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Geometric properties of weakly G-projectively symmetric manifolds

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Abstract

The present paper is devoted to the study of weakly G-projectively symmetric manifolds, denoted by $(WGPS)_n,\ n > 2$. We first investigate several fundamental geometric properties of these manifolds and derive conditions involving the Ricci tensor and the associated 1-forms. We then examine the decomposability of $(WGPS)_n,\ n > 2$ manifolds and establish certain geometric characterizations. Finally, we demonstrate the existence of such manifolds by constructing a non-trivial example.

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1. Introduction

A non-flat Riemannian manifold $(M^n, g), (n > 2)$, is called weakly symmetric by Tamássy and Binh (1989) if the curvature tensor R of the type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\mathscr{D}_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &\quad + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V)\\ &\quad + E(V)R(Y,Z,U,X), \end{aligned} \tag{1.1}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $R(X,Y,Z,U) = g(\mathscr{R}(X,Y)Z,U)$, $\mathscr{R}$ is the curvature tensor of the type $(1,3)$ and A, B, C, D and E are 1-forms and non-zero simultaneously $\mathscr{D}$ is the operator of covariant differentiation with respect

to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$.
De and Bandyopadhyay (1999) established the existence of a $(WS)_n$ by an example proved that in a $(WS)_n$, the associated 1-forms $B = C$ and $D = E$.
Hence, (1.1) reduces to the following form

$$\begin{aligned}(\mathscr{D}_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + D(V)R(Y,Z,U,X). \end{aligned} \tag{1.2}$$

If in (1.2) the 1-form A is replaced by $2A$, B and D are replaced by A, then the manifold (M^n, g) reduces to a pseudo-symmetric manifold in the sense of Chaki (1987).
Again, if $A = B = D = 0$, the manifold defined by (1.2) reduces to a symmetric manifold in the sense of Cartan. Weakly symmetric and pseudo-symmetric manifolds have been studied by several authors.
Let ρ_1, ρ_2 and ρ_3 be the basic vectors corresponding to the 1-forms A, B and D, respectively, that is,

$$g(X,\rho_1) = A(X), \qquad g(X,\rho_2) = B(X) \quad \text{and} \quad g(X,\rho_3) = D(X). \tag{1.3}$$

Gray (1978) introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor, respectively, as follows:

$$(\mathscr{D}_X Ric)(Y,Z) + (\mathscr{D}_Y Ric)(Z,X) + (\mathscr{D}_Z Ric)(X,Y) = 0, \tag{1.4}$$

and

$$(\mathscr{D}_X Ric)(Y,Z) = (\mathscr{D}_Y Ric)(X,Z). \tag{1.5}$$

In 1970, Pokhariyal and Mishra introduced a new tensor field called the W_2-curvature tensor in a semi-Riemannian manifold and studied its properties. According to them, a W_2-curvature tensor in a semi-Riemannian manifold $(M^n, g), (n > 2)$, is defined by the following expression:

$$\begin{aligned} W_2(Y,Z,U,V) = {} & R(Y,Z,U,V) + \\ & \frac{1}{n-1}[Ric(Z,V)g(Y,U) - Ric(Y,V)g(Z,U)], \end{aligned} \tag{1.6}$$

where

$$W_2(Y,Z,U,V) = g(\mathscr{W}_2(Y,Z)U,V).$$

The geometrical and physical properties of the W_2-curvature tensor have been studied fairly widely by many authors on different structures.
Breaking of $W_2(Y,Z,U,V)$ into two parts, viz.,

$$M(Y,Z,U,V) = \frac{1}{2}\left[W_2(Y,Z,U,V) - W_2(Y,Z,V,U)\right], \tag{1.7}$$

and

$$N(Y,Z,U,V) = \frac{1}{2}\left[W_2(Y,Z,U,V) + W_2(Y,Z,V,U)\right],$$

which are, respectively, skew-symmetric and symmetric in U and V.
From (1.6) and (1.7), we have

$$\begin{aligned} M(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{2(n-1)}[&Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V) \\ &+ Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)]. \end{aligned} \tag{1.8}$$

Further, the curvature tensor (1.8) is a curvature-like tensor. Some geometrical properties of this curvature tensor were initiated by Ojha (1986) and called it the M-projective curvature tensor. Many researchers have studied M-projective curvature on various spaces.
In 1971, Pokhariyal and Mishra investigated the following curvature tensor of the type $(0,4)$ as

$$\begin{aligned} W^*(Y,Z,U,V) =& R(Y,Z,U,V) + \frac{1}{n-1}[Ric(Y,V)g(Z,U) \\ &- Ric(Z,V)g(Y,U)]. \end{aligned} \tag{1.9}$$

From (1.9), we notice that $W^*(Y,Z,U,V)$ is skew-symmetric in U and V and also satisfies the cyclic property.
Now, on breaking (1.9) into two parts as follows:

$$G(Y,Z,U,V) = \frac{1}{2}[W^*(Y,Z,U,V) - W^*(Z,Y,U,V)], \tag{1.10}$$

and

$$H(Y,Z,U,V) = \frac{1}{2}\left[W^*(Y,Z,U,V) + W^*(Z,Y,U,V)\right],$$

which are, respectively, skew-symmetric and symmetric in Y and Z.
From (1.9) and (1.10), we get

$$\begin{aligned} G(Y,Z,U,V) = R(Y,Z,U,V) + \frac{1}{2(n-1)}[&Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V) \\ &+ Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)], \end{aligned} \tag{1.11}$$

where

$$G(Y,Z,U,V) = g(\mathscr{G}(Y,Z)U,V).$$

It can be easily seen that $G(Y,Z,U,V)$ has all skew-symmetric and symmetric properties as well as the cyclic property, that is,

$$\begin{aligned} G(Y,Z,U,V) + G(Z,Y,U,V) &= 0, \\ G(Y,Z,U,V) + G(Y,Z,V,U) &= 0, \\ G(Y,Z,U,V) - G(U,V,Y,Z) &= 0, \end{aligned}$$

and

$$G(Y,Z,U,V) + G(Z,U,Y,V) + G(U,Y,Z,V) = 0. \tag{1.12}$$

Since the curvature defined in (1.11) is very close to the M-projective curvature tensor, authors call it the G-projective curvature tensor Kumar (2012), Srivastava (2023) others.

A non-flat semi-Riemannian manifold $(M^n, g), (n > 2)$, whose G-projective curvature satisfies the following condition:

$$\begin{aligned} (\mathscr{D}_X G)(Y,Z,U,V) = {} & A(X)G(Y,Z,U,V) + B(Y)G(X,Z,U,V) \\ & + B(Z)G(Y,X,U,V) + D(U)G(Y,Z,X,V) \\ & + D(V)G(Y,Z,U,X), \end{aligned} \tag{1.13}$$

is called a weakly G-projective symmetric manifold and is denoted by $(WGPS)_n$ and the 1-forms A, B and D are non-zero simultaneously.

We use the following lemma as follows:

Theorem 1.1 [Walker's Lemma (1950)]. *If* a_{ij}, b_i *are numbers satisfying* $a_{ij} = a_{ji}$ *and* $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for $i,j,k = 1,2,\ldots,n$, *then either all* a_{ij} *or all* b_i *are zero.*

In the present paper, we are interested in studying weakly G-projective symmetric manifolds. The paper is organized as follows: After the preliminaries, in Section 3, some properties of $(WGPS)_n, (n > 2)$, are studied. Section 4 deals with the study of Einstein $(WGPS)_n, (n > 2)$. In section 5, we investigate the characteristics of decomposable $(WGPS)_n, (n > 2)$. Finally, in section 6, we have shown that the existence of $(WGPS)_n, (n > 2)$, by a non-trivial example.

2. Preliminaries

Let Ric and r denote the Ricci tensor of the type $(0,2)$ and the scalar curvature, respectively. Let Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric, that is,

$$g(QX, Y) = Ric(X, Y). \tag{2.1}$$

In this section, some basic formulas are derived, which will be useful to the study of $(WGPS)_n, (n > 2)$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$. From (1.11), we have

$$\sum_{i=1}^{n} G(Y, Z, e_i, e_i) = 0 = \sum_{i=1}^{n} G(e_i, e_i, U, V),$$

and

$$\begin{aligned}\sum_{i=1}^{n} G(e_i, Z, U, e_i) &= \sum_{i=1}^{n} G(Z, e_i, e_i, U) \\ &= \frac{3n-4}{2(n-1)}\left[Ric(Z, U) + \frac{r}{(3n-4)} g(Z, U)\right],\end{aligned} \tag{2.2}$$

where $r = \sum_{i=1}^{n} Ric(e_i, e_i)$ is the scalar curvature.
Let

$$G_1(Z, U) = \frac{3n-4}{2(n-1)}\left[Ric(Z, U) + \frac{r}{(3n-4)} g(Z, U)\right]. \tag{2.3}$$

From (2.3), we get

$$G_1(e_i, e_i) = \dot{g} = 2r, \tag{2.4}$$

$$(\mathscr{D}_X \dot{g}) = 2\, dr(X), \tag{2.5}$$

and

$$G_1(Z, \rho) = \frac{3n-4}{2(n-1)} Ric(Z, \rho) + \frac{r}{2(n-1)} g(Z, \rho). \tag{2.6}$$

3. Curvature properties of $(WGPS)_n,\ (n > 2)$

Contracting (1.13) with respect to Y and V, we get

$$\begin{aligned}(\mathscr{D}_X G_1)(Z,U) &= A(X)G_1(Z,U) + G(X,Z,U,\rho_2) + B(Z)G_1(X,U)\\ &\quad + D(U)G_1(Z,X) + G(\rho_3,Z,U,X). \qquad (3.1)\end{aligned}$$

Again, contracting (3.1) with respect to Z and U, we get

$$(\mathscr{D}_X \dot{g}) = A(X)G_1(e_i,e_i) + 2G_1(X,\rho_2) + 2G_1(X,\rho_3). \qquad (3.2)$$

In view of (2.3), (2.4), (2.5) and (3.2), we get

$$\begin{aligned}dr(X) - A(X)r &= \frac{3n-4}{2(n-1)}\left[\left\{Ric(X,\rho_2) + \frac{r}{3n-4}B(X)\right\}\right.\\ &\quad \left. + \left\{Ric(X,\rho_3) + \frac{r}{3n-4}D(X)\right\}\right].\end{aligned}$$

The above equation can be put as

$$dr(X) - A(X)r = \frac{3n-4}{2(n-1)}\left[Ric(X,\rho) + \frac{r}{3n-4}g(X,\rho)\right], \qquad (3.3)$$

where $\rho = \rho_2 + \rho_3,\ \rho_2$ and ρ_3 are defined by (1.3).

Hence, from (3.3), we can state the following theorem:

Theorem 3.1. *Let (M^n, g) be a weakly G-projectively symmetric manifold, denoted by $(WGPS)_n, (n > 2)$, with non-zero scalar curvature r. A necessary and sufficient condition for $\lambda = -\frac{r}{3n-4}$ to be an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector field $\rho = \rho_2 + \rho_3$ is that the scalar curvature r satisfies $dr(X) = r\,A(X)$, for any vector field X, where A is the associated 1-form of the manifold.*

Now, writing (3.1) cyclically and adding them, we get

$$\begin{aligned}&(\mathscr{D}_X G_1)(Z,U) + (\mathscr{D}_Z G_1)(U,X) + (\mathscr{D}_U G_1)(X,Z)\\ &\quad = [A(X) + B(X) + D(X)]G_1(Z,U)\\ &\quad + [A(Z) + B(Z) + D(Z)]G_1(X,U)\\ &\quad + [A(U) + B(U) + D(U)]G_1(X,Z)\\ &\quad + [G(X,Z,U,\rho_2) + G(Z,U,X,\rho_2) + G(U,X,Z,\rho_2)]\\ &\quad + [G(\rho_3,Z,U,X) + G(\rho_3,U,X,Z) + G(\rho_3,X,Z,U)]. \qquad (3.4)\end{aligned}$$

Using (1.12) in (3.4), we get

$$\begin{aligned}&(\mathscr{D}_X G_1)(Z,U) + (\mathscr{D}_Z G_1)(U,X) + (\mathscr{D}_U G_1)(X,Z)\\ &\quad = E(X)G_1(Z,U) + E(Z)G_1(X,U) + E(U)G_1(X,Z), \qquad (3.5)\end{aligned}$$

where

$$E(X) = A(X) + B(X) + D(X).$$

If $(WGPS)_n, (n > 2)$, has G_1-cyclic parallel tensor, then we have from (1.4),

$$(\mathscr{D}_X G_1)(Z,U) + (\mathscr{D}_Z G_1)(U,X) + (\mathscr{D}_U G_1)(X,Z) = 0. \tag{3.6}$$

We also have from (2.3),

$$G_1(Z,U) = G_1(U,Z). \tag{3.7}$$

Using (3.6) in (3.5), we get

$$E(X)G_1(Z,U) + E(Z)G_1(X,U) + E(U)G_1(X,Z) = 0. \tag{3.8}$$

Then, by Walker's Lemma, we can see that either $E(X) = 0$ or $G_1(Z,U) = 0$ for all X. Thus, we have either

$$A(X) + B(X) + D(X) = 0, \tag{3.9}$$

or the manifold is an Einstein manifold.
Thus, we can state the following theorem:

Theorem 3.2. *If in a $(WGPS)_n, (n > 2)$, the tensor is cyclic parallel, then the manifold is either an Einstein manifold or the sum of the associated 1-forms is zero.*
Conversely, suppose that the manifold $(WGPS)_n, (n > 2)$, is not an Einstein manifold. Then $A(X) + B(X) + D(X) = 0$, for all X.
Hence, from (3.5), we get

$$(\mathscr{D}_X G_1)(Z,U) + (\mathscr{D}_Z G_1)(U,X) + (\mathscr{D}_U G_1)(X,Z) = 0, \tag{3.10}$$

which shows that the tensor G is cyclic parallel.
Thus, we can state the following theorem:
Theorem 3.3. *In a $(WGPS)_n, (n > 2)$, the tensor G_1 is cyclic parallel if and only if the sum of the associated 1-forms is zero, provided the manifold is not an Einstein manifold.*
Let us consider $(WGPS)_n, (n > 2)$, admitting the Codazzi type of Ricci tensor. Then from (3.1), we get

$$\begin{aligned} &A(X)G_1(Z,U) + G(X,Z,U,\rho_2) + B(Z)G_1(X,U) + D(U)G_1(Z,X) + G(\rho_3,Z,U,X) \\ &= A(Z)G_1(X,U) + G(Z,X,U,\rho_2) + B(X)G_1(Z,U) \\ &+ D(U)G_1(X,Z) + G_1(\rho_3,X,U,Z). \end{aligned} \tag{3.11}$$

Simplification of (3.11) gives

$$\begin{aligned} &A(X)G_1(Z,U) + 2G_1(X,Z,U,\rho_2) + B(Z)G_1(X,U) + G(\rho_3,Z,U,X) \\ &= A(Z)G_1(X,U) + B(X)G_1(Z,U) + G(\rho_3,X,U,Z). \end{aligned} \tag{3.12}$$

Contracting (3.12) with respect to Z and U, we get

$$2r[A(X) - B(X)] = G(X,\rho_1) - 3G_1(X,\rho_2) - G_1(X,\rho_3). \tag{3.13}$$

In view of (2.6) and (3.13), we get

$$\begin{aligned} 2r[A(X) - B(X)] = \frac{3n-4}{2(n-1)}\Big[&Ric(X,\rho_1 - 3\rho_2 - \rho_3) \\ &+ \frac{r}{3n-4}g(X,\rho_1 - 3\rho_2 - \rho_3)\Big]. \end{aligned} \tag{3.14}$$

In view of (3.14), we can state the following theorem:

Theorem 3.4. *Let* $(WGPS)_n, (n > 2)$, *with non-zero scalar curvature* r. *Then a necessary and sufficient condition for* $-\frac{r}{3n-4}$ *to be an eigenvalue of the Ricci tensor* Ric *corresponding to the eigenvector* $(\rho_1 - 3\rho_2 - \rho_3) = \rho$, *is that* $2r[A(X) - B(X)] = 0$, *where* A *and* B *are the associated* 1*-forms.*

4. Einstein $(WGPS)_n, (n > 2)$

This section deals with a $(WGPS)_n, (n > 2)$, which is an Einstein manifold. Then the Ricci tensor Ric satisfies

$$Ric(Y,Z) = \frac{r}{n}g(Y,Z), \tag{4.1}$$

From which it follows that

$$dr(X) = 0$$

and

$$(\mathscr{D}_X Ric)(Y,Z) = 0, \quad \text{for all } X. \tag{4.2}$$

Using (4.1) and (4.2) in (1.11), we get

$$(\mathscr{D}_X G)(Y,Z,U,V) = (\mathscr{D}_X R)(Y,Z,U,V). \tag{4.3}$$

Using (4.3) in (1.13), we get

$$\begin{aligned}(\mathscr{D}_X R)(Y,Z,U,V) = {} & A(X)G(Y,Z,U,V) + B(Y)G(X,Z,U,V)\\ & + B(Z)G(Y,X,U,V) + D(U)G(Y,Z,X,V)\\ & + D(V)G(Y,Z,U,X).\end{aligned} \tag{4.4}$$

From (1.11) and (4.1), we get

$$\begin{aligned}(\mathscr{D}_X R)(Y,Z,U,V) = {} & A(X)\Big[R(Y,Z,U,V) + \frac{r}{n(n-1)}\{g(Z,U)g(Y,V)\\ & - g(Y,U)g(Z,V)\}\Big] + B(Y)\Big[R(X,Z,U,V)\\ & + \frac{r}{n(n-1)}\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\}\Big]\\ & + B(Z)\Big[R(Y,X,U,V) + \frac{r}{n(n-1)}\{g(X,U)g(Y,V)\\ & - g(Y,U)g(X,V)\}\Big] + D(U)\Big[R(Y,Z,X,V)\\ & + \frac{r}{n(n-1)}\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}\Big]\\ & + D(V)\Big[R(Y,Z,U,X) + \frac{r}{n(n-1)}\{g(Z,U)g(Y,X)\\ & - g(Y,U)g(Z,X)\}\Big].\end{aligned} \tag{4.5}$$

Now, if the Einstein $(WGPS)_n, (n > 2)$, is a $(WS)_n$, then we get from (1.2) and (4.5) as follows:

$$\begin{aligned}& r[A(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} + B(Y)\{g(Z,U)g(X,V)\\ & - g(X,U)g(Z,V)\} + B(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\}\\ & + D(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}\\ & + D(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}] = 0.\end{aligned} \tag{4.6}$$

Contracting (4.6) over Y and V, we get

$$\begin{aligned}& r[(n-1)A(X)g(Z,U) + B(X)g(Z,U) + (n-2)B(Z)g(X,U)\\ & \qquad + D(X)g(Z,U) + (n-2)D(U)g(Z,X)] = 0.\end{aligned} \tag{4.7}$$

Again, contracting (4.7) over Z and U, we get

$$r[n(n-1)A(X) + 2(n-1)B(X) + 2(n-1)D(X)] = 0. \tag{4.8}$$

Similarly, contracting (4.7) over X and Z, we get

$$r[(n-1)A(U)+(n-1)B(U)+(n^2-2n+1)D(U)]=0. \tag{4.9}$$

Replacing U by X in (4.9), we get

$$r[(n-1)A(X)+(n-1)B(X)+(n^2-2n+1)D(X)]=0. \tag{4.10}$$

Again, contracting (4.7) over X and U, we get

$$r[(n-1)A(Z)+(n^2-2n+1)B(Z)+(n-1)D(Z)]=0. \tag{4.11}$$

Replacing Z by X in (4.11), we get

$$r[(n-1)A(X)+(n^2-2n+1)B(X)+(n-1)D(X)]=0. \tag{4.12}$$

Now adding (4.8), (4.10) and (4.12), we get

$$r(n-1)(n+2)[A(X)+B(X)+D(X)]=0. \tag{4.13}$$

Since $(n>2)$, then from (4.13), we get either $r=0$ or $A(X)+B(X)+D(X)=0$. Hence, we have the following theorem:

Theorem 4.1. *If an Einstein $(WGPS)_n, (n>2)$, is a $(WS)_n$, then either the scalar curvature of the manifold vanishes or the sum of the associated 1-forms is zero.*

Again, if in an Einstein $(WGPS)_n, (n>2), r=0$, then using (1.11), (1.13), (4.1) and (4.3), we get

$$\begin{aligned}(\mathscr{D}_X R(Y,Z,U,V) &= A(X)R(Y,Z,U,V)+B(Y)R(X,Z,U,V)\\ &+B(Z)R(Y,X,U,V)+D(U)R(Y,Z,X,V)\\ &+D(V)R(Y,Z,U,X).\end{aligned} \tag{4.14}$$

which is the defining condition of $(WS)_n$.

Hence, we can state the following theorem:

Theorem 4.2. *If in an Einstein $(WGPS)_n, (n>2)$, the scalar curvature vanishes, then it is a $(WS)_n$.*

5. Decomposable $(WGPS)_n, (n>2)$

A semi-Riemannian manifold (M^n, g) is said to be decomposable or a product manifold by Schouten (1954) if it can be expressed as $M_1^p \times M_2^{n-p}$, for $2 \le p \le (n-2)$, that is, in some coordinate neighborhood of the semi-Riemannian manifold (M^n, g) the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{5.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \ldots, x^p$ denoted by $\bar{x}$ $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \ldots, x^n$ denoted by x^*; $a, b, c, \ldots$ run from 1 to p and $\alpha, \beta, \gamma, \ldots$ run from $p+1$ to n. The two parts of (5.1) are the metrics of M_1^p $(p \geq 2)$ and M_2^{n-p} $(n-p \geq 2)$, which are called the decompositions of the decomposable manifold $M^n = M_1^p \times M_2^{n-p},\ 2 \leq p \leq (n-2)$.

Let (M^n, g) be a semi-Riemannian manifold such that M_1^p $(p \geq 2)$ and M_2^{n-p} $(n-p \geq 2)$. Here, throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by 'star' is assumed to be from M_2.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Then in a decomposable semi-Riemannian manifold M_1^p $(p \geq 2)$ and M_2^{n-p} $(n-p \geq 2)$, the following relations hold:

$$\begin{aligned}
R(X^*, \bar{Y}, \bar{Z}, \bar{U}) &= 0 = R(\bar{X}, Y^*, \bar{Z}, U^*) = R(\bar{X}, Y^*, Z^*, U^*),\\
(\mathscr{D}_{X^*}R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) &= 0 = (\mathscr{D}_{\bar{X}}R)(\bar{Y}, Z^*, \bar{U}, V^*) = (\mathscr{D}_{X^*}R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}),\\
R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) &= \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U});\\
R(X^*, Y^*, Z^*, U^*) &= R^*(X^*, Y^*, Z^*, U^*),\\
Ric(\bar{X}, \bar{Y}) &= \bar{Ric}(\bar{X}, \bar{Y});\ Ric(X^*, Y^*) = Ric^*(X^*, Y^*),\\
(\mathscr{D}_{\bar{X}}Ric)(\bar{Y}, \bar{Z}) &= (\bar{\mathscr{D}}_{\bar{X}}Ric)(\bar{Y}, \bar{Z});\\
(\mathscr{D}_{X^*}Ric)(Y^*, Z^*) &= (\mathscr{D}^*_{X^*}Ric)(Y^*, Z^*).
\end{aligned}$$

and $r = \bar{r} + r^*$, where $r, \bar{r}$ and r^* are scalar curvatures of M, M_1 and M_2, respectively.

Let us consider a semi-Riemannian manifold (M^n, g), which is a decomposable $(WGPS)_n$. Then $M^n = M_1^p \times M_2^{n-p},\ 2 \leq p \leq (n-2)$.

Now, from (1.11), we get

$$\begin{aligned}
G(Y^*, \bar{Z}, \bar{U}, \bar{V}) &= 0 = G(\bar{Y}, Z^*, U^*, V^*)\\
&= G(\bar{Y}, Z^*, \bar{U}, \bar{V}) = G(\bar{Y}, \bar{Z}, U^*, \bar{V}),
\end{aligned} \tag{5.2}$$

$$G(\bar{Y}, Z^*, U^*, \bar{V}) = \frac{1}{2(n-1)}\left[Ric(Z^*, U^*)g(\bar{Y}, \bar{V}) + Ric(\bar{Y}, \bar{V})g(Z^*, U^*)\right], \tag{5.3}$$

$$G(Y^*, \bar{Z}, \bar{U}, V^*) = \frac{1}{2(n-1)}\left[Ric(\bar{Z}, \bar{U})g(Y^*, V^*) + Ric(Y^*, V^*)g(\bar{Z}, \bar{U})\right], \tag{5.4}$$

$$G(Y^*,\bar{Z},U^*,\bar{V}) = -\frac{1}{2(n-1)}\left[Ric(Y^*,U^*)g(\bar{Z},\bar{V}) + Ric(\bar{Z},\bar{V})g(Y^*,U^*)\right], \quad (5.5)$$

$$G(\bar{Y},Z^*,\bar{U},V^*) = -\frac{1}{2(n-1)}\left[Ric(\bar{Y},\bar{U})g(Z^*,V^*) + Ric(Z^*,V^*)g(\bar{Y},\bar{U})\right], \quad (5.6)$$

and

$$(\mathscr{D}_{X^*}G)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0 = (\mathscr{D}_{\bar{X}}G)(Y^*,Z^*,U^*,V^*). \quad (5.7)$$

Again, from (1.13), we get

$$\begin{aligned}(\mathscr{D}_{\bar{X}}G)(\bar{Y},\bar{Z},\bar{U},\bar{V}) &= A(\bar{X})G(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(\bar{Y})G(\bar{X},\bar{Z},\bar{U},\bar{V})\\ &+ B(\bar{Z})G(\bar{Y},\bar{X},\bar{U},\bar{V}) + D(\bar{U})G(\bar{Y},\bar{Z},\bar{X},\bar{V})\\ &+ D(\bar{V})G(\bar{Y},\bar{Z},\bar{U},\bar{X}), \quad (5.8)\end{aligned}$$

$$A(X^*)G(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0, \quad (5.9)$$

$$B(Y^*)G(\bar{X},\bar{Z},\bar{U},\bar{V}) = 0, \quad (5.10)$$

$$B(Z^*)G(\bar{Y},\bar{X},\bar{U},\bar{V}) = 0, \quad (5.11)$$

$$D(U^*)G(\bar{Y},\bar{Z},\bar{X},\bar{V}) = 0, \quad (5.12)$$

$$D(V^*)G(\bar{Y},\bar{Z},\bar{U},\bar{X}) = 0, \quad (5.13)$$

$$B(Y^*)G(\bar{X},Z^*,U^*,\bar{V}) + B(Z^*)G(Y^*,\bar{X},U^*,\bar{V}) = 0, \quad (5.14)$$

$$B(\bar{Y})G(X^*,\bar{Z},\bar{U},V^*) + B(\bar{Z})G(\bar{Y},X^*,\bar{U},V^*) = 0, \quad (5.15)$$

$$\begin{aligned}(\mathscr{D}_{\bar{X}}G)(\bar{Y},Z^*,U^*,\bar{V}) =& A(\bar{X})G(\bar{Y},Z^*,U^*,\bar{V}) + B(\bar{Y})G(\bar{X},Z^*,U^*,\bar{V})\\ &+ D(\bar{V})G(\bar{Y},Z^*,U^*,\bar{X}), \quad (5.16)\end{aligned}$$

and

$$\begin{aligned}(\mathscr{D}_{X^*}G)(Y^*,\bar{Z},\bar{U},V^*) =& A(X^*)G(Y^*,\bar{Z},\bar{U},V^*) + B(Y^*)G(X^*,\bar{Z},\bar{U},V^*)\\ &+ D(V^*)G(Y^*,\bar{Z},\bar{U},X^*). \quad (5.17)\end{aligned}$$

Also, from (1.13), we obtain

$$\begin{aligned}(\mathscr{D}_{X^*}G)(Y^*,Z^*,U^*,V^*) =& A(X^*)G(Y^*,Z^*,U^*,V^*)+B(Y^*)G(X^*,Z^*,U^*,V^*)\\ &+B(Z^*)G(Y^*,X^*,U^*,V^*)+D(U^*)G(Y^*,Z^*,X^*,V^*)\\ &+D(V^*)G(Y^*,Z^*,U^*,X^*),\end{aligned} \tag{5.18}$$

$$A(\bar{X})G(Y^*,Z^*,U^*,V^*)=0, \tag{5.19}$$

$$B(\bar{Y})G(X^*,Z^*,U^*,V^*)=0, \tag{5.20}$$

$$B(\bar{Z})G(Y^*,X^*,U^*,V^*)=0, \tag{5.21}$$

$$D(\bar{U})G(Y^*,Z^*,X^*,V^*)=0, \tag{5.22}$$

and

$$D(\bar{V})G(Y^*,Z^*,U^*,X^*)=0. \tag{5.23}$$

Using the relations from (5.9) to (5.13), we conclude that either

I) $A=B=D=0$ on M_2, or

II) M_1 is G-projectively-flat.

First, we consider case (I). Then, from (5.17), it follows that

$$(\mathscr{D}_{X^*}G)(Y^*,\bar{Z},\bar{U},V^*)=0,$$

which implies by virtue of (5.4), that

$$(\mathscr{D}_{X^*}Ric)(Y^*,V^*)=0, \tag{5.24}$$

and hence, the decomposition M_2 is Ricci symmetric. Also, from (5.18), we have

$$(\mathscr{D}_{X^*}G)(Y^*,Z^*,U^*,V^*)=0,$$

and hence

$$\begin{aligned}&(\mathscr{D}_{X^*}R)(Y^*,Z^*,U^*,V^*)+\frac{1}{2(n-1)}\Big[(\mathscr{D}_{X^*}Ric)(Z^*,U^*)g(Y^*,V^*)\\ &-(\mathscr{D}_{X^*}Ric)(Y^*,U^*)g(Z^*,V^*)\\ &+(\mathscr{D}_{X^*}Ric)(Y^*,V^*)g(Z^*,U^*)-(\mathscr{D}_{X^*}Ric)(Z^*,V^*)g(Y^*,U^*)\Big]=0,\end{aligned}$$

which yields by virtue of (5.24), that

$$(\mathscr{D}_{X^*}R)(Y^*,Z^*,U^*,V^*)=0,$$

that is, the decomposition M_2 is locally symmetric.

Secondly, we assume that M_1 is G-projectively flat. Then we have

$$\begin{aligned}R(\bar{Y},\bar{Z},\bar{U},\bar{V}) = -\frac{1}{2(n-1)}[&Ric(\bar{Z},\bar{U})g(\bar{Y},\bar{V})-Ric(\bar{Y},\bar{U})g(\bar{Z},\bar{V})\\ &+Ric(\bar{Y},\bar{V})g(\bar{Z},\bar{U})-Ric(\bar{Z},\bar{V})g(\bar{Y},\bar{U})].\end{aligned} \tag{5.25}$$

Contracting (5.25) over $\bar{Y}$ and $\bar{V}$, we obtain

$$Ric(\bar{Z},\bar{U})=-\frac{\bar{r}}{(2n+p-4)}\,g(\bar{Z},\bar{U}). \tag{5.26}$$

In view of (5.26), (5.25) yields

$$R(\bar{Y},\bar{Z},\bar{U},\bar{V})=-\frac{r}{(2n+p-4)(n-1)}[g(\bar{Y},\bar{V})g(\bar{Z},\bar{U})-g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})].$$

that is, the decomposition M_1 is a manifold of constant curvature.
Thus, we can state the following theorem:

Theorem 5.1 *Let (M^n,g) be a semi-Riemannian manifold such that $M=M_1^p\times M_2^{n-p}, 2\leq p\leq n-2$. If M^n is a $(WGPS)_n$, then either (I) or (II) holds.*

- *(I) $A=B=D=0$ on M_2, (resp. M_1) and hence M_2, (resp. M_1) is Ricci symmetric as well as locally symmetric.*
- *(II) M_1 (resp. M_2) is G-projectively-flat and hence M_1, (resp. M_2) is a manifold of constant curvature.*

6. Example of a $(WGPS)_4$.

Let us consider a Lorentzian manifold (M^4,g) endowed with the Lorentzian metric g is

$$ds^2=g_{ij}dx^idx^j=(dx^1)^2+4(\sin x^1)^2(dx^2)^2+(\sin x^2)^2(dx^3)^2-(dx^4)^2, \tag{6.1}$$

where, $i,j=1,2,3,4$ and $\sin x^1, \sin x^2$ are non-zero.
Then, the only non-vanishing components of the Christoffel symbols are

$$\begin{aligned}&\Gamma^1_{22}=-4\sin x^1\cos x^1, \Gamma^2_{33}=-\frac{\sin x^2\cos x^2}{(2\sin x^1)^2},\\ &\Gamma^2_{12}=\cot x^1=\Gamma^2_{21}, \Gamma^3_{23}=\cot x^2=\Gamma^3_{32}.\end{aligned} \tag{6.2}$$

The curvature tensor of the type $(1,3)$ is defined by

$$\mathscr{R}^{\alpha}_{ijk} = -\frac{\partial}{\partial x^k}\Gamma^{\alpha}_{ij} + \frac{\partial}{\partial x^j}\Gamma^{\alpha}_{ik} + \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{ik} - \Gamma^{\alpha}_{\beta k}\Gamma^{\beta}_{ij}. \tag{6.3}$$

In view of (6.2) and (6.3), we get

$$\mathscr{R}^{1}_{332} = -\frac{1}{2}\cot x^1 \sin 2x^2. \tag{6.4}$$

The curvature tensor of the type $(0,4)$ is defined by

$$R_{hijk} = g_{h\alpha}\mathscr{R}^{\alpha}_{ijk}. \tag{6.5}$$

In view of (6.4) and (6.5), we get

$$R_{1332} = -\frac{1}{2}\cot x^1 \sin 2x^2. \tag{6.6}$$

The non-vanishing components of the Ricci tensor is

$$(Ric)_{12} = \cot x^1 \cot x^2, \tag{6.7}$$

We shall now show that this M^4 is a weakly G-projectively symmetric manifold, i.e., it satisfies the defining relation (1.11). Now, the only non-vanishing components for G and its covariant derivatives are given by

$$G_{1332} = -\frac{5}{12}\cot x^1 \sin 2x^2,$$

$$G_{1332,1} = \frac{5}{12}\sin 2x^2\left(\frac{1+\cos^2 x^1}{\sin^2 x^1}\right),$$

$$G_{1332,2} = \frac{5}{6}\cot x^1. \tag{6.8}$$

Let us choose the associated 1-form as follows:

$$A_i = \begin{cases} -\operatorname{cosec} x^1 \sec x^1, & \text{if } i = 1,\\ 0, & \text{otherwise,}\end{cases}$$

$$B_i = \begin{cases} -\cot x^1, & \text{if } i = 1,\\ 0, & \text{otherwise,}\end{cases}$$

$$D_i = \begin{cases} -\text{cosec } x^2 \sec x^2, & \text{if } i = 2, \\ \\ 0, & \text{otherwise,} \end{cases} \tag{6.9}$$

at any point $x \in (M^4, g)$. In (M^4, g), after using (6.8) and (6.9) in RHS of (1.13), we obtain

$$\begin{aligned} &A_1 G_{1332} + B_1 G_{1332} + B_3 G_{1132} + D_3 G_{1312} + D_2 G_{1331} \\ &= (-\text{cosec } x^1 \sec x^1 - \cot x^1) \left(-\frac{5}{12} \cot x^1 \sin 2x^2 \right) \\ &= \frac{5}{12} \sin 2x^2 \left(\frac{1 + \cos^2 x^1}{\sin^2 x^1} \right). \end{aligned} \tag{6.10}$$

In view of (6.8) and (6.10), we get

$$G_{1332,1} = A_1 G_{1332} + B_1 G_{1332} + B_3 G_{1132} + D_3 G_{1312} + D_2 G_{1331}. \tag{6.11}$$

Similarly, we can prove that

$$G_{1332,2} = A_2 G_{1332} + B_1 G_{2332} + B_3 G_{1232} + D_3 G_{1322} + D_2 G_{1332}. \tag{6.12}$$

Hence, it is verified that the equations (6.11) and (6.12) are true. Thus, the manifold under consideration is a weakly G-projectively symmetric manifold, that is $(WGPS)_4$.

Theorem 6.1. *Let (M^4, g) be a semi-Riemannian manifold endowed with the metric g given by*

$$ds^2 = g_{ij} dx^i dx^j = (dx^1)^2 + 4(\sin x^1)^2 (dx^2)^2 + (\sin x^2)^2 (dx^3)^2 - (dx^4)^2.$$

where, $i, j = 1, 2, 3, 4$ and $\sin x^1, \sin x^2$ are non-zero. Then (M^4, g) is a $(WGPS)_4$.

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