ISSN: 0976-4933 (Print)
Journal of Progressive Science
A Peer-reviewed Research Journal
Vol.17, No.01, pp 64-86 (2026)
https://doi.org/10.21590/jps.16.01.05

Geometrical properties of quasi-conharmonic curvature tensor on a semi-Riemannian manifold

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Abstract

This paper examines the quasi-conharmonic curvature tensor $\tilde{H}$ which generalizes the concept of a conharmonic curvature tensor. Initially, we acquire some geometrical features. Subsequently, we examine pseudo quasi-conharmonic symmetric manifolds. The Divergence-free quasi-conharmonic curvature tensor is derived from the Gray's decomposition. Additionally, we also examine the nature of Einstein $(PQ\tilde{H}S)_n, (n > 2)$,manifolds. Finally, we construct a non-trivial Lorentzian metric of $(PQ\tilde{H}S)_4$.

2010 Mathematics Subject Classification: 53C25, 53D15.

Keywords and Phrases: Quasi-conformal curvature tensor $\tilde{C}$, quasi-conharmonic curvature tensor $\tilde{H}$, pseudo-projective curvature tensor $\tilde{P}$, quasi-concircular curvature tensor, $\tilde{L}$, pseudo-symmetric manifold.

1. Introduction

The study of semi-Riemannian manifolds relies heavily on identifying tensors that remain invariant under specific geometric mappings. The conformal curvature tensor $\bar{C}$ is perhaps the most prominent of these, characterized by its total invariance under any conformal transformation. When such a transformation specifically preserves the integrity of harmonic functions, it is classified as a conharmonic curvature tensor $\bar{H}$ which remains the fundamental invariant. Furthermore, the relationship

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between a manifold's geodesic structure and its flatness is governed by the projective curvature tensor $\bar{P}$ it is a well-established principle that a semi-Riemannian manifold is locally projectively flat if and only if this tensor vanishes. Beyond these conformal and projective frameworks, the concircular curvature tensor $\bar{L}$ serves as an additional, essential tool for characterizing the underlying geometry and symmetry of the semi-Riemannian setting.

These fundamental curvature tensors of type $(0,4)$ on a semi-Riemannian manifold, namely the conformal curvature tensor $\bar{C}$, the conharmonic curvature tensor $\bar{H}$, the projective curvature tensor $\bar{P}$ and the concircular curvature tensor $\bar{L}$, are given by Yano and Kon (1984) of the type $(0,4)$ as follows:

$$\begin{aligned}\bar{C}(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{1}{(n-2)}[Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V)\\ &+ Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)]\\ &+ \frac{r}{(n-1)(n-2)}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.1}$$

$$\begin{aligned}\bar{H}(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{1}{(n-2)}[Ric(Z,U)g(Y,V)-\\ & Ric(Y,U)g(Z,V) + Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)],\end{aligned} \tag{1.2}$$

$$\begin{aligned}\bar{P}(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{1}{(n-1)}[Ric(Z,U)g(Y,V)\\ &- Ric(Y,U)g(Z,V)],\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\bar{L}(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{r}{n(n-1)}[g(Z,U)g(Y,V)\\ &- g(Y,U)g(Z,V)],\end{aligned} \tag{1.4}$$

where Ric is the Ricci tensor of type $(0,2)$ defined by $Ric(X,Y) = g(\bar{Q}X,Y), \bar{Q}$ is the Ricci operator of type $(1,1)$ and r is the scalar curvature. The conformal curvature tensor $\bar{C}$, the conharmonic curvature tensor $\bar{H}$, the concircular curvature tensor $\bar{L}$ are curvature-like tensor, whereas the projective curvature tensor $\bar{P}$ is not a curvature-like tensor.

The quasi-conformal curvature tensor $\tilde{C}$, quasi-conharmonic curvature tensor $\tilde{H}$ and pseudo-projective curvature tensor $\tilde{P}$ and quasi-concircular curvature tensor $\tilde{L}$ were introduced, respectively, by Yano and Sawaki (1968) Lal, Srivastava and Prasad

(2013) Prasad (2002) and Prasad and Maurya (2007). The corresponding curvature tensors of the type $(0,4)$ are given by the following equations:

$$\begin{aligned}\tilde{C}(Y,Z,U,V) = aR(Y,Z,U,V) &+ b[Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V) \\ &+ Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)] - \\ &\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.5}$$

$$\begin{aligned}\tilde{H}(Y,Z,U,V) = aR(Y,Z,U,V) &+ b[Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V)] \\ &+ c[Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)] \\ &- \frac{r}{n}\left[\frac{2a}{n-2} + b + c\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.6}$$

$$\begin{aligned}\tilde{P}(Y,Z,U,V) = aR(Y,Z,U,V) &+ b[Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V)] \\ &- \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.7}$$

and

$$\tilde{L}(Y,Z,U,V) = \ aR(Y,Z,U,V) - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],$$

where a, b and c are three arbitrary constants, not simultaneously zero. Numerous researchers have explored generalizations of classical curvature invariants, specifically focusing on quasi-conformal curvature tensor, quasi-conharmonic curvature tensor, pseudo-projective curvature tensor and quasi-concircular curvature tensor. These studies investigate various structural aspects of manifolds, examining how these tensors characterize symmetry conditions and geometric properties across different manifold settings. Through these investigations, the impact of such tensors on the classification of semi-Riemannian spaces has been extensively refined.
In particular, setting $a = 1$ and $b = c = \frac{-1}{n-2}$, the quasi-conformal curvature tensor $\tilde{C}$ and the quasi-conharmonic curvature tensor $\tilde{H}$ reduce to the standard conformal curvature tensor $\bar{C}$ and the conharmonic curvature tensor $\bar{H}$,respectively. Similarly, by taking $a = 1$ and $b = \frac{-1}{n-1}$, the pseudo-projective curvature tensor $\tilde{P}$ and the quasi-concircular curvature tensor $\tilde{L}$ reduce to the classical projective curvature tensor $\bar{P}$ and the concircular curvature tensor $\bar{L}$,respectively.

A semi-Riemannian manifold (M^n, g) is said to be quasi-conharmonically symmetric in the sense of Cartan if the quasi-conharmonic curvature tensor $\tilde{H}$ is parallel,i.e.,

$$(D_X\tilde{H})(Y, Z, U, V) = 0. \tag{1.8}$$

According to Chaki (1987), a non-flat semi-Riemannian manifold $(M^n, g)(n > 2)$ is said to be a pseudo-symmetric if its curvature tensor R of the type $(0, 4)$ satisfies the condition:

$$\begin{aligned}(D_X R)(Y, Z, U, V) =& 2A(X)R(Y, Z, U, V)+\\ & A(Y)R(X, Z, U, V) + A(Z)R(Y, X, U, V)\\ & + A(U)R(Y, Z, X, V) + A(V)R(Y, Z, U, X),\end{aligned} \tag{1.9}$$

where A is a non-zero 1-form and ρ is a vector field defined by

$$g(X, \rho) = A(X), \tag{1.10}$$

for all $X \in T\tilde{M}$ and $R(Y, Z, U, V) = g(\mathscr{R}(Y, Z)U, V)$, where R and $\mathscr{R}$ denote the curvature tensor of type $(0, 4)$ and $(1, 3)$, respectively. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a locally symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

Numerous investigators have explored the existence and properties of pseudo-symmetric manifolds across a diverse range of geometric environments, including Sasakian, Kählerian, and Kenmotsu spaces,and how they interact with other curvature properties such as Ricci solitons, eta-Ricci solitons, almost eta-Ricci solitons etc.

In the present paper, we study a semi-Riemannian manifold $(M^n, g)(n > 2)$,whose quasi-conharmonic curvature tensor $\tilde{H}$ satisfies

$$\begin{aligned}(D_X\tilde{H})(Y, Z, U, V) =& 2A(X)\tilde{H}(Y, Z, U, V)\\ & + A(Y)\tilde{H}(X, Z, U, V) + A(Z)\tilde{H}(Y, X, U, V)\\ & + A(U)\tilde{H}(Y, Z, X, V) + A(V)\tilde{H}(Y, Z, U, X).\end{aligned} \tag{1.11}$$

The manifold stated above is referred to as a pseudo-quasi-conharmonically symmetric manifold and is designated by the symbol $(PQ\tilde{H}S)_n, (n > 2)$, where P stands for pseudo, Q stands for quasi, $\tilde{H}$ stands for conharmonic and S stands for symmetric.

This paper is organized as follows:

After the introduction, section 2 discusses the basic geometric properties of quasi-conharmonic curvature tensor. In section 3, a classification theorem is presented for semi-Riemannian or Riemannian manifolds admitting divergence-free quasi-conharmonic curvature tensor using Gray's decomposition of the covariant derivative of the Ricci tensor . Section 4 studies the $(PQ\tilde{H}S)_4, (n > 2)$, manifold satisfying Bianchi's second identity. Section 5 to 8, examine special types of Ricci tensors such as Ricci symmetric, Codazzi type, Pseudo cyclic and Ricci recurrent. Section 9 focuses Einstein $(PQ\tilde{H}S)_n$, manifold for $n > 2$. Finally, section 10 construct a non-trivial Lorentzian metric for $(PQ\tilde{H}S)_4$.

2. Elementary properties of quasi-conharmonic curvature tensor $(\tilde{H})$

In this section, some basic formulas are derived which will be useful for the study of $(PQ\tilde{H}S)_n, (n > 2)$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$. In a semi-Riemannian manifold, the Ricci tensor Ric is defined by

$$Ric(X, Y) = \sum_{i=1}^{n} R(X, e_i, e_i, Y).$$

From (1.6), we have,

$$\tilde{H}(Y, Z, U, V) + \tilde{H}(Z, Y, U, V) = 0, \tag{2.1}$$

$$\begin{aligned}\tilde{H}(Y, Z, U, V) + \tilde{H}(Y, Z, V, U) =&(b - c)[Ric(Z, U)g(Y, V) - Ric(Y, U)g(Z, V)\\ &+ Ric(Z, V)g(Y, U) - Ric(Y, V)g(Z, U)],\end{aligned} \tag{2.2}$$

$$\begin{aligned}\tilde{H}(Y, Z, U, V) - \tilde{H}(U, V, Y, Z) =&(b - c)[Ric(Z, U)g(Y, V)\\ &- Ric(V, Y)g(U, Z)],\end{aligned} \tag{2.3}$$

$$\tilde{H}(Y, Z, U, V) + \tilde{H}(Z, U, Y, V) + \tilde{H}(U, Y, Z, V) = 0 \tag{2.4}$$

and

$$\begin{aligned}
&(D_X\tilde{H})(Y,Z,U,V)+(D_Y\tilde{H})(Z,X,U,V)+(D_Z\tilde{H})(X,Y,U,V)\\
&=b[\{(D_XRic)(Z,U)-(D_ZRic)(X,U)\}g(Y,V)\\
&+\{(D_YRic)(X,U)-(D_XRic)(Y,U)\}g(Z,V)\\
&+\{(D_ZRic)(Y,U)-(D_YRic)(Z,U)\}g(X,V)]\\
&+c[\{(D_XRic)(Y,V)-(D_YRic)(X,V)\}g(Z,U)\\
&+\{(D_ZRic)(X,V)-(D_XRic)(Z,V)\}g(Y,U)\\
&+\{(D_YRic)(Z,V)-(D_ZRic)(Y,V)\}g(X,U)]\\
&-\frac{1}{n}\left(\frac{2a}{n-2}b+c\right)[\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}dr(X)\\
&+\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\}dr(Y)\\
&+\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}dr(Z)]. \qquad (2.5)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{i=1}^{n}\tilde{H}(Y,e_i,e_i,V)=&Ric(Y,V)[a-b+(n-1)c]-\\
&\frac{r}{n(n-2)}[2(n-1)a-(n-2)b+(n-1)(n-2)c]g(Y,V)\\
&=\tilde{H}_1(Y,V), \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n}\tilde{H}(e_i,Z,U,e_i)=&Ric(Z,U)[a+(n-1)b-c]\\
&-\frac{r}{n(n-2)}[2(n-1)a-(n-1)(n-2)b+(n-2)c]g(Z,U)\\
&=\tilde{H}_2(Z,U), \qquad (2.7)
\end{aligned}$$

$$\sum_{i=1}^{n}\tilde{H}(Y,Z,e_i,e_i)=0=\sum_{i=1}^{n}\tilde{H}(e_i,e_i,Y,Z). \qquad (2.8)$$

Hence, in view of the above explanation, we can state the following theorems:

Theorem 2.1 *The quasi-conharmonic curvature tensor $\tilde{H}$ on a semi-Riemannian manifold (M^n,g) possesses the following algebraic properties:*

(i) skew-symmetric in the first pair of slots,

(ii) skew-symmetric in the last pair of slots if and only if it is an Einstein manifold,

(iii) symmetric in pair of slots if and only if it is an Einstein manifold,

(iv) Bianchi's first identity exists,

(v) Bianchi's second identity exists if the tensor is of Codazzi type,

(vi) Ricci tensor is symmetric.

Covariant differentiation of (1.6) gives,

$$\begin{aligned}(D_X\tilde{H})(Y,Z,U,V) = {} & a(D_XR)(Y,Z,U,V) + b[(D_XRic)(Z,U)g(Y,V) \\ & - (D_XRic)(Y,U)g(Z,V)] + c[(D_XRic)(Y,V)g(Z,U) \\ & - (D_XRic)(Z,V)g(Y,U)] - \frac{dr(X)}{n}\left[\frac{2a}{n-2} + b + c\right] \times \\ & [g(Z,U)g(Y,V) - g(Y,U)g(Z,V)]. \end{aligned} \tag{2.9}$$

Here, we assume that the quasi-conharmonic curvature tensor $\tilde{H}$ is parallel. Hence,from (1.8) and (2.9), we get

$$\begin{aligned}& (D_XR)(Y,Z,U,V) + b[(D_XRic)(Z,U)g(Y,V) \\ & - (D_XRic)(Y,U)g(Z,V)] + c[(D_XRic)(Y,V)g(Z,U) \\ & - (D_XRic)(Z,V)g(Y,U)] - \frac{dr(X)}{n}\left[\frac{2a}{n-2} + b + c\right] \times \\ & [g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] = 0.\end{aligned}$$

Putting e_i for Y and V in the above equation, we get

$$\begin{aligned}& (D_XRic)(Z,U) + b(n-1)(D_XRic)(Z,U) + c[(dr(X)g(Z,U) \\ & - (D_XRic)(Z,U)] - \frac{dr(X)}{n}\left[\frac{2a}{n-2} + b + c\right](n-1)g(Z,U) = 0.\end{aligned}$$

Again, putting e_i for Z and U in the above equation, we get

$$dr(X)[a(n-2) + (n-1)(n-3)(b+c)] = 0. \tag{2.10}$$

From (2.10), we can state the following theorem:

Theorem 2.2 *If the quasi-conharmonic curvature tensor $\tilde{H}$ on a semi-Riemannian manifold (M^n, g), is parallel, then the scalar curvature is constant, provided $[a(n-2)+(n-1)(n-3)(b+c)] \neq 0$.*

Further, we assume that the quasi-conharmonic curvature tensor is flat.That is,

$$\tilde{H}(Y,Z,U,V) = 0. \tag{2.11}$$

Here, in view of (1.6) and (2.11), we get

$$\begin{aligned} &aR(Y,Z,U,V) + b[Ric(Z,U)g(Y,V) - Ric(Y,U)g(Z,V)] \\ &+ c[Ric(Y,V)g(Z,U) - Ric(Z,V)g(Y,U)] \\ &- \frac{r}{n}\left[\frac{2a}{n-2} + b + c\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] = 0. \end{aligned} \tag{2.12}$$

Now, putting e_i for Z and U in (2.12), we get

$$Ric(Y,V) = \frac{[2(n-1)a - (n-2)b + (n-1)(n-2)c]r}{n(n-2)[a-b+(n-1)c]} g(Y,V). \tag{2.13}$$

Further, contraction of (2.13) yields,

$$r = 0, \quad \text{provided } a \neq 0.$$

Hence, we can state the following theorem:

Theorem 2.3 *If a quasi-conharmonic curvature tensor $\tilde{H}$ is flat on a semi-Riemannian manifold, then its scalar curvature vanishes, provided $a \neq 0$.*

Contracting equation (2.9) with respect to X and V, we get

$$\begin{aligned} (div\tilde{H})(Y,Z)U =& (a+b)(divR)(Y,Z)U + \left[\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times \\ & [g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \end{aligned} \tag{2.14}$$

We also know that

$$\begin{aligned} (div\bar{H})(Y,Z)U = (a+b)\left(\frac{n-3}{n-2}\right)(divR)(Y,Z)U \\ - \frac{1}{2(n-2)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we get

$$(div\tilde{H})(Y,Z)U = \left[(a+b)\left(\frac{n-3}{n-2}\right)\right](div\bar{H})(Y,Z)U \\ + \left[\frac{a+b}{2(n-3)} + \frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times \\ [g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \tag{2.16}$$

Thus, we have the following theorem:

Theorem 2.4 *For quasi-conharmonic curvature tensor, $(div\tilde{H})(Y,Z)U$ is equivalent to $(div\bar{H})(Y,Z)U$, if and only if $dr(X) = 0$.*

3. Divergence of quasi-conharmonic curvature tensor $\tilde{H}$ under Gray's decomposition

Considering the action of the orthogonal group on the space of tensors with the symmetries of the covariant derivative of the Ricci curvature, Gray decomposed this space into irreducible components Gray (1978). Gray proposed that the curvature derivative of the Ricci tensor, that is, $D.Ric$ can be decomposed into $O(X)$-invariant terms. According to him, the covariant derivative of the Ricci tensor Ric can be converted into $O(X)$-invariant terms as follows Mantica, Molinari, Suh and Shenawy (2019):

$$(D_X Ric)(Y,Z) = \hat{R}(X,Y)Z + \frac{nD_X r}{(n-1)(n+2)}g(Y,Z) + \frac{(n-2)D_Y r}{2(n-1)(n+2)}g(X,Z) \\ + \frac{(n-2)D_Z r}{2(n-1)(n+2)}g(X,Y), \tag{3.1}$$

for all vector fields X, Y, Z and $\hat{R}(X,Y)Z = \hat{R}(X,Z)Y$ is a tensor with zero trace that can be written as a sum of its orthogonal components:

$$\hat{R}(X,Y)Z = \frac{1}{3}[\hat{R}(X,Y)Z + \hat{R}(Y,Z)X + \hat{R}(Z,X)Y] + \frac{1}{3}[\hat{R}(X,Y)Z - \hat{R}(Y,X)Z] \\ + \frac{1}{3}[\hat{R}(X,Y)Z - \hat{R}(Z,X)Y]. \tag{3.2}$$

The decompositions (3.1) and (3.2) yield $O(X)$-invariant subspace, which is characterized by a linear invariant equation in $(D_X Ric)(Y,Z)$. Therefore, the relation between $(D_X Ric)(Y,Z)$ and the divergence of conformal curvature tensor can be given by the equation

$$(divC)(X,Y)Z = \left(\frac{n-3}{n-2}\right)[R(X,Y)Z - R(Y,X)Z], n > 3. \tag{3.3}$$

The subspaces in Gray's decomposition are as follows:

i. The trivial subspace is given by $(D_X Ric)(Y,Z) = 0$.

ii. The subspace is characterized by $R(X,Y)Z = 0, i.e.,$

$$\begin{aligned}(D_X Ric)(Y,Z) =& \frac{nD_X r}{(n-1)(n+2)}g(Y,Z) + \frac{(n-2)D_Y r}{2(n-1)(n+2)}g(X,Z)\\ &+ \frac{(n-2)D_Z r}{2(n-1)(n+2)}g(X,Y).\end{aligned} \tag{3.4}$$

Manifolds satisfying equation (3.4) are called Sinyukov manifolds,Sinyukov (1979).

iii. The orthogonal complements $\mathscr{I}'$, which is also called subspace $\mathscr{A}$, is defined by

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0, \tag{3.5}$$

which yields that the scalar curvature r is constant. Also, the Ricci tensor is a Killing tensor if equation (3.5) is satisfied.

iv. In the subspaces $\mathscr{B}'$ and $\mathscr{B}$, the Ricci tensor is of Codazzi type, i.e.,

$$(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z) = 0. \tag{3.6}$$

v. In the subspace $\mathscr{I} \oplus \mathscr{A}$, the Ricci tensor satisfies the following cyclic condition:

$$\begin{aligned}(D_X Ric)(Y,Z)+&(D_Y Ric)(Z,X) + (D_Z Ric)(X,Y)\\ &= 2\frac{dr(X)}{n+2}g(Y,Z) + 2\frac{dr(Y)}{n+2}g(Z,X) + 2\frac{dr(Z)}{n+2}g(X,Y),\end{aligned} \tag{3.7}$$

that is, the Ricci tensor is conformal Killing.

vi. The Ricci tensor fulfills the following Codazzi condition in the subspace $\mathscr{I} \oplus \mathscr{B}$,

$$(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U) = \frac{1}{2(n-1)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)], \tag{3.8}$$

which gives $divC = 0$.

vii. In the subspace $\mathscr{A} \oplus \mathscr{B}$, the scalar curvature is covariant constant.

Let us consider each of these seven cases separately.
Case(i): Contraction of (2.9) with respect to X and V, we get

$$(div\tilde{H})(Y,Z)U = (a+b)(divR)(Y,Z)U + \left[\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times [g(Z,U)dr(Y) - g(Y,U)dr(Z)], \tag{3.9}$$

with condition (i),i.e., $(D_X Ric)(Y,Z) = 0$, and using (3.9), we obtain
The quasi-conharmonic curvature tensor $\tilde{H}$ is divergence-free. But the converse is not generally true.
Case(ii): A semi-Riemannian manifold (M^n, g) is in the subspace $\mathscr{I}$. Then in view of (3.4) and (3.9), we get

$$(div\tilde{H})(Y,Z)U = \left[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times [g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \tag{3.10}$$

Assuming that $(div\tilde{H})(Y,Z)U = 0$ and contracting with respect to Z and U, we get

$$\left[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] dr(Y) = 0.$$

Hence, we have the following theorem:

Theorem 3.1 *If (M^n, g) be a semi-Riemannian manifold whose Ricci tensor Ric satisfies the equation* (3.4)*. Then, the curvature tensor $\tilde{H}$ is divergence-free, if and only if either the scalar curvature r is constant or*

$$\left[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] = 0.$$

Case(iii): A semi-Riemannian manifold (M^n, g) is in the subspace $\mathscr{A}$ if the Ricci tensor is Killing i.e.,

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0.$$

Thus, the scalar curvature is constant,i.e., $dr(X) = 0$. Using this property in (3.9), we get

$$(div\tilde{H})(Y,Z)U = (a+b)[(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U)]. \tag{3.11}$$

Conversely, if (3.11) exists, then from (3.9), we get $dr(X) = 0$.
Hence, we have the following theorem:

Theorem 3.2 *If* (M^n, g) *be a semi-Riemannian manifold whose Ricci tensor Ric is Killing, then* $(div\tilde{H})(Y, Z)U = (a + b)[(D_Y Ric)(Z, U) - (D_Z Ric)(Y, U)]$*, if and only if the scalar curvature is constant.*

Case(iv): The next subspace $\mathscr{B}$ and $\mathscr{B}'$ contains a manifold whose Ricci tensor is of Codazzi type,i.e., $(D_X Ric)(Y, Z) = (D_Y Ric)(X, Z)$. Then, equation (3.9) becomes

$$(div\tilde{H})(Y, Z)U = \left[\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times [g(Z, U)dr(Y) - g(Y, U)dr(Z)]. \tag{3.12}$$

Hence, from (3.12), we have the following theorem:

Theorem 3.3 *If* (M^n, g) *be a semi-Riemannian manifold whose Ricci tensor Ric is Codazzi type. Then, the curvature tensor* $\tilde{H}$ *is divergence-free, if and only if either the scalar curvature* r *is constant or*

$$\left[\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] = 0.$$

Case(v): A semi-Riemannian manifold (M^n, g) is in the subspace $\mathscr{I} \oplus \mathscr{A}$. Then from (3.7) and (3.9), we get

$$\begin{aligned}(div\tilde{H})(Y, Z)U =& (a + b)[-2(D_Z Ric)(U, Y) - (D_U Ric)(Z, U) \\ &+ \frac{2}{n+2}\{dr(Y)g(U, Z) + dr(Z)g(U, Y) + dr(U)g(Y, Z)\}] \\ &+ \left[\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times \\ &[g(Z, U)dr(Y) - g(Y, U)dr(Z)].\end{aligned} \tag{3.13}$$

Here, we assume that $(div\tilde{H})(Y, Z)U = 0$, then from (3.13), we get, after contraction with respect to Z and U,

$$\left[\frac{a+b}{2} + (n-1)\left\{\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right\}\right] dr(Y) = 0.$$

Hence, we can state the following theorem:

Theorem 3.4 *If (M^n, g) be a semi-Riemannian manifold whose Ricci tensor satisfies the conformal Killing tensor, the curvature tensor $\tilde{H}$ is divergence-free, if and only if either*

$$\left[\frac{a+b}{2} + (n-1)\left\{\frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right\}\right] = 0,$$

or scalar curvature is constant.

Case(vi): Let the curvature tensor $\tilde{H}$ belongs to the class $\mathscr{I} \oplus \mathscr{B}$. Then

$$\begin{aligned}(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U) =& \frac{1}{2(n-1)}[g(Z,U)dr(Y) \\ & - g(Y,U)dr(Z)]. \end{aligned} \tag{3.14}$$

Then, divergence of $\tilde{H}$ becomes

$$\begin{aligned}(div\tilde{H})(Y,Z)U = & \left[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{2a}{n-2} + b + c\right)\right] \times \\ & [g(Z,U)dr(Y) - g(Y,U)dr(Z)]\,. \end{aligned} \tag{3.15}$$

Hence from (3.14), we have the following theorem:

Theorem 3.5 *Let (M^n, g) be a semi-Riemannian manifold whose Ricci tensor satisfies* (3.14)*. The curvature tensor $\tilde{H}$ is divergence-free, if and only if either* $[a(n-4) - b(n-2) + c(n-1)(n-2)] = 0$ *or scalar curvature is constant.*

Case(vii): In the subspace $\mathscr{A} \oplus \mathscr{B}$, the scalar curvature is covariant constant i.e. $dr(X) = 0$. Then in view of (3.9), we get

Theorem 3.6 *Let (M^n, g) be a semi-Riemannian manifold whose scalar curvature is constant. The curvature tensor $\tilde{H}$ is divergence-free, if and only if either*

$$a\left\{\frac{1}{2} - \frac{2(n-1)}{n(n-2)}\right\} + b\left\{\frac{1}{2} - \frac{n-1}{n}\right\} + c\left\{\frac{n-1}{2} - \frac{n-1}{n}\right\} = 0, \tag{3.16}$$

or scalar curvature is constant.

4. Bianchi's second identity for $(PQ\tilde{H}S)_n, (n > 2)$

In the view of (1.11) and (2.4), we get

$$\begin{aligned}&(D_X\tilde{H})(Y,Z,U,V) + (D_Y\tilde{H})(Z,X,U,V) + (D_Z\tilde{H})(X,Y,U,V) = \\ &A(V)[\tilde{H}(X,Y,U,Z) + \tilde{H}(Y,Z,U,X) + \tilde{H}(Z,X,U,Y)].\end{aligned}$$

Using (1.6) in the above equation, we get

$$(D_X\tilde{H})(Y,Z,U,V)+(D_Y\tilde{H})(Z,X,U,V)+(D_Z\tilde{H})(X,Y,U,V)=0. \tag{4.1}$$

Thus, we have the following theorem:

Theorem 4.1 *The quasi-conharmonic curvature tensor in* $(PQ\tilde{H}S)_n, n>2$, *satisfies Bianchi's second identity.*

From (2.5), we have,

$$\begin{aligned}
&(D_X\tilde{H})(Y,Z,U,V)+(D_Y\tilde{H})(Z,X,U,V)+(D_Z\tilde{H})(X,Y,U,V)=\\
&b[\{(D_XRic)(Z,U)-(D_ZRic)(X,U)\}g(Y,V)+\\
&\{(D_YRic)(X,U)-(D_XRic)(Y,U)\}g(Z,V)\\
&+\{(D_ZRic)(Y,U)-(D_YRic)(Z,U)\}g(X,V)]+\\
&c[\{(D_XRic)(Y,V)-(D_YRic)(X,V)\}g(Z,U)\\
&+\{(D_ZRic)(X,V)-(D_XRic)(Z,V)\}g(Y,U)\\
&+\{(D_YRic)(Z,V)-(D_ZRic)(Y,V)\}g(X,U)]\\
&-\frac{1}{n}\left(\frac{2a}{n-2}+b+c\right)[\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}dr(X)\\
&+\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\}dr(Y)\\
&+\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}dr(Z)].
\end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we have

$$\begin{aligned}
&b[\{(D_XRic)(Z,U)-(D_ZRic)(X,U)\}g(Y,V)+\\
&\{(D_YRic)(X,U)-(D_XRic)(Y,U)\}g(Z,V)\\
&+\{(D_ZRic)(Y,U)-(D_YRic)(Z,U)\}g(X,V)]+\\
&c[\{(D_XRic)(Y,V)-(D_YRic)(X,V)\}g(Z,U)\\
&+\{(D_ZRic)(X,V)-(D_XRic)(Z,V)\}g(Y,U)\\
&+\{(D_YRic)(Z,V)-(D_ZRic)(Y,V)\}g(X,U)]\\
&-\frac{1}{n}\left(\frac{2a}{n-2}+b+c\right)[\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}dr(X)\\
&+\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\}dr(Y)\\
&+\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}dr(Z)]=0.
\end{aligned} \tag{4.3}$$

Using Codazzi type Ricci tensor, i.e., $(D_X Ric)(Z,U) = (D_Z Ric)(X,U)$ in (4.3), and contracting in resulting equation with respect to Z and U, we get

$$[2a + (n-2)(b+c)]\, dr(X) = 0. \tag{4.4}$$

Hence, in view of above discussion, we have the following theorem:

Theorem 4.2 *Let $(PQ\tilde{H}S)_n, n > 2$, be an n-dimensional manifold (M^n, g). If the Ricci tensor Ric is of Codazzi type and the manifold satisfies the Bianchi's second identity, then the scalar curvature is constant, provided*

$$2a + (n-2)(b+c) \neq 0.$$

5. Ricci-symmetric $(PQ\tilde{H}S)_n, (n > 2)$

In this section, we assume that $(PQ\tilde{H}S)n, (n > 2)$, manifold is Ricci-symmetric, i.e., $(D_X Ric)(Y,Z) = 0$. This implies that the scalar curvature r is constant.
Now in view of (1.11) and (2.9), we get

$$\begin{aligned}
&a(D_X R)(Y,Z,U,V) + b[(D_X Ric)(Z,U)g(Y,V) - (D_X Ric)(Y,U)g(Z,V)] \\
&+ c[(D_X Ric)(Y,V)g(Z,U) - (D_X Ric)(Z,V)g(Y,U)] \\
&- \frac{dr(X)}{n}\left[\frac{2a}{n-2} + b + c\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] = \\
&2A(X)\tilde{H}(Y,Z,U,V) + A(Y)\tilde{H}(X,Z,U,V) + A(Z)\tilde{H}(Y,X,U,V) \\
&+ A(U)\tilde{H}(Y,Z,X,V) + A(V)\tilde{H}(Y,Z,U,X).
\end{aligned} \tag{5.1}$$

Putting e_i for Y and V, we get

$$\begin{aligned}
&a[(D_X Ric)(Z,U)] + b[(D_X Ric)(Z,U)(n-1)] + c[dr(X)g(Z,U) - (D_X Ric)(Z,U)] \\
&- \frac{dr(X)}{n}\left[\frac{2a}{n-2} + b + c\right](n-1)g(Z,U) = 2A(X)\tilde{H}_2(Z,U) + \tilde{H}(X,Z,U,\rho) \\
&- A(Z)\tilde{H}_2(X,U) + A(U)\tilde{H}_2(Z,X) + \tilde{H}(\rho,Z,U,X).
\end{aligned}$$

Since, the manifold is Ricci symmetric, the above equation reduces to

$$\begin{aligned}
&2A(X)\tilde{H}_2(Z,U) + \tilde{H}(X,Z,U,\rho) - A(Z)\tilde{H}_2(X,U) \\
&+ A(U)\tilde{H}_2(Z,X) + \tilde{H}(\rho,Z,U,X) = 0.
\end{aligned} \tag{5.2}$$

Again, we put e_i for Z and U in (5.2), we get

$$Ric(X,\rho) = \left[\frac{(n^2+4n-4)a + (n-2)^2(b+c)r}{n\{2a+(n-2)(b+c)\}(n-2)}\right] g(X,\rho). \tag{5.3}$$

Hence, we have the following theorem:

Theorem 5.1 *If a $(PQ\tilde{H}S)_n, n > 2$, manifold is Ricci-symmetric, then* $\left[\frac{a(n^2+4n-4)+(n-2)^2(b+c)}{\{2a+(n-2)(b+c)\}n(n-2)}\right] r$ *is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ, provided $2a + (n-2)(b+c) \neq 0$.*

6. A $(PQ\tilde{H}S)_n, (n > 2)$ admitting Codazzi type Ricci tensor

In $(PQ\tilde{H}S)_n, n > 2$, we have from (1.11)

$$(D_X\tilde{H}_2)(Z,U) = 2A(X)\tilde{H}_2(Z,U) + \tilde{H}(X,Z,U,\rho) + A(Z)\tilde{H}_2(X,U) + A(U)\tilde{H}_2(Z,X) + \tilde{H}(\rho,Z,U,X). \tag{6.1}$$

Since $(PQ\tilde{H}S)_n, n > 2$ admits the Codazzi type Ricci tensor, therefore from (6.1), we get

$$\begin{aligned} &2A(X)\tilde{H}_2(Z,U) + \tilde{H}(X,Z,U,\rho) + A(Z)\tilde{H}_2(X,U) + A(U)\tilde{H}_2(Z,X) + \tilde{H}(\rho,Z,U,X) \\ &= 2A(Z)\tilde{H}_2(X,U) + \tilde{H}(Z,X,U,\rho) + A(X)\tilde{H}_2(Z,U) \\ &+ A(U)\tilde{H}_2(X,Z) + \tilde{H}(\rho,X,U,Z). \end{aligned} \tag{6.2}$$

Solving equation (6.2), we get

$$A(X)\tilde{H}_2(Z,U) - A(Z)\tilde{H}_2(X,U) + 2\tilde{H}(X,Z,U,\rho) + \tilde{H}(\rho,Z,U,X) - \tilde{H}(\rho,X,U,Z) = 0. \tag{6.3}$$

Contracting (6.3) with respect to Z and U, we get

$$Ric(X,\rho) = \lambda_1 g(X,\rho),$$

where

$$\lambda_1 = \frac{[an^2 - n(b-c)(n-2)]r}{n[2a - b(n+2) + c(3n-2)](n-2)}.$$

Theorem 6.1 *In a $(PQ\tilde{H}S)_n, n > 2$, admitting a Codazzi type Ricci tensor, then λ_1 is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ, provided*

$$\lambda_1 = \frac{[an^2 - n(b-c)(n-2)]r}{n[2a - b(n+2) + c(3n-2)](n-2)}.$$

7. A $(PQ\tilde{H}S)_n, (n > 2)$ admitting a pseudo cyclic Ricci tensor

In this section, we assume that the Ricci tensor of $(PQ\tilde{H}S)_n, n > 2$, admits a pseudo cyclic Ricci tensor.
A Riemannian (M^n, g) is said to have pseudo cyclic Ricci tensor Shaikh and Hui (2009), if its Ricci tensor satisfies

$$\begin{aligned}&(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) =\\ &2A(X)Ric(Y,Z) + A(Y)Ric(Z,X) + A(Z)Ric(X,Y).\end{aligned} \tag{7.1}$$

In view of (6.1), we have

$$\begin{aligned}(D_X\tilde{H}_2)(Z,U) =& 2A(X)\tilde{H}_2(Z,U) + \tilde{H}(X,Z,U,\rho) + A(Z)\tilde{H}_2(X,U)\\ &+ A(U)\tilde{H}_2(Z,X) + \tilde{H}(\rho,Z,U,X),\end{aligned} \tag{7.2}$$

$$\begin{aligned}(D_Z\tilde{H}_2)(U,X) =& 2A(Z)\tilde{H}_2(U,X) + \tilde{H}(Z,U,X,\rho) + A(U)\tilde{H}_2(Z,X)\\ &+ A(X)\tilde{H}_2(U,Z) + \tilde{H}(\rho,U,X,Z),\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}(D_U\tilde{H}_2)(X,Z) =& 2A(U)\tilde{H}_2(X,Z) + \tilde{H}(U,X,Z,\rho) + A(X)\tilde{H}_2(U,Z)\\ &+ A(Z)\tilde{H}_2(X,U) + \tilde{H}(\rho,X,Z,U).\end{aligned} \tag{7.4}$$

Hence, we assume that $(PQ\tilde{H}S)_n, n > 2$, admits a pseudo cyclic Ricci tensor.
Thus, from (7.1), (7.2), (7.3) and (7.4), we have

$$\begin{aligned}&2A(X)\tilde{H}_2(Z,U) + 3A(Z)\tilde{H}_2(X,U) + 3A(U)\tilde{H}_2(Z,X)\\ &+ \Big[\tilde{H}(X,Z,U,\rho) + \tilde{H}(Z,U,X,\rho) + \tilde{H}(U,X,Z,\rho)\Big]\\ &+ \Big[\tilde{H}(\rho,Z,U,X) + \tilde{H}(\rho,U,X,Z) + \tilde{H}(\rho,X,Z,U)\Big] = 0.\end{aligned} \tag{7.5}$$

Putting e_i for Z and U in (7.5), we get

$$2A(X)\tilde{H}_2(e_i,e_i) + 5\tilde{H}_2(X,\rho) + \tilde{H}_1(X,\rho) = 0. \tag{7.6}$$

In view of (2.6), (2.7) and (7.6), we get

$$Ric(X,\rho) = \left[\frac{[2an^2 - n(n-2)(b-c)]r}{n\{6a + b(5n-6) + c(n-6)\}(n-2)}\right] g(X,\rho). \tag{7.7}$$

Hence, from (7.7), we have the following theorem:

Theorem 7.1 *For a $(PQ\tilde{H}S)_n$, $n > 2$, admitting pseudo cyclic Ricci tensor, λ_2 is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ, where*

$$\lambda_2 = \left[\frac{[2an^2 - n(n-2)(b-c)]r}{n\{6a + b(5n-6) + c(n-6)\}(n-2)}\right].$$

8. A $(PQ\tilde{H}S)_n, (n > 2)$, admitting Ricci recurrent tensor

Patterson (1952) introduced the concept of Ricci recurrent manifold as

$$(D_X Ric)(Y, Z) = A(X) Ric(Y, Z), \tag{8.1}$$

for arbitrary vector fields X, Y and Z.
Here, we assume that $(PQ\tilde{H}S)_n, (n > 2)$, admits Ricci recurrent manifold. Hence, in view of (8.1), we have

$$(D_X \tilde{H}_2)(Z, U) = A(X)\tilde{H}_2(Z, U). \tag{8.2}$$

Thus, from (7.2) and (8.2), we get

$$\begin{aligned} &A(X)\tilde{H}_2(Z, U) + \tilde{H}(X, Z, U, \rho) + A(Z)\tilde{H}_2(X, U) \\ &+ A(U)\tilde{H}_2(Z, X) + \tilde{H}(\rho, Z, U, X) = 0. \end{aligned} \tag{8.3}$$

Under suitable contraction of (8.3), we get

$$A(X)\tilde{H}_2(e_i, e_i) + 2\tilde{H}_2(X, \rho) + 2\tilde{H}_1(X, \rho) = 0. \tag{8.4}$$

In view of (2.6) and (8.4), we get

$$Ric(X, \rho) = \left[\frac{\{a(n^2 + 8n - 8) + 2(b+c)(n-2)\}r}{2n\{2a + (b+c)(n-2)\}(n-2)}\right] g(X, \rho). \tag{8.5}$$

Equation (8.5) can be put as

$$Ric(X, \rho) = \lambda_3 g(X, \rho),$$

where

$$\lambda_3 = \left[\frac{\{a(n^2 + 8n - 8) + 2(b+c)(n-2)\}r}{2n\{2a + (b+c)(n-2)\}(n-2)}\right].$$

Hence, in view of (8.5), we can state the following theorem:

Theorem 8.1 *For $(PQ\tilde{H}S)_n, n > 2$, admitting Ricci recurrent tensor, λ_3 is an eigenvalue of the Ricci tensor Ric, corresponding to the eigenvector ρ, where*

$$\lambda_3 = \left[\frac{\{a(n^2 + 8n - 8) + 2(b+c)(n-2)\}r}{2n\{2a + (b+c)(n-2)\}(n-2)}\right].$$

9. Einstein $(PQ\tilde{H}S)_n, (n > 2)$

This section deals with an Einstein $(PQ\tilde{H}S)_n, (n > 2)$. Then, the Ricci tensor satisfies

$$Ric(U, V) = \frac{r}{n} g(U, V), \tag{9.1}$$

From which it follows that

$$dr(U) = 0 \quad \text{and} \quad (D_Z Ric)(U, V) = 0, \tag{9.2}$$

for all vector fields U, V and Z.
From (2.6), we have

$$\tilde{H}_2(e_i, e_i) = -\left(\frac{n}{n-2}\right) ar. \tag{9.3}$$

Equation (9.3), can be put as

$$\omega = -\left(\frac{n}{n-2}\right) ar,$$

which gives

$$D_X \omega = -\frac{dr(X) \cdot an}{n-2}. \tag{9.4}$$

Putting e_i for Z and U in equation (6.1), we get

$$(D_X \tilde{H}_2)(e_i, e_i) = 2A(X)\tilde{H}_2(e_i, e_i) + 2\tilde{H}_2(X, \rho) + 2\tilde{H}_1(X, \rho). \tag{9.5}$$

Hence, (9.4) and (9.5) gives

$$\begin{aligned}
-\frac{an\, dr(X)}{n-2} =& -2A(X)\frac{arn}{n-2} + 2[a + b(n-1) - c]Ric(X, \rho) \\
&+ 2[a - b + c(n-1)]Ric(X, \rho) \\
&- 2A(X)\frac{r}{n(n-2)}\left[2(n-1) + b(n-1)(n-2) - c(n-2)\right] \\
&- 2A(X)\frac{r}{n(n-2)}\left[2a(n-1) - b(n-2) + c(n-1)(n-2)\right].
\end{aligned} \tag{9.6}$$

Hence, in view of (9.1), (9.2) and (9.6), we get

$$an(n-2)r \cdot A(X) = 0. \tag{9.7}$$

Now, from (9.7), we see that $A(X) \neq 0$. Thus $r = 0$, provided $an(n-2) \neq 0$.

Theorem 9.1 *An Einstein* $(PQ\tilde{H}S)_n, n > 2$, *is of zero scalar curvature, provided* $an(n-2) \neq 0$.

If possible, let $(PQ\tilde{H}S)_n, n > 2$, be a space of constant curvature. Then, we have

$$\mathscr{R}(X,Y)Z = K[g(Y,Z)X - g(X,Z)Y], \tag{9.8}$$

where K is a constant. Contracting X in (9.8), we have

$$Ric(Y,Z) = K(n-1)g(Y,Z). \tag{9.9}$$

Again, contracting Y and Z in (9.9), we get

$$r = K \cdot n(n-1). \tag{9.10}$$

Using (9.10) in (9.8), we get

$$\mathscr{R}(X,Y)Z = \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{9.11}$$

Since every space of constant curvature is an Einstein manifold, then from Theorem 9.1 we have $r = 0$. Hence, from (9.11) it follows that $\mathscr{R}(X,Y)Z = 0$, which is inadmissible by definition. This leads to the following corollary of the above theorem under the assumption $an(n-2) \neq 0$:

Corollary 9.2 *A* $(PQ\tilde{H}S)_n, n > 2$, *cannot be of constant curvature.*

10. Example of a $(PQ\tilde{H}S)_4$

Let us consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g is

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (\sin x^1)^2(dx^2)^2 + (\sin x^2)^2(dx^3)^2 - (dx^4)^2, \tag{10.1}$$

where, $i, j = 1, 2, 3, 4$ and $\sin x^1, \sin x^2$ are non-zero.
Then, the only non-vanishing components of the Christoffel symbols are

$$\begin{aligned} &\Gamma^1_{22} = -\sin x^1 \cos x^1, \Gamma^2_{33} = -\frac{\sin x^2 \cos x^2}{(\sin x^1)^2}, \\ &\Gamma^2_{12} = \cot x^1 = \Gamma^2_{21}, \Gamma^3_{23} = \cot x^2 = \Gamma^3_{32}. \end{aligned} \tag{10.2}$$

The curvature tensor of the type $(1,3)$ is defined by

$$\mathscr{R}^\alpha_{ijk} = -\frac{\partial}{\partial x^k}\Gamma^\alpha_{ij} + \frac{\partial}{\partial x^j}\Gamma^\alpha_{ik} + \Gamma^\alpha_{\beta j}\Gamma^\beta_{ik} - \Gamma^\alpha_{\beta k}\Gamma^\beta_{ij}. \tag{10.3}$$

In view of (10.2) and (10.3), we get

$$\mathscr{R}^1_{332} = -\frac{1}{2}\cot x^1 \sin 2x^2. \tag{10.4}$$

The curvature tensor of the type $(0,4)$ is defined by

$$R_{hijk} = g_{h\alpha}\mathscr{R}^{\alpha}_{ijk}. \tag{10.5}$$

In view of (10.4) and (10.5), we get

$$R_{1332} = -\frac{1}{2}\cot x^1 \sin 2x^2. \tag{10.6}$$

The non-vanishing components of the Ricci tensor is

$$(Ric)_{12} = -\cot x^1 \cot x^2, \tag{10.7}$$

and scalar curvature is

$$r = 0.$$

We shall now show that this M^4 is a pseudo-generalized quasi-conharmonic symmetric spacetime, i.e., it satisfies the defining relation (1.11). Now, the only non-vanishing components for $\tilde{H}$ and its covariant derivatives are given by

$$\begin{aligned}
\tilde{H}_{1332} &= -\left(\frac{a+c}{2}\right) cotx^1 sin2x^2, \\
\tilde{H}_{1332,1} &= \left(\frac{a+c}{2}\right)\left(cosec^2x^1 sin2x^2\right), \\
\tilde{H}_{1332,2} &= \left(\frac{a+c}{2}\right)\left(4cos^2x^2 - 2cos2x^2\right) cotx^1.
\end{aligned} \tag{10.8}$$

Let us choose the associated 1-form as

$$A_i = \begin{cases} -\dfrac{1}{3sinx^1cosx^1}, & \text{for } i = 1, \\[2ex] \dfrac{1}{3}\left(\dfrac{2cos2x^1}{sin2x^2} - 2cotx^2\right), & \text{for } i = 2, \\[2ex] 0, & \text{for } i = 3, 4. \end{cases} \tag{10.9}$$

at any point $x \in (M^4, g)$. In (M^4, g), after using (10.8)and (10.9) in R.H.S. of (1.11), we obtain

$$\begin{aligned}
&2A_1\tilde{H}_{1332} + A_1\tilde{H}_{1332} + A_3\tilde{H}_{1132} + A_3\tilde{H}_{1312} + A_2\tilde{H}_{1331} \\
&= -3\left(\frac{-1}{3sinx^1cosx^1}\right)\left(\frac{a+c}{2}\right)cotx^1sin2x^2 \\
&= \left(\frac{a+c}{2}\right)cosec^2x^1sin2x^2. \qquad (10.10)
\end{aligned}$$

In view of (10.8) and (10.10), we get

$$\tilde{H}_{1332,1} = 2A_1\tilde{H}_{1332} + A_1\tilde{H}_{3332} + A_3\tilde{H}_{1132} + A_3\tilde{H}_{1312} + A_2\tilde{H}_{1331}. \qquad (10.11)$$

Similarly, we can prove that

$$\tilde{H}_{1332,2} = 2A_2\tilde{H}_{1332} + A_1\tilde{H}_{2332} + A_3\tilde{H}_{1232} + A_3\tilde{H}_{1322} + A_2\tilde{H}_{1332}. \qquad (10.12)$$

Hence, it is verified that the equations (10.11) and (10.12) are true. Thus, the manifold under consideration is a pseudo-generalized quasi-conharmonic symmetric spacetime, that is $(PQ\tilde{H}S)_4$.

Theorem 10.1 *Let (M^n, g) be semi-Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (sinx^1)^2(dx^2)^2 + (sinx^2)^2(dx^3)^2 - (dx^4)^2.$$

where, $i,\ j = 1, 2, 3, 4$ and $\sin x^1, \sin x^2$ are non-zero. Then (M^n, g) is a $(PQ\tilde{H}S)_4$.

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Received: 10.06.2026, Revised: 03.07.2026, Accepted: 07.07.2026