (2.2) is called a $\tilde{S}\tilde{S}\tilde{C}$. Further from (2.1), we have

$$(\bar{D}_X g)(Y,Z) = -2a\eta(X)g(Y,Z) - (b+c)[\eta(Y)g(X,Z) + \eta(Z)g(X,Y)]. \quad (2.3)$$

Hence, a linear connection $\bar{D}$ defined by (2.1) satisfies (2.2) and (2.3) and hence authors called $\bar{D}$ as $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}$. The existence of such a connection on an n-dimensional Riemannian manifold in their paper. In particular this connection includes $\tilde{S}\tilde{S}\tilde{M}\tilde{C}$and $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}$ (see Ram and Gupta, 2022) .

Particular cases:

S. No.	Conditions	$\bar{D}_X Y = D_X Y + a.\pi(X)Y + b.\pi(Y)X + c.g(X,Y)\xi$	References
1.	$a = 0,\ b = 1$ and $c = -1$	$\bar{D}_X Y = D_X Y + \pi(Y)X - g(X,Y)\xi$	Yano (1970)
2.	$a = b = 1$ and $c = 0$	$\bar{D}_X Y = D_X Y + \pi(X)Y + \pi(Y)X$	Yano (1970) and Samanda (1981)
3.	$a = b = 1$ and $c = -1$	$\bar{D}_X Y = D_X Y + \pi(X)Y + \pi(Y)X - g(X,Y)\xi$	Yano (1970)
4.	$a = b = -\frac{1}{2}$ and $c = \frac{1}{2}$	$\bar{D}_X Y = D_X Y - \frac{1}{2}[\pi(X)Y + \pi(Y)X - g(X,Y)\xi]$	Folland (1970)
5.	$a = -1, \quad b = c = 0$	$\bar{D}_X Y = D_X Y - \pi(X)Y$	Liang (1994)
6.	$a = 1,$ and $b = c = 0$	$\bar{D}_X Y = D_X Y + \pi(X)Y$	Melhotra (2014)
7.	$a = 0,$ and $b = c = 1$	$\bar{D}_X Y = D_X Y + \pi(Y)X + g(X,Y)\xi$	De and Biswas (1996/97)
8.	$a = -1, b = 0$ and $c = 1$	$\bar{D}_X Y = D_X Y - \pi(X)Y + g(X,Y)\xi$	Barua and Mukhopadhya (1992)
9.	$a = c = 0,$ and $b = 1$	$\bar{D}_X Y = D_X Y + \pi(Y)X$	Mangla and Chafle (1992)
10.	$a = 0,$ and $b = c = -1$	$\bar{D}_X Y = D_X Y - \pi(Y)X - g(X,Y)\xi$	Kumar and Chaubey (2010)
11.	$c = 0$	$\bar{D}_X Y = D_X Y + a.\pi(X)Y + b.\pi(Y)X$	Chaturvedi and Pandey (2009)
12.	$a = -\frac{1}{2}, b = \frac{1}{2}$ and $c = 0$	$\bar{D}_X Y = D_X Y - \frac{1}{2}\pi(X)Y + \frac{1}{2}\pi(Y)X$	Chaubey and Yieldiz (2019)

13.	$a = -1, b = 1$ and $c = -1$	$\overline{D}_X Y = D_X Y - \pi(X)Y + \pi(Y)X - g(X,Y)\xi$	Prasad and Yadav (2025)

A $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\tilde{C}\ \overline{D}$ with torsion (2.2) is defined as a $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$ if

$$(\overline{D}_X\pi)(Y) = A(X)\pi(Y), \tag{2.4}$$

for arbitrary vector fields X and Y, where A is non-zero 1-form and Q is vector field satisfying $g(X,Q) = A(X)$.

By virtue of (2.1), we have

$$\begin{aligned}\overline{R}(X,Y)Z = {} & R(X,Y)Z + a[D_X\pi)(Y) - (D_Y\pi)(X)]Z + \\ & b[(D_X\pi)(Z)Y - (D_Y\pi)(Z)X] + c[g(Y,Z)(D_X\xi) - g(X,Z)(D_Y\xi)] \\ & +b^2[\pi(Y)X - \pi(X)Y]\pi(Z) + c^2[g(Y,Z)\pi(X) - g(X,Z)\pi(Y)]\xi \\ & +bc[g(Y,Z)X - g(X,Z)Y]\pi(\xi),\end{aligned} \tag{2.5}$$

where

$$\overline{R}(X,Y)Z = \overline{D}_X\overline{D}_Y Z - \overline{D}_Y\overline{D}_X Z - \overline{D}_{[X,Y]} Z,$$

and

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

be the curvature tensor of the connection $\overline{D}$ and D respectively.
From (2.1) and (2.4), we get

$$(\overline{D}_X\pi)(Y) = A(X)\pi(Y) + (a+b)\pi(X)\pi(Y) + cg(X,Y)\pi(\xi). \tag{2.6}$$

From (2.1), we get

$$\overline{D}_X\xi = D_X\xi + (a+c)\pi(X)\xi + b.\pi(\xi)X. \tag{2.7}$$

It is given that from (2.3)

$$(\overline{D}_X g)(\xi,Z) = -2a\pi(X)\pi(Z) - (b+c)[\pi(\xi)g(X,Z) + \pi(Z)\pi(X)]$$

$$\Longrightarrow \quad (\overline{D}_X\pi)Z - g(\overline{D}_X\xi,Z) = -2a\pi(X)\pi(Z) - (b+c)[\pi(\xi)g(X,Z) + \pi(Z)\pi(X)]. \tag{2.8}$$

From (2.4) and (2.8), we get

$$A(X)g(Z,\xi) - g(\overline{D}_X\xi,Z) = -(2a+b+c)\pi(X)g(Z,\xi) - (b+c)\pi(\xi)g(X,Z).$$

Above equation can be put as

$$\overline{D}_X\xi = A(X)\xi + (2a+b+c)\pi(X)Z + (b+c)\pi(\xi)X. \tag{2.9}$$

In view of (2.7) and (2.9), we have

$$D_X\xi = A(X)\xi + c\pi(\xi)X + (a+b)\pi(X)\xi. \tag{2.10}$$

By virtue of (2.5), (2.6) and (2.10), we get

$$\begin{aligned}\overline{R}(X,Y)Z = {} & R(X,Y)Z + a[A(X)\pi(Y) - A(Y)\pi(X)]Z + \\ & b[A(X)Y - A(Y)X]\pi(Z) + c[g(Y,Z)A(X) - g(X,Z)A(Y)]\xi \\ & -ab[\pi(Y)X - \pi(X)Y]\pi(Z) - bc[g(Y,Z)X - g(X,Z)Y]\pi(\xi)\end{aligned}$$

$$-ac[g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]\xi. \tag{2.11}$$

Then from (2.11), we get

$$'\bar{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) + a[A(X)\pi(Y)-A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W)-A(Y)g(X,W)]\pi(Z) + c[g(Y,Z)A(X)-g(X,Z)A(Y)]\pi(W) -ab[\pi(Y)g(X,W)-\pi(X)g(Y,W)]\pi(Z) - bc[g(Y,Z)g(X,W) -g(X,Z)g(Y,W)\pi(\xi)] - ac[g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]\pi(W), \tag{2.12}$$

where

$$'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W) \text{ and } {}'R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

In particular, if $a=0, b=1$ and $c=-1$, then equation (2.12) reduces in

$$'\bar{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) + [g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\pi(\xi)] +A(X)[\pi(Z)g(Y,W)-g(Y,Z)\pi(W)] -A(Y)[g(X,W)\pi(Z)-g(X,Z)\pi(W)].$$

This results was obtained by De and Ghosh (1994) in their paper.

From (2.12), we get

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = 0. \tag{2.13}$$

Moreover,

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) \neq 0,$$

and

$$'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) \neq 0.$$

Using (2.12) and the first Bianchi identity with respect to the Levi-Civita connection D, we have

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,Z,X,W) + {}'\bar{R}(Z,X,Y,W) = (a+b)[\{A(Y)\pi(X)-A(X)\pi(Y)\}g(Z,W) +\{A(Z)\pi(Y)-A(Y)\pi(Z)\}g(X,W) +\{A(X)\pi(Z)-A(Z)\pi(X)\}g(Y,W)]. \tag{2.14}$$

We refer to equation (2.14) as the first Bianchi identity with respect to $\tilde{S}\tilde{S}\tilde{N}\pi\tilde{M}\tilde{R}\tilde{C}\ \bar{D}$.

In particular, if $a=0, b=1$ and $c=-1$ in (2.14) , then we get

$$'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,Z,X,W) + {}'\bar{R}(Z,X,Y,W) = [g(Y,W)\{A(X)\pi(Z)-A(Z)\pi(X)\} +g(Z,W)\{A(Y)\pi(X)-A(X)\pi(Y)\} +g(X,W)\{A(Z)\pi(Y)-A(Y)\pi(Z)\}].$$

De and Ghosh (1994) established this results and termed it the first Bianchi identity with respect to $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$.

Let $\bar{R}ic(Y,Z)$ and $Ric(Y,Z)$ be the Ricci tensor of the connection D and $\bar{D}$ respectively. Also, let $\bar{r}$ and r be the scalar curvature of the connection D and $\bar{D}$ respectively. Putting e_i for X and W in (2.12), where $\{e_i\}, i = 1,2,3,\dots,n$ is an orthonormal basis of the tangent space at a point and 1 is summed for $1 \le i \le n$, we get

$$\bar{R}ic\ (Y,Z) = Ric(Y,Z) + [a + b(n-1) + c]\pi(Z)A(Y) + a.A(Z)\pi(Y)$$
$$-a[b(n-1) + c]\pi(Y)A(Z) - c[a + b(n-1) + c]g(Y,Z)\pi(\xi)$$
$$+cg(Y,Z)A(\xi). \tag{2.15}$$

If we put $a = 0, b = 1$ and $c = -1$ in (2.15) , then it becomes

$$\bar{R}ic\ (Y,Z) = Ric(Y,Z) + (n-1)g(Y,Z)\pi(\xi) - (n-2)\pi(Z)A(Y) - g(Y,Z)A(\xi).$$

This $\bar{R}ic$ tensor was gives De and Ghosh (1994) in their paper.

Similarly, if we put for Y and Z in (2.15), we get

$$\bar{r} = r - (n-1)[ab + nbc + ac]\pi(\xi) - (n-1)(b-c)A(\xi). \tag{2.16}$$

Further (2.16) satisfies for $a = 0, b = 1$ and $c = -1$ and this was shown by De and Ghosh in 1994.

From the above discussion, we can state the following theorem:

Theorem (2.1): For a Riemannian manifold M^n admitting $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$,

(i) the curvature tensor $\bar{R}$ of $\bar{D}$ in given by (2.11),

(ii) the curvature tensor $\bar{R}$ of $\bar{D}$ is skew symmetric in first pair of sots where as not symmetric in last pair slots,

(iii) the curvature tensor $\bar{R}$ of $\bar{D}$ satisfies Bianchi's 1st identity if $(a+b) = 0$,

(iv) the Ricci tenor $\bar{R}ic(Y,Z)$ and scalar curvature $\bar{r}$ of $\bar{D}$ are given by the expression (2.15) and (2.16), respectively,

(v) the scalar curvature of $\bar{r}$ of $\bar{D}$ is equal to the scalar curvature of r of D, if and only if $(ab + nbc + ac)\pi(\xi) = (b-c)A(\xi)$,

(vi) the Ricci tenor $\bar{R}$ic of $\bar{D}$ is symmetric if and only if $A(Z)\pi(Y) = A(Y)\pi(Z)$, provided $(2a + (n-1)b + c) \neq 0$,

(vii) the Ricci tenor $\bar{R}$ic of $\bar{D}$ is skew-symmetric if and only if
$$Ric(Y,Z) = \frac{(n-2)}{2}[A(Y)\pi(Z) + A(Z)\pi(Y)] - [(n-1)\pi(\xi) - A(\xi)].$$

3. Conformal curvature tensor equipped with $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$ on a Riemannian manifold

Let $'\bar{C}(X,Y,Z,W)$ and $'C(X,Y,Z,W)$ be the conformal curvature tensor of the connection $\bar{D}$ and D, respectively. Then

$$'\bar{C}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{1}{n-2}[\bar{R}ic(Y,Z)g(X,W) - \bar{R}(X,Z)g(Y,W)$$
$$+g(Y,Z)\bar{R}ic(X,W) - g(X,Z)\bar{R}ic(Y,W)]$$
$$+\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{3.1}$$

and

$$'C(X,Y,Z,W) = 'R(X,Y,Z,W) - \frac{1}{n-2}[Ric(Y,Z)g(X,W) - R(X,Z)g(Y,W)$$
$$+g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)]$$
$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \quad (3.2)$$

where $'\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$ and $'C(X,Y,Z,W) = g(C(X,Y)Z,W)$.

In consequences of (2.12), (2.15), (2.16), (3.1), and (3.2), we get

$$'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W)$$
$$+ b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z)$$
$$+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W)$$
$$-ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z)$$
$$-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) -$$
$$\frac{1}{n-2}[\{a - ab(n-1) - ac\} \times$$
$$\{\big(\pi(X)g(Y,Z) - g(X,Z)\pi(Y)\big)\pi(W) +$$
$$\big(\pi(Y)g(X,W) - \pi(X)g(Y,W)\big)\pi(Z)\} -$$
$$(a + b(n-1) + c)\{(A(Y)g(X,W)$$
$$-g(Y,W)A(X)\big)\pi(Z) +$$
$$\big(A(X)g(Y,Z) - g(X,Z)A(Y)\big)\pi(W)\}] -$$
$$\frac{1}{n-2}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$$
$$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (3.3)$$

Hence, from (3.3), we have the following theorem:

Theorem (3.1): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$, then its conformal curvature tensor $\bar{D}$ is same as the conformal curvature tensor of the manifold D, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) +$$
$$c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) -$$
$$ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-2}[\{a - ab(n-1) - ac\} \times$$
$$\{\big(\pi(X)g(Y,Z) - g(X,Z)\pi(Y)\big)\pi(W) + \big(\pi(Y)g(X,W) - \pi(X)g(Y,W)\big)\pi(Z)\}$$
$$-(a + b(n-1) + c)\{\big(A(Y)g(X,W) - g(Y,W)A(X)\big)\pi(Z) + (A(X)g(Y,Z)$$
$$-g(X,Z)A(Y)\big)\pi(W)\}] - \frac{1}{n-2}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$$
$$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

In particular , if $a = 0$, $b = 1$ and $c = -1$ in (3.3), we get

$$'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W).$$

Hence, we have the following theorem:

Theorem (De and Ghosh, 1994): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$, then its conformal curvature tensor $\bar{D}$ is same as the conformal curvature tensor of the manifold D.

Now, suppose the Ricci tensor of $\bar{D}$ of the $\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$ vanishes identically, then we have

$$\bar{R}ic(X,Y) = 0. \tag{3.4}$$

That is

$$\bar{r} = 0. \tag{3.5}$$

Using (3.4) and (3.5) in (3.1), we get

$$'\bar{C}(X,Y,Z,W) = '\bar{R}(X,Y,Z,W). \tag{3.6}$$

From (3.3) and (3.6), we get

$$\begin{aligned}'\bar{R}(X,Y,Z,W) = {} & 'C(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) \\ & + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ & + c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) \\ & - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) \\ & - ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \\ & \frac{1}{n-2}[\{a - ab(n-1) - ac\} \times \\ & \{\big(\pi(X)g(Y,Z) - g(X,Z)\pi(Y)\big)\pi(W) + \\ & \big(\pi(Y)g(X,W) - \pi(X)g(Y,W)\big)\pi(Z)\} - \\ & (a + b(n-1) + c)\{(A(Y)g(X,W) \\ & - g(Y,W)A(X)\big)\pi(Z) + \\ & \big(A(X)g(Y,Z) - g(X,Z)A(Y)\big)\pi(W)\}] - \\ & \frac{1}{n-2}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times \\ & [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].\end{aligned} \tag{3.7}$$

Hence, from (3.7), we have the following theorem:

Theorem (3.2): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{N}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$ whose Ricci tensor vanishes, then the curvature tensor of the connection $\bar{D}$ is equal to the conformal curvature tensor of the manifold D, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) +$$

$c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) -$

$ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-2}[\{a - ab(n-1) - ac\} \times$

$\{(\pi(X)g(Y,Z) - g(X,Z)\pi(Y))\pi(W) + (\pi(Y)g(X,W) - \pi(X)g(Y,W))\pi(Z)\}$

$-(a + b(n-1) + c)\{(A(Y)g(X,W) - g(Y,W)A(X))\pi(Z) + (A(X)g(Y,Z)$

$-g(X,Z)A(Y))\pi(W)\}] - \frac{1}{n-2}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$

$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$

In particular , if $a = 0$, $b = 1$ and $c = -1$ in (3.7), we get

$'C(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W).$

Hence, we have the following theorem due to De and Ghosh (1994}.

Theorem (De and Ghosh, 1994): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$, whose Ricci tensor vanishes, then the conformal curvature tensor of the connection $\bar{D}$ is equal to the conformal curvature tensor of the manifold D.

Further, if the curvature tensor of the $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$ vanishes, that is,

$'\bar{R}(X,Y,Z,W) = 0.$ (3.8)

Therefore, from (3.7) and (3.8), we get

$'C(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z)$

$+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) -$

$ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-2}[\{a - ab(n-1) - ac\} \times$

$\{(\pi(X)g(Y,Z) - g(X,Z)\pi(Y))\pi(W) + (\pi(Y)g(X,W) - \pi(X)g(Y,W))\pi(Z)\}$

$-(a + b(n-1) + c)\{(A(Y)g(X,W) - g(Y,W)A(X))\pi(Z) + (A(X)g(Y,Z)$

$-g(X,Z)A(Y))\pi(W)\}] - \frac{1}{n-2}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$

$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$ (3.9)

Hence, from (3.9), we have the following theorem:

Theorem (3.3): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$, whose curvature tensor vanishes, then the manifold is conformally flat, if and only if

$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) +$

$c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) -$

$ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-2}[\{a - ab(n-1) - ac\} \times$

$\{(\pi(X)g(Y,Z) - g(X,Z)\pi(Y))\pi(W) + (\pi(Y)g(X,W) - \pi(X)g(Y,W))\pi(Z)\}$

$$-(a+b(n-1)+c)\{(A(Y)g(X,W)-g(Y,W)A(X))\pi(Z)+(A(X)g(Y,Z)$$
$$-g(X,Z)A(Y))\pi(W)\}]-\frac{1}{n-2}[(ab+nbc+ac)\pi(\xi)+(b-c)A(\xi)]\times$$
$$[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]=0.$$

In particular, if $a=0,\ b=1$ and $c=-1$ in (3.9), we get

$$'C(X,Y,Z,W)=0.$$

Consequently, we have the following results due to De and Ghosh (1994).

Theorem (De and Ghosh, 1994): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$ whose curvature tensor vanishes, then the manifold is conformally flat.

4. Projective curvature tensor equipped with $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$ on a Riemannian manifold

Let $'\bar{P}(X,Y,Z,W)$ and $'P(X,Y,Z,W)$ be the projective curvature tensor of the connection $\bar{D}$ and D, respectively. Then

$$'\bar{P}(X,Y,Z,W)={}'\bar{R}(X,Y,Z,W)-\frac{1}{n-1}[\bar{R}ic(Y,Z)g(X,W)-\bar{R}(X,Z)g(Y,W)]\,, \tag{4.1}$$

and

$$'P(X,Y,Z,W)={}'R(X,Y,Z,W)-\frac{1}{n-1}[Ric(Y,Z)g(X,W)-R(X,Z)g(Y,W)], \tag{4.2}$$

where $'\bar{P}(X,Y,Z,W)=g(\bar{P}(X,Y)Z,W)$ and $'P(X,Y,Z,W)=g(P(X,Y)Z,W)$.

In consequences of (2.12), (2.15), (4.1), and (4.2), we get

$$'\bar{P}(X,Y,Z,W)={}'P(X,Y,Z,W)+a[A(X)\pi(Y)-A(Y)\pi(X)]g(Z,W)$$
$$+c[A(X)g(Y,Z)-g(X,Z)A(Y)]\pi(W)$$
$$-ac[\pi(X)g(Y,Z)-\pi(Y)g(X,Z)]\pi(W)-$$
$$\frac{1}{n-1}\big[\{a(A(Z)\pi(Y)-A(Y)\pi(Z))+$$
$$c(A(\xi)g(Y,Z)-\pi(Z)A(Y))-$$
$$ac(\pi(\xi)g(Y,Z)-\pi(Y)\pi(Z))\}g(X,W)$$
$$-\{a(A(Z)\pi(X)-A(X)\pi(Z))+$$
$$c(A(\xi)g(X,Z)-A(X)\pi(Z))-$$
$$ac(g(X,Z)\pi(\xi)-\pi(X)\pi(Z))\}g(Y,W)]. \tag{4.3}$$

Hence, from (4.3), we have the following theorem:

Theorem (4.1): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$, then its projective curvature tensor $'\bar{P}$ is same as the projective curvature tensor of the manifold D, if and only if

$$a[A(X)\pi(Y)-A(Y)\pi(X)]g(Z,W)+c[A(X)g(Y,Z)-g(X,Z)A(Y)]\pi(W)$$

$-ac[\pi(X)g(Y,Z)-\pi(Y)g(X,Z)]\pi(W)-\frac{1}{n-1}\Big[\big\{a\big(A(Z)\pi(Y)-A(Y)\pi(Z)\big)+$

$c\big(A(\xi)g(Y,Z)-\pi(Z)A(Y)\big)-ac(\pi(\xi)g(Y,Z)-\pi(Y)\pi(Z))\big\}g(X,W)$

$-\big\{a\big(A(Z)\pi(X)-A(X)\pi(Z)\big)\pi(Z)+c\big(A(\xi)g(X,Z)-A(X)\pi(Z)\big)$

$-ac\big(g(X,Z)\pi(\xi)-\pi(X)\pi(Z)\big)\big\}g(Y,W)\Big]=0.$

In particular, if $a=0,\ b=1$ and $c=-1$ in (4.3) becomes,

$$'\bar{P}(X,Y,Z,W)='P(X,Y,Z,W)-[A(X)g(Y,Z)-g(X,Z)A(Y)]\pi(W)$$
$$\frac{1}{n-1}\Big[\big(A(\xi)g(X,Z)-A(X)\pi(Z)\big)g(Y,W)-$$
$$\big(A(\xi)g(Y,Z)-\pi(Z)A(Y)\big)g(X,W)\Big].$$

Hence, Theorem (4.1) can be re-written as follows:

Theorem (4.2): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$, then its projective curvature tensor $'\bar{P}$ is same as the projective curvature tensor of the manifold D, if and only if

$$[A(X)g(Y,Z)-g(X,Z)A(Y)]\pi(W)-\frac{1}{n-1}\Big[\big(A(\xi)g(X,Z)-A(X)\pi(Z)\big)g(Y,W)-$$
$$\big(A(\xi)g(Y,Z)-\pi(Z)A(Y)\big)g(X,W)\Big]=0.$$

Let us assume that Ricci tensor of $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$ vanishes, then we have

$$\bar{R}ic(X,Y)=0. \tag{4.4}$$

Hence, from (4.1) and (4.4), we get

$$'\bar{P}(X,Y,Z,W)='\bar{R}(X,Y,Z,W). \tag{4.5}$$

Thus, from (4.3) and (4.5), we get

$$'\bar{R}(X,Y,Z,W)='P(X,Y,Z,W)+a[A(X)\pi(Y)-A(Y)\pi(X)]g(Z,W)$$
$$+c[A(X)g(Y,Z)-g(X,Z)A(Y)]\pi(W)$$
$$-ac[\pi(X)g(Y,Z)-\pi(Y)g(X,Z)]\pi(W)-$$
$$\frac{1}{n-1}\Big[\big\{a\big(A(Z)\pi(Y)-A(Y)\pi(Z)\big)+$$
$$c\big(A(\xi)g(Y,Z)-\pi(Z)A(Y)\big)-$$
$$ac(\pi(\xi)g(Y,Z)-\pi(Y)\pi(Z)\}g(X,W)$$
$$-\big\{a\big(A(Z)\pi(X)-A(X)\pi(Z)\big)+$$
$$c\big(A(\xi)g(X,Z)-A(X)\pi(Z)\big)-$$
$$ac\big(g(X,Z)\pi(\xi)-\pi(X)\pi(Z)\big)\big\}g(Y,W)\Big]. \tag{4.6}$$

Hence, we can state the following theorem:

Theorem (4.3): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \overline{D}$ whose Ricci tensor vanishes, then the curvature tensor of the connection $\overline{D}$ is equal to the projective curvature tensor of the manifold D, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W)$$
$$-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-1}\big[\big\{a\big(A(Z)\pi(Y) - A(Y)\pi(Z)\big) +$$
$$c\big(A(\xi)g(Y,Z) - \pi(Z)A(Y)\big) - ac(\pi(\xi)g(Y,Z) - \pi(Y)\pi(Z)\big\}g(X,W)$$
$$-\big\{a\big(A(Z)\pi(X) - A(X)\pi(Z)\big) + c\big(A(\xi)g(X,Z) - A(X)\pi(Z)\big)$$
$$-ac\big(g(X,Z)\pi(\xi) - \pi(X)\pi(Z)\big)\big\}g(Y,W)\big] = 0.$$

In particular, if $a = 0,\ b = 1$ and $c = -1$, then from (4.6) and Theorem (4.3), we have the following theorem:

Theorem (4.4): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{M}\pi\tilde{R}\tilde{C}\ \overline{D}$ whose Ricci tensor vanishes, then the curvature tensor of the connection $\overline{D}$ is equal to the projective curvature tensor of the manifold D, if and only if

$$[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \frac{1}{n-1}\big[\big(A(\xi)g(Y,Z) - \pi(Z)A(Y)\big)g(X,W) -$$
$$\big(A(X)\pi(Z) - A(\xi)g(X,Z)\big)g(Y,W)\big] = 0.$$

Now, if the curvature tensor of the $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \overline{D}$ vanishes, that is,

$$'\overline{R}(X,Y,Z,W) = 0. \tag{4.7}$$

In view of (4.6) and (4.8), we get

Theorem (4.5): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \overline{D}$ whose curvature tensor vanishes, then the manifold is projectively flat, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W)$$
$$-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) - \frac{1}{n-1}\big[\big\{a\big(A(Z)\pi(Y) - A(Y)\pi(Z)\big) +$$
$$c\big(A(\xi)g(Y,Z) - \pi(Z)A(Y)\big) - ac(\pi(\xi)g(Y,Z) - \pi(Y)\pi(Z)\big\}g(X,W)$$
$$-\big\{a\big(A(Z)\pi(X) - A(X)\pi(Z)\big) + c\big(A(\xi)g(X,Z) - A(X)\pi(Z)\big)$$
$$-ac\big(g(X,Z)\pi(\xi) - \pi(X)\pi(Z)\big)\big\}g(Y,W)\big] = 0.$$

In particular, if $a = 0,\ b = 1$ and $c = -1$, then from Theorem (4.5), we have the following theorem:

Theorem (4.6): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{M}\pi\tilde{R}\tilde{C}\ \overline{D}$ whose curvature tensor vanishes, then the manifold is projectively flat, if and only if

$$[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \frac{1}{n-1}\big[\big(\pi(Z)A(Y) - A(\xi)g(Y,Z)\big)$$
$$-A(X)\pi(Z) - A(\xi)g(X,Z)\big] = 0.$$

5. Concircular curvature tensor equipped with $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$ on a Riemannian manifold

Let $'\bar{L}(X,Y,Z,W)$ and $'L(X,Y,Z,W)$ be the concircular curvature tensor of the connection $\bar{D}$ and D, respectively. Then

$$'\bar{L}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{5.1}$$

and

$$'L(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{5.2}$$

where $'\bar{L}(X,Y,Z,W) = g(\bar{L}(X,Y)Z,W)$ and $'L(X,Y,Z,W) = g(L(X,Y)Z,W)$.

In view of (2.12), (2.16), (5.1), and (5.2), we get

$$\begin{aligned} '\bar{L}(X,Y,Z,W) = {}&'L(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) \\ &+ b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ &+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) \\ &-ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) \\ &-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) + \\ &\frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times \\ &[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{5.3}$$

Hence, from (5.3), we have the following theorem:

Theorem (5.1): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$, then its concircular curvature tensor $\bar{D}$ is same as the concircular curvature tensor of the manifold D, if and only if

$$\begin{aligned} &a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ &+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) \\ &-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) + \frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times \\ &[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \end{aligned}$$

In particular , if $a = 0,\ b = 1$ and $c = -1$, then from (5.3), we have

$$\begin{aligned} '\bar{L}(X,Y,Z,W) = {}&'L(X,Y,Z,W) + [A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ &-[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \\ &\frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{5.4}$$

Hence, from (5.4), we have the following theorem:

Theorem (5.2): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{M}\pi\tilde{R}\tilde{C}\ \bar{D}$, then its concircular curvature tensor $\bar{D}$ is same as the concircular curvature tensor of the manifold D, if and only if

$$[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) - [A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

Let us consider that the scalar curvature of $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$ vanishes, then

$$\bar{r} = 0. \tag{5.5}$$

Therefore, in view of (5.1) and (5.5), we get

$$'\bar{L}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W). \tag{5.6}$$

Using (5.3) in (5.6), we get

$$\begin{aligned} '\bar{R}(X,Y,Z,W) = {}&'L(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) \\ &+ b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ &+ c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) \\ &- ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) \\ &- ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) + \\ &\frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times \\ &[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{5.7}$$

Hence, from (5.7), we have the following theorem:

Theorem (5.3): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{N}\tilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$, whose scalar curvature vanishes, then the curvature tensor $\bar{D}$ is equal to the concircular curvature tensor of the manifold D, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) + c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) - ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) + \frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

In particular , if $a = 0$, $b = 1$ and $c = -1$, then from (5.7), we have

$$\begin{aligned} '\bar{R}(X,Y,Z,W) = {}&'L(X,Y,Z,W) + [A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) \\ &- [A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \\ &\frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{5.8}$$

Hence, from (5.8), we have the following theorem:

Theorem (5.4): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\tilde{M}\pi\tilde{R}\tilde{C}$ $\bar{D}$, whose scalar curvature vanishes, then the curvature tensor $\bar{D}$ is equal to the concircular curvature tensor of the manifold D, if and only if

$$[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) - [A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) + \frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

Let us assume that the curvature tensor of the $\tilde{S}\tilde{S}\tilde{N}\widetilde{M}\pi\tilde{R}\tilde{C}\ \ \overline{D}$ vanishes, that is,

$$'\bar{R}(X,Y,Z,W) = 0. \tag{5.9}$$

In view of (5.7) and (5.9), we get

$$'L(X,Y,Z,W) + a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W)$$
$$+ b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z)$$
$$+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W)$$
$$-ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z)$$
$$-ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) +$$
$$\frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$$
$$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \tag{5.10}$$

Hence, from (5.10), we have the following theorem:

Theorem (5.5): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}\ \ \overline{D}$, whose curvature tensor vanishes, then the manifold is concircularly flat, if and only if

$$a[A(X)\pi(Y) - A(Y)\pi(X)]g(Z,W) + b[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z)$$
$$+c[A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) - ab[\pi(Y)g(X,W) - \pi(X)g(Y,W)]\pi(Z) -$$
$$ac[\pi(X)g(Y,Z) - \pi(Y)g(X,Z)]\pi(W) + \frac{1}{n}[(ab + nbc + ac)\pi(\xi) + (b - c)A(\xi)] \times$$
$$[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

If, we use $a = 0$, $b = 1$ and $c = -1$ in (5.10), then we have

$$'L(X,Y,Z,W) + [A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) - [A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W)$$
$$+\frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \tag{5.11}$$

Hence, in view of (5.8), we have the following theorem:

Theorem (5.6): If a Riemannian manifold M^n admits $\tilde{S}\tilde{S}\widetilde{M}\pi\tilde{R}\tilde{C}\ \ \overline{D}$, whose curvature tensor vanishes, then the manifold is concircularly flat if and only if

$$[A(X)g(Y,W) - A(Y)g(X,W)]\pi(Z) - [A(X)g(Y,Z) - g(X,Z)A(Y)]\pi(W) +$$
$$\frac{1}{n}[-n\pi(\xi) + 2A(\xi)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0.$$

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