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A study of nearly Ricci recurrent Trans-Sasakian Manifold

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Abstract

The object of the present paper is to study nearly Ricci-recurrent trans-Sasakain manifold and its various properties. Finally, we construct a non-trivial example of nearly Ricci recurrent trans-Sasakain manifold.

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1. Introduction

In 1985, a trans-Sasakain manifold of type (α, β) which is of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ known as the cosymplectic, $\alpha-$Sasakian and $\beta-$Kenmotsu manifolds, respectively, was introduced by Oubina (1985). If in particular case, $\alpha = 1, \beta = 0$ and $\alpha = 0,\ \beta = 1$, the trans-Sasakian manifold becomes a Sasakian manifold and Kenmotsu manifold respectively. The geometry of trans-Sasakian manifold have been studied many worker such as Kim et al. (2002), De and Tripathi (2003), De and De (2012), Deshmukh and Tripathi (2013), Shukla and Jaiswal (2014), Singh and Yadav (2016), Singh (2018), Chaubey and De (2021), Danish (2018), Pahan (2020), Yadav, Prasad and Pathak (2025) and many others.
In 1952, Patterson introduced Ricci recurrent manifolds whose Ricci tensor S follows:

$$(D_X S)(Y, Z) = A(X)S(Y, Z), \tag{1.1}$$

for some 1-form A, where D and S denotes the operator of covariant differentiation with respect to metric tensor g and Ricci tensor, respectively. Ricci recurrent man-

ifold was denoted by him as $\mathscr{R}_n$. Ricci recurrent manifold has been developed by many authors such as Chaki (1956), Prakash (1962), Yamaguchi and Matsumoto (1968), Roter (1974) and many others.
De, Guha and Kamilya (1995) introduced and studied generalized Ricci recurrent manifold whose Ricci tensor follows:

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \tag{1.2}$$

where A and B are two non-zero 1-forms. Generalized Ricci recurrent manifold were denoted by them as $\mathscr{G}\mathscr{R}_n$. Many investigators investigated generalized Ricci-recurrent manifold on various aspects.
In 2023, Prasad and Yadav introduced nearly recurrent with two non-zero 1-forms A and B as follows:

$$\begin{aligned}(D_U R)(X, Y)Z =& [A(U) + B(U)]R(X, Y)Z+ \\ & B(U)[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{1.3}$$

Such a manifold have been denoted by them as $(\mathscr{N}\mathscr{R})_n$. The name nearly recurrent Riemannian manifold was chosen because if $B = 0$ in (1.3) then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name "Nearly recurrent manifold" for the manifold defined by (1.3) and the use of the symbol $(\mathscr{N}\mathscr{R})_n$ for it. $(\mathscr{N}\mathscr{R})_n$ have been studied by Prasad and Yadav (2024) and many other workers.
Currently, in 2024, Yadav and Prasad investigated a super hyper-generalized recurrent manifold with four non-zero 1-forms A, B, C and D' as follows:

$$\begin{aligned}(D_U R)(X, Y)Z =& [A(U) + B(U)]R(X, Y)Z+ \\ & C(U)g(Y, Z)X + D'(U)g(X, Z)Y.\end{aligned} \tag{1.4}$$

Such a manifold have been denoted by them as $(\mathscr{S}\mathscr{H}\mathscr{G}\mathscr{R})_n$.
The beauty of such $(\mathscr{S}\mathscr{H}\mathscr{G}\mathscr{R})_n$ space is that it has the flavor of:

(i) symmetric space for $A = B = C = D' = 0$ Kobayashi and Nomizu (1963), Desai and Amur (1975),

(ii) recurrent space for $B = C = D' = 0$ Ruse (1946),

(iii) semi-generalized recurrent space for $B = D' = 0$ Prasad (2000) (25),

(iv) generalized recurrent space for $B = 0$ and D' is replaced by $-C$ De and Guha (1991),

(v) extended recurrent space for $B = 0$ Singh and Prasad (2020),

(vi) nearly recurrent space if C is replaced by B and D' is replaced by $-B$ Prasad and Yadav (2023).

This justifies the name "Super hyper-generalized recurrent Riemannian manifold" for the manifold (1.4) and the use of the symbol $(\mathscr{SHGR})_n$, $(n > 2)$ for it.
Recently Prasad and Yadav (2021) introduced and studied nearly Ricci recurrent manifold whose Ricci tensor S follows:

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z), \tag{1.5}$$

$\forall X, Y \in \chi(M)$, where A and B are two non-zero 1-forms and ρ_1 and ρ_2 are two vector fields such that

$$A(X) = g(\rho_1, X), \qquad and \quad B(X) = g(\rho_3, X). \tag{1.6}$$

Such a manifold denoted by them as $\mathscr{N}\{\mathscr{R}(\mathscr{R}_n)\}$. The name nearly Ricci recurrent Riemannian manifold was chosen because if $B = 0$ in (1.5) then the manifold reduces to a Ricci recurrent manifold which is very close to Ricci recurrent space. This justifies the name "Nearly Ricci recurrent manifold" for the manifold defined by (1.5) and the use of the symbol $\mathscr{N}\{\mathscr{R}(\mathscr{R}_n)\}$ for it. The nearly Ricci recurrent manifold have been studied by Yadav and Prasad (2023, 2023).

2. Preliminaries

A (2n+1) dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if it admits a tensor ϕ of type (1,1), ξ is a contravariant vector fields of type (0,1) and 1-form η is a covariant tensor of the type (1,0) satisfying Blair (1976, 2002):

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta(\phi X) = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X) \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) + g(\phi X, Y) = 0. \tag{2.3}$$

An almost contact metric structure (ϕ, ξ, η, g) on M^{2n+1} is called a trans-Sasakain structure Oubina (1985) if $(M \times \mathscr{R}, J, G)$ belongs to the new class W_4 (1980) where J is the almost complex structure on $M \times \mathscr{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X, -f\xi, \eta(X)f\frac{d}{dt}\right),$$

for all vector fields X on M^{2n+1} and smooth functions on $M \times \mathscr{R}$ and G is the product metric on $M \times \mathscr{R}$. This may be expressed by the condition Blair and Oubina (1990) :

$$(D_X\phi)(Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

for some smooth functions α and β on M^{2n+1} and hence we say that the trans-Sasakain structure is of the type (α, β).
We have, from Blair and Oubina (1990), Prasad and Srivastava (2013) and Singh (2018)

$$D_X\xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.5}$$

$$(D_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.6}$$

$$\begin{aligned} R(X,Y)\xi =& (\alpha^2-\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.7}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)\eta(X), \tag{2.8}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.9}$$

$$S(X,\xi) = [2n(\alpha^2-\beta^2) - \xi\beta)]\eta(X) - (2n-1)X\beta + (\phi X)\alpha, \tag{2.10}$$

$$S(\xi,\xi) = 2n[(\alpha^2-\beta^2) - \xi\beta)], \tag{2.11}$$

$$Q\xi = [2n(\alpha^2-\beta^2) - \xi\beta)]\xi - (2n-1)grad\beta + \phi(grad\alpha), \tag{2.12}$$

where R, S and Q are the curvature tensor, $\mathscr{R}T$ and Ricci operator of trans-Sasakian manifold respectively.
When $\phi(grad\alpha) = (2n-1)grad\beta$, then equations (2.10), (2.11) and (2.12) becomes Prasad and Srivastava (2013) and Singh (2018)

$$S(X,\xi) = 2n(\alpha^2-\beta^2))\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = 2n(\alpha^2-\beta^2), \tag{2.14}$$

$$Q\xi = 2n(\alpha^2-\beta^2))\xi, \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{2.16}$$

$\forall\ X, Y \in \chi(M)$.
After introduction and preliminaries, in section 3 we have examined the character of 1-forms A and B on nearly Ricci recurrent trans-Sasakian manifold. Section 4 deals nearly Ricci recurrent trans-Sasakian manifold with cyclic Ricci tensor. Finally, the existence of nearly Ricci recurrent trans-Sasakian manifold is ensured by a non trivial example.

3. Nearly Ricci recurrent Trans-Sassakian manifold

A trans-Sasakian manifold (M^{2n+1}, g) is called nearly Ricci recurrent (2021) if its Ricci tensor S follows (1.5)

$$(D_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + B(X)g(Y,Z). \tag{3.1}$$

In view of equation (3.1), we obtain

$$[A(X)+B(X)]S(Y,Z)+B(X)g(Y,Z) = XS(Y,Z)-S(D_X Y,Z)-S(Y,D_X Z). \tag{3.2}$$

Putting ξ for Y and Z in (3.2), we get

$$[A(X) + B(X)]S(\xi,\xi) + B(X)g(\xi,\xi) = XS(\xi,\xi) - S(D_X\xi,\xi) - S(\xi,D_X\xi). \tag{3.3}$$

In view of (2.3), (2.5), (2.11) and (3.3), we have

$$\begin{aligned} B(X) = & \left[\frac{2n}{1+2n(\alpha^2-\beta^2-\xi\beta)}\right][X(\alpha^2-\beta^2-\xi\beta)-(\alpha^2-\beta^2-\xi\beta)A(X)]- \\ & \left[\frac{1}{1+2n(\alpha^2-\beta^2-\xi\beta)}\right][2(2n-1)(\alpha\phi X+\beta\phi^2 X)\beta \\ & +2(\alpha\phi^2 X-\beta\phi X)\alpha]. \end{aligned} \tag{3.4}$$

Hence, we have the following:

Theorem 3.1 *Let (M^{2n+1}, g) be a nearly Ricci recurrent trans-Sasakian manifold. Then the 1-forms A and B are related by (3.4), provided $1 + 2n(\alpha^2 - \beta^2 - \xi\beta) \neq 0$.*

The Particular cases of Theorem 3.1 as follows

S.No.	Case	Manifold	Relation between 1-form A, B and C	Ref.
1.	$\alpha \neq 0, \beta = 0$	nearly Ricci recurrent α-Sasakian manifold	$B(X) = -\frac{2n\alpha^2}{2n\alpha^2+1}A(X)$, provided $(2n\alpha^2 + 1) \neq 0$	Yadav, Prasad and Pathak (2025)
2.	$\alpha = 1, \beta = 0$	nearly Ricci recurrent Sasakian manifold	$B(X) = -\frac{2n}{2n+1}A(X)$, provided $(2n + 1) \neq 0$	Yadav, Prasad and Pathak (2025)
3.	$\alpha = 0, \beta = f$	nearly Ricci recurrent normal almost cosymplectic $f-$structure or $f-$Kenmotsu structure	$B(X) = -\frac{2n[X(f^2+\xi f)-(f^2+\xi f)A(X)]}{1-2n(f^2+\xi f)} + \frac{2(2n-1)(f\phi^2 X)f}{1-2n(f^2+\xi f)}$, $1-2n(f^2+\xi f \neq 0$	Yadav, Prasad and Pathak (2025)
4.	$\alpha = 0, \beta =$ constant	nearly Ricci recurrent β-Kenmotsu manifold	$B(X) = \frac{2n\beta^2}{1-2n\beta^2}A(X)$, provided $(1 - 2n\beta^2) \neq 0$	Yadav, Prasad and Pathak (2025)
5.	$\alpha = 0, \beta = 1$	nearly Ricci recurrent Kenmotsu manifold	$B(X) = \frac{2n}{2n-1}A(X)$, provided $(2n - 1) \neq 0$	Yadav, Prasad and Pathak (2025)
6.	$\alpha \neq 0, \beta \neq 0$ and $B = 0$	Ricci recurrent trans-Sasakian manifold	$A(X) = \frac{2nX\left(\alpha^2-\beta^2-\xi\beta\right)}{\alpha^2-\beta^2-\xi\beta} - \frac{2(2n-1)\left(\alpha\phi X+\beta\phi^2 X\right)\beta-2\left(\alpha\phi^2 X-\beta\phi X\right)\alpha}{\alpha^2-\beta^2-\xi\beta}$, $\alpha^2 - \beta^2 - \xi\beta \neq 0$	Yadav, Prasad and Pathak (2025)

7.	$\alpha \neq 0, \beta = 0$ and $B = 0$	Ricci recurrent α-Sasakian manifold	$A(X) = \frac{2nX\alpha^2 - 2(\alpha\phi^2 X)\alpha}{\alpha^2}, \alpha^2 \neq 0$	Yadav, Prasad and Pathak (2025)
8.	$\alpha = 1, \beta = 0$ and $B = 0$	Ricci recurrent Sasakian manifold	$A(X) = -2\phi^2 X$	Yadav, Prasad and Pathak (2025)
9.	$\alpha = 0, \beta = f$ and $B = 0$	Ricci recurrent normal almost cosymplectic $f-$structure or $f-$Kenmotsu structure	$A(X) = \frac{2nX(f^2+\xi f) - 2(2n-1)(f\phi^2 X)f}{f^2+\xi f}$, provided $(f^2 + \xi f) \neq 0$	Yadav, Prasad and Pathak (2025)
10.	$\alpha = 0, \beta =$ constant, $A \neq 0$ and $B = 0$	Ricci recurrent β-Kenmotsu manifold	$A(X) = 0$, i.e. Ricci symmetric	Yadav, Prasad and Pathak (2025)

We know that the locally a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or α-Sasakian manifold or β-Kenmotsu manifold Marrero (1962). Hence, in view of S.No.1 and 4, we have the following Theorem:

Theorem 3.2 *Let (M^{2n+1}, g) be a nearly Ricci recurrent trans-Sasakian manifold of dimension ≥ 5 . Then*

(i) either M^{2n+1} is Ricci recurrent if $B = 0$,

(ii) or $B + 2n\alpha^2(A + B) = 0$,

(iii) or $B - 2n\beta^2(A + B) = 0$, where α and β are non-zero constant.

Moreover, putting ξ for Z in (3.2) and using (2.2), we get

$$[A(X) + B(X)]S(Y, \xi) + B(X)\eta(Y) = D_X S(Y, \xi) - S(D_X Y, \xi) - S(Y, D_X \xi). \quad (3.5)$$

Using (2.5), (2.13) in (3.5), we have

$$[2n(\alpha^2-\beta^2)\{A(X)+B(X)\}+B(X)]\eta(Y)=4n[\alpha d\alpha(X)-\beta d(\beta(X))]\eta(Y)$$
$$-\beta S(X,Y)-2n\alpha(\alpha^2-\beta^2)g(\phi X,Y)+2n\beta(\alpha^2-\beta^2)g(X,Y)+\alpha S(Y,\phi X). \quad (3.6)$$

Putting ξ for Y in (3.6) and using (2.1), (2.3) and (2.13), we get

$$2n(\alpha^2-\beta^2)\{A(X)+B(X)\}+B(X)=4n[\alpha d\alpha(X)-\beta d(\beta(X))]. \quad (3.7)$$

After use of (3.7) in (3.6), we obtain

$$\alpha S(Y,\phi X)-\beta S(X,Y)=2n\alpha(\alpha^2-\beta^2)g(\phi X,Y)-2n\beta(\alpha^2-\beta^2)g(X,Y). \quad (3.8)$$

Mutually interchange X and Y in (3.8), we get

$$\alpha S(X,\phi Y)-\beta S(Y,X)=2n\alpha(\alpha^2-\beta^2)g(\phi Y,X)-2n\beta(\alpha^2-\beta^2)g(Y,X). \quad (3.9)$$

Adding (3.8) and (3.9) and then using (2.3) in the resulting equation, we get

$$\alpha[S(X,\phi Y)+S(Y,\phi X)]-2\beta S(X,Y)=-4n\beta(\alpha^2-\beta^2)g(Y,X). \quad (3.10)$$

Replacing Y by ϕY in (2.16) and using (2.1) and (2.13), we get

$$S(X,\phi Y)+S(\phi X,Y)=0. \quad (3.11)$$

From (3.10) and (3.11), we have

$$\beta[S(X,Y)-2n(\alpha^2-\beta^2)g(Y,X)]=0.$$

This gives

$$S(X,Y)=2n(\alpha^2-\beta^2)g(Y,X)=0, \ \textit{provided}\ \beta\neq 0. \quad (3.12)$$

Hence, we have the following theorem:

Theorem 3.3 *An M^{2n+1}-dimensional nearly Ricci recurrent trans-Sasakian manifold with condition $\phi(grad\alpha)=(2n-1)grad\beta$ is an Einstein manifold, provided $\beta\neq 0$.*

4. An nearly Ricci-recurrent trans-Sasakian manifolds with cyclic Ricci tensor

A Riemannian manifold is said to be cyclic Ricci tensor if Prasad and Yadav (2021)

$$(D_X S)(Y,Z) + (D_Y S)(Z,X) + (D_Z S)(X,Y) = 0. \tag{4.1}$$

From (3.1) and (4.1), we get

$$\begin{aligned}&[A(X)+B(X)]S(Y,Z) + [A(Y)+B(Y)]S(Z,X) + [A(Z)+B(Z)]S(X,Y)\\ &+ B(X)g(Y,Z) + B(Y)g(Z,X) + B(Z)g(X,Y) = 0.\end{aligned} \tag{4.2}$$

Putting ξ for Z in (4.2) and using (2.10) and (3.4), we get

$$\begin{aligned}
&\frac{A(\xi)+2n\,\xi(C)}{K}S(X,Y) =\\
&2n\Bigg[C\bigg(\frac{A(Y)+2nY(C)-2(2n-1)(\alpha\phi Y+\beta\phi^2 Y)\beta-2(\alpha\phi^2 Y-\beta\phi Y)\alpha}{K}\bigg)\\
&-\xi(C)\Bigg]g(X,Y)+(2n-1)\times\\
&\Bigg[\bigg(\frac{A(X)+2nX(C)-2(2n-1)(\alpha\phi X+\beta\phi^2 X)\beta-2(\alpha\phi^2 X-\beta\phi X)\alpha}{K}\bigg)Y\beta\\
&+\bigg(\frac{A(Y)+2nY(C)-2(2n-1)(\alpha\phi Y+\beta\phi^2 Y)\beta-2(\alpha\phi^2 Y-\beta\phi Y)\alpha}{K}\bigg)X\beta\Bigg]\\
&-(2n-1)(\xi\beta)\times\\
&\Bigg[\bigg(\frac{A(X)+2nX(C)-2(2n-1)(\alpha\phi X+\beta\phi^2 X)\beta-2(\alpha\phi^2 X-\beta\phi X)\alpha}{K}\bigg)\eta(Y)\\
&+\bigg(\frac{A(Y)+2nY(C)-2(2n-1)(\alpha\phi Y+\beta\phi^2 Y)\beta-2(\alpha\phi^2 Y-\beta\phi Y)\alpha}{K}\bigg)\eta(X)\Bigg]\\
&+\bigg(\frac{A(X)+2nX(C)-2(2n-1)(\alpha\phi X+\beta\phi^2 X)\beta-2(\alpha\phi^2 X-\beta\phi X)\alpha}{K}\bigg)(\phi Y)\alpha\\
&+\bigg(\frac{A(Y)+2nY(C)-2(2n-1)(\alpha\phi Y+\beta\phi^2 Y)\beta-2(\alpha\phi^2 Y-\beta\phi Y)\alpha}{K}\bigg)(\phi X)\alpha\\
&-2n\Big[\eta(X)Y(C)+\eta(Y)X(C)\Big]\\
&+2(2n-1)\Big[\eta(X)(\alpha\phi Y+\beta\phi^2 Y)\beta+\eta(Y)(\alpha\phi X+\beta\phi^2 X)\beta\Big]\\
&+\Big[\eta(X)(\alpha\phi^2 Y-\beta\phi Y)\alpha+\eta(Y)(\alpha\phi^2 X+\beta\phi X)\alpha\Big],
\end{aligned}\tag{4.3}$$

where $C = \alpha^2 - \beta^2 - \xi\beta, \qquad K = 1 + 2nC$. Hence, we have the following theorem:

Theorem 4.1 *In an M^{2n+1}-dimensional nearly Ricci recurrent trans-Sasakian manifold with cyclic Ricci tensor, the Ricci tensor is given by (4.3).*

The particular case of Theorem 4.1 can be restated as follows:

S.No.	Case	Manifold	Nature of $\mathscr{R}\mathscr{R}$	Ref.
1.	$\alpha \neq 0, \beta = 0$	nearly Ricci recurrent α-Sasakian manifold	$A(\xi)S(X,Y) = 2n\alpha^2 A(\xi)g(X,Y)$	Yadav, Prasad and Pathak (2025)
2.	$\alpha = 1, \beta = 0$	nearly Ricci recurrent Sasakian	$A(\xi)S(X,Y) = 2nA(\xi)g(X,Y)$	Yadav, Prasad and Pathak (2025)
3.	$\alpha = 0, \beta = 0, A \neq 0$ and $B = 0$	cosymplectic	$A(\xi)S(X,Y) = 0$	Yadav, Prasad and Pathak (2025)

Hence, in view of the above, we can state the following:

Theorem 4.2 *Let M^{2n+1} be a nearly Ricci recurrent trans-Sasakian manifold with cyclic Ricci tensor. If M^{2n+1} is one of Sasakian, $\alpha-$Sasakian with non-zero $A(\xi)$ everywhere, then M^{2n+1} is an Einstein manifold.*

5. Example

In 2025, Yadav, Prasad and Pathak taking the 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$ defined as

$$e_1 = e^{kz}\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \; e_2 = e^{kz}\frac{\partial}{\partial y}, \; e_3 = \frac{\partial}{\partial z}, \tag{5.1}$$

which is linearly independent at each point of M^3, where k is any real number.
Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{5.2}$$

Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for any $U \in \chi(M)$. Let ϕ be tensor field of type (1,1) defined by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0. \tag{5.3}$$

$$\eta(e_3) = 1, \quad \phi^2 U = -U + \eta(U)e_3 \quad g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W), \tag{5.4}$$

$\forall\ U, W \in \chi(M)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M^3.
Let D be the Levi-Civita connection with respect to the Riemannian metric g. Then, from equation (5.1), we have

$$[e_1, e_2] = kye^{kz}e_2 - e^{2kz}e_3, \quad [e_1, e_3] = -ke_1, \quad [e_2, e_3] = -ke_2. \tag{5.5}$$

In view of (5.2),(5.5) and Koszul's formula , we get

$$\begin{aligned}
&D_{e_1}e_3 = -ke_1 + \frac{e^{2kz}}{2}e_2, \quad D_{e_2}e_3 = -ke_2 - \frac{e^{2kz}}{2}e_1, \quad D_{e_3}e_3 = 0,\\
&D_{e_1}e_2 = -\frac{e^{2kz}}{2}e_3, \qquad D_{e_2}e_2 = ke_3 + kye^{kz}e_1, \qquad D_{e_3}e_2 = -\frac{e^{2kz}}{2}e_1,\\
&D_{e_1}e_1 = ke_3, \quad D_{e_2}e_1 = -kye^{kz}e_2 + \frac{e^{2kz}}{2}e_3, \quad D_{e_3}e_1 = \frac{e^{2kz}}{2}e_2.
\end{aligned} \tag{5.6}$$

From (5.6), it can be easily seen that $M^3(\phi,\ \xi,\ \eta,\ g)$ satisfies (2.5) for $X = e_1,\ e_2,\ \alpha = -\frac{e^{2kz}}{2},\ \ \beta = -k$ and $e_3 = \xi$. Hence, the manifold M^3 is trans-Sasakian manifold with $\alpha = -\frac{e^{2kz}}{2},\ \ \beta = -k$.
Use of (5.5) and (5.6), we can easily calculate the curvature tensor as follows:

$$\begin{aligned}
&R(e_1, e_2)e_1 = \left(k^2 + kye^{3kz} + 4e^{kz}\right)e_2, \quad R(e_1, e_2)e_2 = -\left(k^2 + e^{4kz}\right)e_1,\\
&R(e_1, e_2)e_3 = -\frac{3}{2}kye^{3kz}e_1 - \frac{ke^{2kz}}{2}e_3, R(e_2, e_3)e_1 = \left(kye^{3kz} - ky\frac{e^{4kz}}{2}\right)e_1 + \frac{ke^{2kz}}{2}e_3,\\
&R(e_2, e_3)e_2 = -\frac{ke^{3kz}}{2}e_3 + \left(k^2 - \frac{e^{4kz}}{4}\right)e_3, R(e_2, e_3)e_3 = \left(\frac{e^{4kz}}{4} - k^2\right)e_3,\\
&R(e_1, e_3)e_1 = -2kye^{3kz}e_2 + \left(k^2 + ke^{2kz} - \frac{e^{4kz}}{2}\right)e_3,\\
&R(e_1, e_3)e_2 = yke^{2kz}e_1, \quad R(e_1, e_3)e_3 = \left(-k^2 + \frac{e^{4kz}}{4}\right)e_1 + \frac{ke^{2kz}}{2}e_2.
\end{aligned} \tag{5.7}$$

In view of (5.7), we obtain

$$\begin{aligned}
&S(e_1,e_1) = -2k^2 + \frac{5}{4}e^{4kz}, \qquad S(e_1,e_2) = \frac{ke^{2kz}}{2},\\
&S(e_2,e_2) = -\left(k^2 - \frac{ke^{2kz}}{2} + kye^{3kz} + e^{4kz}\right),\\
&S(e_3,e_3) = -k^2 + \frac{ke^{2kz}}{2} + \frac{e^{4kz}}{4}.
\end{aligned} \tag{5.8}$$

Since $\{e_1, e_2, e_3\}$ forms a basis for trans-Sasakian manifold, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3, \quad Y = a_2e_1 + b_2e_2 + c_2e_3,$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2, 3$.
This implies that

$$S(X,Y) = l \qquad g(X,Y) = m. \tag{5.9}$$

In view of (5.9), we get

$$(D_{e_i}S)(X,Y) = D_{e_i}S(X,Y) - S(D_{e_i}X, Y) - S(X, D_{e_i}Y),$$

which gives

$$(D_{e_1}S)(X,Y) = p_1, \quad (D_{e_2}S)(X,Y) = p_2, \quad \text{and} \quad (D_{e_3}S)(X,Y) = p_3,$$

where

$$\begin{aligned}
l =&k^2(-2a_1b_1 + a_2b_2 - a_3b_3) + \frac{k}{2}(a_1b_2 - a_2b_1 + a_3b_3)e^{2kz} +\\
&kya_2b_2e^{3kz} + ka_2b_2e^{3kz} + \left(\frac{5}{4}a_1b_1 + a_2b_2 + \frac{a_3b_3}{4}\right)e^{4kz},
\end{aligned}$$

$$m = a_1a_2 + b_1b_2 + c_1c_2,$$

$$\begin{aligned}
p_1 =&[k^2y(a_1b_2 - a_2b_1) + ky(5a_1b_1 + 4a_2b_2 + a_3b_3) - 2a_3b_1k^2 + (-2a_1b_3 + a_3b_1)k^3] +\\
&\frac{1}{2}[-a_2b_3 - (a_3b_3 + a_2b_2 + a_3b_1)k + (a_3b_3 - 2a_3b_2 - a_1b_3)k^2]e^{2kz} + a_3b_3e^{3kz}\\
&+ \frac{1}{4}[(5a_1b_3 - a_3b_3) - (2a_3b_3 + 6a_3b_1 - a_3b_2)k + 12a_2b_2y^2k^2]e^{4kz} -\\
&\frac{1}{2}a_3b_2e^{5kz} - \frac{1}{8}[a_2b_3 + 4a_3b_2]e^{6kz},
\end{aligned}$$

$$\begin{aligned}p_2 =&[k^3y(-a_1b_2+2a_2b_3+a_3b_2)]+\frac{1}{2}[-a_1b_3k+(-2ya_1b_1+2ya_1b_2+2a_1b_3-a_3b_3+\\&4a_3b_1-a_3b_2-2a_3b_1-a_1a_3)k^2]e^{2kz}+\frac{1}{2}[a_2b_2-2ka_1b_2-2a_2b_2-2a_3b_2+\\&2a_1b_1-2a_2b_1-2a_2b_3]yk^2e^{3kz}+\frac{1}{4}[-4a_3b_2+(4a_2b_2-4ya_1b_2-a_1b_3-a_2b_3\\&+a_3b_2+a_1a_3-4a_2b_3)k-4a_2b_1k^2y^2]e^{4kz}-\frac{9ky}{4}a_2b_1e^{5kz}+\frac{1}{8}[-a_1b_3+5a_3b_1]e^{6kz},\end{aligned}$$

and

$$\begin{aligned}p_3 =&\frac{1}{2}[-3a_2b_1+a_1b_2+a_3b_3)]k^2e^{2kz}+3k^2a_2b_2ye^{3kz}+\frac{1}{4}[20a_1b_1+17a_2b_2+a_3b_3+\\&(-a_1b_1-a_1b_2-a_2b_1-a_2b_2)k]e^{4kz}+\frac{ky}{2}[a_2b_2+a_2b_1]e^{5kz}+\frac{1}{8}[14a_1b_2+9a_2b_1]e^{6kz}.\end{aligned}$$

Consequently, the manifold under consideration is neither Ricci symmetric nor Ricci recurrent. Let us now consider 1-forms A and B non-vanishing

$$\begin{aligned}A\left(e_1\right) &= \frac{6(l+m)-(p_1+q_1)}{k_2}, B\left(e_1\right) = \frac{6}{1+6k_1}\left[e_1k_1-k_1\left\{\frac{6(l+m)-(p_1+q_1)}{k_2}\right\}-\frac{q_1}{6}\right],\\A\left(e_2\right) &= \frac{6(l+m)-(p_2+q_2)}{k_2}, B\left(e_2\right) = \frac{6}{1+6k_1}\left[e_2k_1-k_1\left\{\frac{6(l+m)-(p_2+q_2)}{k_2}\right\}-\frac{q_2}{6}\right],\\A\left(e_3\right) &= \frac{6(l+m)-(p_3+q_3)}{k_2}, B\left(e_3\right) = \frac{6}{1+6k_1}\left[e_3k_1-k_1\left\{\frac{6(l+m)-(p_3+q_3)}{k_2}\right\}-\frac{q_3}{6}\right],\end{aligned}$$

at any point $x \in M^3$. From (3.1), we have

$$\left(D_{e_i}S\right)(X,Y) = [A(e_i)+B(e_i)]S(X,Y)+B(e_i)g(X,Y), \quad j=1,2,3, \tag{5.10}$$

where

$$\begin{aligned}&q_i = 10(\alpha\phi e_i+\beta^2\phi^2e_i)+2(\alpha\phi^2e_i-\beta\phi e_i)\alpha,\ i=1,2,3,\\&k_1 = (\alpha^2-\beta^2-e_3\beta), \quad k_2 = 6m(\alpha^2-\beta^2-e_3\beta)-l,\\&\alpha = -\frac{e^{2kz}}{2}, \quad \beta=-k^2, \quad \phi e_1=e_2, \quad \phi e_2=-e_1, \quad \phi e_3=0.\end{aligned}$$

It can be easily seen that the manifold with 1-forms satisfy the relation (5.10). Hence, the manifold under consideration is a nearly Ricci recurrent trans-Sasakian manifold (M^3,g), which is neither Ricci recurrent nor Ricci symmetric. Thus, we have the following theorem:

Theorem 5.1 *There exist a nearly Ricci recurrent trans-Sasakian manifold* (M^3,g), *which is neither Ricci recurrent nor Ricci symmetric.*

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