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Totally r–Contact–Umbilical semi–invariant busmanifold of a r-Sasakian

Geeta Verma[1]
Department of Applied Sciences & Humanities, Azad Institute of Engineering & Technology
Natkur, Lucknow -226002, India

Abstract

This paper gives a characterization of totally r- contact-umbilical semi-invariant submanifolds of a r-Sasakian manifold.

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1. Preliminaries: Let N be a (2n+r)-dimensional r-Sasakian manifold with structure tensors (ϕ, ξ, η, g). Then they satisfy

$$\phi^2 X = -X + \eta^p(X)\xi_p \ , \ \phi\xi_p = 0 \ , \ \eta^p(\phi X) = 0 \ , \ \eta^p(\xi_p) = 1 \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta^p(X)\eta^p(Y) \quad , \quad \eta^p(X) = g(X, \xi_p) \tag{1.2}$$

for any vector fields X and Y tangent to N. We denote by $\overline{\nabla}$ the Levi-Civita connection on N and $\overline{R}$ the curvature tensor corresponding to $\overline{\nabla}$. Then we have Yano and Kon (1984)

$$(\overline{\nabla}_X \phi)Y = g(X,Y)\xi_p - \eta^p(Y)X \quad , \quad \overline{\nabla}_X \xi_p = -\phi X \tag{1.3}$$

$$\overline{R}(X,Y)\phi Z = \phi\overline{R}(X,Y)Z + g(\phi X, Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y - g(\phi Y, Z)X \tag{1.4}$$

$$g(\overline{R}(\phi X, \phi Y)\phi Z, \phi W) = g(\overline{R}(X,Y)Z, W) - \eta^p(Y)\eta^p(Z)g(X,W) \tag{1.5}$$

$$-\eta^p(X)\eta^p(W)g(Y,Z) + \eta^p(Y)\eta^p(W)g(X,Z) \quad + \eta^p(X)\eta^p(Z)g(Y,W)$$

$$\overline{R}(X, \xi_p)Y = -(\overline{\nabla}_X \phi)Y = -g(X,Y)\xi_p + \eta^p(Y)X \tag{1.6}$$

for any vector fields X, Y,Z and W tangent to N.

An m-dimensional submanifold M of N is said to be a semi-invariant submanifold if there exists a pair of orthogonal distributions $(D, D^{\perp})$ satisfying the conditions Bejancu and Papaghuic(1981)

(i) $TM = D \oplus D^{\perp} \oplus \{\xi_p\}$

E-mail – geeta_verma1@yahoo.com

(ii) The distribution D is invariant by ϕ , i .e. $\phi(D_x) = D_x, x \in M$.

(iii) the distribution $D^{\perp}$ is anti-invariant , i.e. $\phi(D_x^{\perp}) \subset T_x M$, $x \in M$

where TM and $TM^{\perp}$ denote the tangent bundle and normal bundle to M respectively . It follows that the normal bundle splits as $TM^{\perp} = \phi D^{\perp} \oplus v$, where v is an invariant sub-bundle of $TM^{\perp}$ by ϕ. If $D = \{0\}(resp. D^{\perp} = \{0\})$ then M is said to be an anti invariant (resp. invariant) submanifold. We say that M is proper if it is neither invariant nor anti-invariant.

For any vector bundle S over M we denote by $\Gamma(S)$ the module of all differentiable sections on S. Let ∇ be the induced Levi-Civita connection on M and $\nabla^{\perp}$ the induced normal connection on $TM^{\perp}$. The Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

for any X, Y $\in \Gamma(TM)$ and $\varsigma \in \Gamma(TM^{\perp})$, where h is the second fundamental form of M and the shape operator A_ς is related to h by

$$g(A_\varsigma X, Y) = g(h(X,Y), \varsigma)$$

The projection morphism of TM on D and $D^{\perp}$ are denoted by P and Q respectively . For $\varsigma \in \Gamma(TM^{\perp})$ we denote $t\varsigma$ the tangential part and $f\varsigma$ the normal part of $\phi\varsigma$ respectively. Also , we put $\psi = \phi \circ P$ and $\omega = \phi \circ Q$. Then we have Bejancu (1986)

$$(\nabla_X \psi)Y = th(X,Y) + A_{\omega Y} X + g(X,Y)\xi_p - \eta^p(Y)X \tag{1.7}$$

$$(\nabla_X \omega)Y = fh(X,Y) - h(X, \psi Y) \tag{1.8}$$

$$(\nabla_X f)\varsigma = -h(X, t\varsigma) - \omega A_\varsigma X \tag{1.9}$$

$$h(X, \xi_p) = -\omega X \ , \quad \nabla_X \xi_p = -\psi X \tag{1.10}$$

for any X, Y $\in \Gamma(TM)$ and $\varsigma \in \Gamma(TM^{\perp})$.

Now we recall the definition of a locally conformal Kaehler manifold. Let M be a Hermitian manifold with complex structure J. Then M is called a locally conformal Kaehler manifold if there exists a closed 1-form τ , called the Lee form , on M such that

$$d\Omega = \tau \Lambda \Omega$$

or equivalently,

$$(\nabla_X J)Y = \frac{1}{2}\{\theta(Y)X - \tau(Y)JX - \Omega(X,Y)B - g(X,Y)A\} \tag{1.11}$$

for X, Y $\in \Gamma(TM)$, where $\Omega(X,Y) = g(X, JY)$, B is the Lee vector field such that $g(B,X) = \tau(X), \theta = \tau \circ J$ is the anti -Lee 1-form and A = - J B is the anti-Lee vector field. Moreover , a generalized Hopf manifold is a locally conformal Kaehler manifold whose Lee form is parallel , i.e., $\nabla_\tau = 0$. (cf.Vaisman 1982).

2. Geometry of Totally r-Contact - umbilical Semi- invariant Submanifolds

A submanifold M is said to be totally umbilical if h (X, Y) = g(X, Y) $\overline{H}$, for all X, Y $\in \Gamma(TM)$, where $\overline{H} = \frac{1}{m}(trace$ of h), is the mean curvature vector of M. If the mean curvature vector $\overline{H}$ = 0 then M is called a totally geodesic submanifold .

A semi-invariant submanifold M is said to be totally r- contact-umbilical if

$$h(X,Y) = g(\phi X,\phi Y)H + \eta^p(Y)h(X,\xi_p) + \eta^p(X)h(Y,\xi_p) \tag{2.1}$$

$$= \{g(X,Y) - \eta^p(X)\eta^p(Y)\}H - \eta^p(Y)\omega X - \eta^p(X)\omega Y$$

or equivalently,

$$A_\varsigma X = g(H,\varsigma)X - \{\eta^p(X)g(H,\varsigma) + g(\omega X,\varsigma)\}\xi_p + \eta^p(X)t\varsigma \tag{2.2}$$

for any X, Y $\in \Gamma(TM)$ and $\varsigma \in \Gamma(TM^\perp)$, where H is normal vector field on M. If H $\equiv 0$ then M is called a totally r- contact –geodesic submanifold. We have

Theorem (2.1)

Any totally r-contact –umbilical proper semi-invariant submanifold of a r-Sasakian manifold N is a totally r-contact-geodesic submanifold. In the rest of this section, suppose M is a connected non-totally r-contact-geodesic, totally r-contact –umbilical proper semi-invariant submanifold of a r-Sasakian manifold N. It follows from Theorem (2.1) that $\dim D^\perp = 1$.

Now, we establish the following Lemmas

Lemma (2.1)

$H \in \Gamma(\phi D^\perp)$.

Proof: By putting Y = X $\in \Gamma(D)$ in (1.8) and taking account of (2.1) we obtain

$$-\omega\nabla_X X = g(X,X)fH$$

Note that the left side and the right side of the above equation is respectively in $\Gamma(\phi D^\perp)$ and $\Gamma(v)$, hence f H = 0 or $H \in \Gamma(\phi D^\perp)$.

Lemma (2.2)

$$\nabla^\perp_X H \in \Gamma(\phi D^\perp), \text{ for any } X \in \Gamma(TM).$$

Proof: By putting $\varsigma = H$ in (1.9) and taking account of the fact that f H = 0, we obtain

$$-f\nabla^\perp_X H = -h(X,tH) - \omega A_H X$$

Note that the left side of this equation is in $\Gamma(v)$ while the right side is in $\Gamma(\phi D^\perp)$ by virtue of (2.1) and Lemma (2.1). It follows that $f\nabla^\perp_X H = 0$ and so $\nabla^\perp_X H \in \Gamma(\phi D^\perp)$.

Lemma (2.3)

$$[\overline{R}(X,Y)W]^\perp = \{g(Y,W) - \eta^p(Y)\eta^p(W)\}\nabla^\perp_X H - \{g(X,W) - \eta^p(X)\eta^p(W)\}\nabla^\perp_Y H$$

$$-g(\psi Y,W)\omega X + g(\psi X,W)\omega Y + 2g(\psi X,Y)\omega W$$

for any $X,Y,W \in \Gamma(TM)$.

Proof: For any X, Y,W $\in \Gamma(TM)$,by using (1.8) , (1.10) and (2.1) we obtain

$$(\nabla_X h)(Y,W) = \{g(Y,W) - \eta^p(Y)\eta^p(W)\}\nabla_X^{\perp} H - \{(\nabla_X \eta^p)Y\eta^p(W) + \eta^p(Y)(\nabla_X \eta^p)W\}H$$

$$- (\nabla_X \eta^p)Y.\omega W - \eta^p(Y)(\nabla_X \omega)W - (\nabla_X \eta^p)W.\omega Y - \eta^p(W)(\nabla_X \omega)Y$$

$$= \{g(Y,W) - \eta^p(Y)\eta^p(W)\}\nabla_X^{\perp} H + \{g(Y,\psi X)\eta^p(W) + \eta^p(Y)g(W,\psi X)\}H$$

$$+ g(Y,\psi X)\omega W - \eta^p(Y)\{fh(X,W) - h(X,\psi W)\} + g(W,\psi X)\omega Y$$

$$- \eta^p(W)\{fh(X,Y) - h(X,\psi Y)\}$$

It follows from (2.1) and Lemma (2.1) that this equation reduces to

$$(\nabla_X h)(Y,W) = \{g(Y,W) - \eta^p(Y)\eta^p(W)\}\nabla_X^{\perp} H + g(Y,\psi X)\omega W + g(W,\psi X)\omega Y$$

Exchanging X and Y in the above equation , we have

$$(\nabla_Y h)(X,W) = \{g(X,W) - \eta^p(X)\eta^p(W)\}\nabla_Y^{\perp} H + g(X,\psi Y)\omega W + g(W,\psi Y)\omega X$$

From these equations and the Codazzi equation we obtain the Lemma.

Since M is non-totally r-contact-geodesic, we may choose a connected open set G on M such that H is nowhere zero on G. For the moment, we restrict our arguments on such an open set G. Define a unit vector field Z in $D^{\perp}$ by Z = $-\frac{1}{\mu}\phi H$, where $\mu = \|H\|$. Then we have the following

Lemma (2.4)

$\nabla_X Z = \mu\psi X$, for any $X \in \Gamma(TM)$.

Proof: For any $X \in \Gamma(TM)$, we have .

$$g(\nabla_X Z, Z) = 0 \quad \text{and} \quad g(\nabla_X Z, \xi_p) = -g(Z, \nabla_X \xi_p) = g(Z, \psi X) = 0$$

Next, by using (1.7) we obtain

$$-\psi\nabla_X Z = th(X,Z) + A_{\omega Z} X + g(X,Z)\xi_p \quad .$$

By applying ψ to this equation and taking account of (2.2) we get

$$\nabla_X Z = \psi A_{\omega Z} X = g(H,\omega Z)\psi X = \mu\psi X$$

Lemma (2.5)

The normal vector field H is parallel

Proof: Let $Y \in \Gamma(D)$ be unit vector field.

Then from (1.6) and Lemma (2.3)

$$\nabla_{\xi_p}^{\perp} H = [\overline{R}(\xi_p, Y)Y] = 0$$

Now, consider a unit vector field $X \in \Gamma(D)$ with $g(X,Y) = g(Y,\psi Y) = 0$ .Then by (1.4) we have

$$\overline{R}(\phi Z, X)\phi^2 X = \phi\overline{R}(\phi Z, X)\phi X - \phi Z.$$

By taking inner product with Y we get

$$g(\overline{R}(\phi Z, X)X, Y) = g(\overline{R}(\phi Z, X)\phi X, \phi Y)$$

$$g(\overline{R}(Y, X)X, \phi Z) = g(\overline{R}(\phi Y, \phi X)X, \phi Z).$$

Together with Lemma (2.3), we obtain

$$g(\nabla_Y^{\perp} H, \phi Z) = 0.$$

Next , by making use of (1.5) we obtain .

$g(\overline{R}(Z,Y)Y,\phi Z) = g(\overline{R}(\phi Z,\phi Y)\phi Y,\phi^2 Z) = -g(\overline{R}(\phi Z,\phi Y)\phi Y,Z).$

On the other hand, it follows from Lemma (2.3) that we obtain

$g(\overline{R}(Z,Y)Y,\phi Z) = g(\overline{R}(\phi Z,\phi Y)\phi Y,Z) = g(\nabla_Z^{\perp} H,\phi Z).$

These two equations imply that $g(\nabla_Z^{\perp} H,\phi Z) = 0.$ Also this amount to say that $\nabla_X^{\perp} H \in \Gamma(v)$, for all $X \in \Gamma(TM).$ Together with Lemma (2.2), we obtain that H is parallel.

It follows from Lemma (2.5) that μ is constant on G. Since M is connected, μ is a nonzero constant on M. Hence we have

Lemma (2.6)

Z is a unit vector field defined on the whole of M.

3. Characterization of Totally r-Contact –umbilical Semi-invariant Submanifolds

Theorem (3.1)

Let M be a connected proper, non totally r-contact –geodesic, totally r-contact –umbilical m –dimensional semi-invariant submanifold of a r- Sasakian manifold N. Then it is a generalized Hopf manifold.

Proof: From our assumption and Theorem (2.1) for any $X \in \Gamma(TM)$, we may put

$X = PX + \alpha(X)Z + \eta^p(X)\xi_p = -\psi^2 X + \alpha(X)Z + \eta^p(X)\xi_p$

where $\alpha(X) = g(X,Z).$ Now we define tensor field J of type (1,1) on M by

$$JX = \psi X + \alpha(X)\xi_p - \eta^p(X)Z \tag{3.1}$$

It is clear that J is an almost complex structure on M. Furthermore , we define a vector field B and a 1-form τ on M by

$$B = 2(\mu\xi_p + Z), \qquad \tau(X) = g(B,X) = 2(\alpha(X) + \mu\eta^p(X)) \tag{3.2}$$

for any $X \in \Gamma(TM)$.

It follows from (1.10) , (3.2) and Lemma (2.4) that , we have $(\nabla_X \tau)Y = 0$ for any $X,Y \in \Gamma(TM)$.

Hence, τ is parallel (and so is closed).

Finally, we shall show that (1.11) holds. For any $X,Y \in \Gamma(TM)$, it follows from (1.7), (1.10), (3.1) and Lemma (2.4) that

$$\begin{aligned}(\nabla_X J)Y &= (\nabla_X \psi)Y + (\nabla_X \alpha)Y.\xi_p + \alpha(Y)\nabla_X \xi_p - (\nabla_X \eta^p)Y.Z - \eta^p(Y)\nabla_X Z \\ &= th(X,Y) + \alpha(Y)A_{\omega Z}X + g(X,Y)\xi_p - \eta^p(Y)X + \mu g(\psi X,Y)\xi_p - \alpha(Y)\psi X \\ &+ g(\psi X,Y)Z - \mu\eta^p(Y)\psi X\end{aligned}$$

Now, from (2.1) and (2.2) the above equation becomes

$$\begin{aligned}(\nabla_X J)Y &= -\{g(X,Y) - \eta^p(X)\eta^p(Y)\}\mu Z + \eta^p(X)\alpha(Y)Z + \eta^p(Y)\alpha(X)Z \\ &\alpha(Y)\{\mu X - \mu\eta^p(X)\xi_p - \eta^P(X)Z - \alpha(X)\xi_p\} + g(X,Y)\xi_p - \eta^p(Y)X \\ &\mu g(\psi X,Y)\xi_p - \alpha(Y)\psi X + g(\psi X,Y)Z - \mu\eta^p(Y)\psi X\end{aligned}$$

This, together with (3.1) and (3.2) give

$$(\nabla_X J)Y = \frac{1}{2}\{g(X,Y)JB - g(JB,Y)X + g(JX,Y)B - g(B,Y)JX\}.$$

$$= \frac{1}{2}\{g(X,Y)JB - g(X,JY)B + \tau(JY)X - \tau(Y)JX\}$$

This completes the proof of the Theorem.
As an immediate consequence of Theorem (2.1) and Theorem (3.1), we obtain the following

Theorem (3.2)
Let M be a connected totally r-contact-umbilical m-dimensional semi-invariant submanifold of a r-Sasakian manifold N. Then either
(i) M is totally r-contact-geodesic; or
(ii) M is anti-invariant; or
(iii) M is a generalized Hopf manifold

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