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On Einstein pseudo conformally bi-symmetric smooth Riemannian manifold G (PCBS)n

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Abstract

In this paper, we have studied on generalized pseudo conformally bi-symmetric smooth Riemannian manifold and results related to Einstein $(PCBS)_n$ have been investigated.

Keywords: Pseudo symmetric, Pseudo conformal, Riemannian manifold, conformal curvature tensor, Einstein manifold.

1. Introduction

The notion of a pseudo conformally symmetric manifold was introduced by De and Biswas (1994) . A non- flat smooth Riemannian manifold $(M_n,<,>)$ $(n>3)$ was called pseudo conformally symmetric smooth Riemannian manifold, if the conformal curvature tensor C defined by relation

$$C(X, Y, Z) = R(X, Y, Z) - \frac{1}{(n-2)} [< Y, Z > QX - < X, Z > QY \tag{1.1}$$

$$+ T(Y, Z) X - T(X, Z)Y] + \frac{r}{(n-1)(n-2)} [< Y, Z > X - < X, Z > Y],$$

where R is the curvature tensor of type (1,3), T is Ricci tensor ,<, > is metric tensor, r is the scalar curvature and Q is the symmetric endomorphism corresponding to the Ricci tensor T defined as

$$T(X,Y) = <QX,Y> , \quad 1.2)$$

satisfies the condition

$$(D_XC)(Y,Z,V)= 2A(X)C(Y,Z,V)+A(Y)C(X,Z,V)+A(Z) C(X,Y,V) + A(V) C (Y,Z,X) + <C(Y,Z,V), X >L, \tag{1.3}$$

where A is non - zero 1- form D_X denote the operator of covariant differentiation with respect to the metric < >, $X, Y, Z, L \in \mathcal{X}(M_n)$ and L is a vector field given by

$$<X,L> = A(X), \qquad \text{for all } X \in \mathcal{X}(M_n). \tag{1.4}$$

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A non- flat smooth Riemannian manifold is called pseudo bi - symmetric by Singh and Sinha (2004) , if curvature tensor R satisfies the relation

$$(D_W D_X R)(Y,Z,V) = 2A(W,X)R(Y,Z,V) + A(W,Y)R(X,Z,V) + A(W,Z) R(Y,X,V) + A(W,V) R(Y,Z,X) + \langle R(Y,Z,V),(W,X)\rangle L , \quad (1.5)$$

where L is vector field given by

$$\langle (W,X) , 2L\rangle = A(W,X) , \qquad \text{for } W, X \in \mathcal{X}(M_n) .$$

This smooth Riemannian manifold is converted in to a special weakly bi - symmetric smooth Riemannian manifold (Singh and Sinha, 2004), if it satisfy the relation

$$(D_W D_X R)(Y,Z,V) = 2\alpha(W,X)R(Y,Z,V) + \alpha(W,Y)R(X,Z,V)+ \alpha(W,Z) R(Y,X,Z) +\alpha(W,V) R(Y,Z,X) , \quad (1.6)$$

where α is non - zero 2- form defined as

$$\alpha(W,X) = \langle (W,X), L\rangle, \quad (1.7)$$

where L is a vector field .

In section 2, we have studied generalized pseudo conformally bi- symmetric manifold $(GPCBS)_n$. In section 3, we have studied Einstein generalized pseudo conformally bi- symmetric manifold . In the last section of this paper, we have studied transformation of $(GPCBS)_n$ and obtained some important results .

2 . Generalized pseudo conformally bi-symmetric smooth Riemannian manifold $G(PCBS)_n$

A non - flat smooth Riemannian manifold is called conformally pseudo bi-symmetric, if C satisfies the relation

$$(D_W D_X C)(Y,Z,V) = 2A(W,X)C(Y,Z,V) + A(W,Y)C(X,Z,V) + A(W,Z) C(Y,X,V) + A(W,V) C(Y,Z,X) + \langle C(Y,Z,V) ,(W,X)\rangle L . \quad (2.1)$$

The objective of this section is to study a type of non- flat smooth Riemannian manifold $(M_n,\langle , \rangle)$, whose conformal curvature tensor C satisfies the condition

$$(D_W D_X C)(Y,Z,V) = 2A(W,X)C(Y,Z,V) + B(W,Y)C(X,Z,V) \quad (2.2)$$
$$+ D(W,Z) C(Y,X,V) +E(W,V) C(Y,Z,X) + \langle C(Y,Z,V), (W,X)\rangle L ,$$

where A, B, D, E are non - zero 2- forms and L is a vector field given by

$$\langle (W,X) , L\rangle = A(W,X) , \quad \forall \; W \text{ and } X\in \mathcal{X}(M_n) . \quad (2.3)$$

Such a manifold will be called a generalized pseudo conformally bi - symmetric smooth Riemannian manifold and will be denoted as $G(PCBS)_n$.

Let

$$\langle (W,X), \lambda\rangle = B(W,X), \quad (2.4)$$

$< (W,Z),\ \mu> \ = \ D(W,Z),$

$< (W,V),\ \nu> \ = \ E(W,V).$

Then $\lambda, \mu, \nu \in \mathcal{X}(M_n)$ will be called the basic vector fields of G(PCBS $)_n$ corresponding to the associated 2-forms A,B,D,E, respectively. If in particular A = B = D = E , then the smooth Riemannian manifold reduce to conformally pseudo bi-symmetric smooth Riemannian manifold .

3. Einstein G $(PCBS)_n$ (n>3)

In this section, we assume that equation (2.2) in holds a G$(PCBS)_n$ to be an Einstein. Then the Ricci tensor satisfies following relation:

$$T(X,Y) = \frac{r}{n} < X, Y >, \tag{3.1}$$

from which it follows that

$$dr(X) = 0 \ \text{ and } \ (D_W D_Z T)(X,Y) = 0. \tag{3.2}$$

From $(D_W D_Z T)(X,Y) = 0$ by contraction, we get

$(D_W D_Z Q)(Y) = 0,$

where Q is defined by equation (1.2).

In consequences of (1.1), (2.2), (3.1), (3.2) and Bianchi identity, we obtained

$$\begin{aligned} &3A(R(Y,Z,V),W) + B(R(Y,Z,V),W) + (R(Y,Z,V),W) \\ &+ [2T(Z,V) - \frac{2r}{(n-1)} < Z, W >] A(W,Y) \\ &+ [\frac{(n-2)}{n(n-1)} < Y, V > 2T(Y,V)]A(W,Z) - \frac{r}{n(n-1)} B(W,Y) < Z, V > \\ &- \frac{r}{n(n-1)} B(W,Z) < Y, V > - \frac{r}{n(n-1)} D(W,Y) < Z, W > \\ &- \frac{r}{n(n-1)} D(W,Z) < Y, W > - \frac{r}{n(n-1)} E(W,V) < Z, Y > = 0. \end{aligned} \tag{3.3}$$

Putting $Y = Z = e_i$ in equation (3.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold , we get

$$r [A(W,X) + n E(W,V)] = 0 . \tag{3.4}$$

Hence if $r = 0$ in equation (3.4), it follows that G$(PCBS)_n$ is G$(PBS)_n$. Thus , we can state the following theorem :

Theorem 3.1

An Einstein G$(PCBS)_n$ is G$(PBS)_n$, if

$(A(W,V) + n E(W,V)) \neq 0$.

Next, we suppose that in an Einstein G$(PCBS)_n$, vector field L defined by equation (2.3) is parallel vector field . Then

$$(D_W D_X L) = 0, \qquad \text{for all } X \in \mathcal{X}(M_n). \tag{3.5}$$

Applying Ricci identity in equation (3.5), we get

$$R(W,X,Y)L = 0 . \tag{3.6}$$

From equation (3.6), it follows that

$'R(X,Y,Z,W)\,L = 0,$ (3.7)

where $'R(X,Y,Z,W)L = \langle 'R(X,Y,Z,W), L\rangle$

By virtue of equation (3.7), we get

$T((X,W),L) = 0$. (3.8)

Now by equations (3.5) and (3.8),

$(D_W D_X T)(Y,L) = D_W D_X T(Y,L) - T(D_W D_x Y,L) - T(Y, D_W D_X L) = 0.$ (3.9)

Further, we have

$$(D_W D_X T)(Z, V) = B(R'(X, Y, Z, V), W) - \frac{r}{n(n-1)}[\langle (Y, Z), V\rangle B(W, X) - \langle (X,Z),V\rangle B(W,Z)] . \quad (3.10)$$

Putting V = L in equation (3.10) and applying equations (3.6) and (3.8), we get

$$\frac{r}{n(n-1)}[A(W, Z)\, B(X, Y) - A(W, X)\, B(Z, Y)] = 0 , \quad (3.11)$$

if $A(W,Z)\, B(X,Y) \neq A(W,X)\, B(Z,Y)$, we get $r = 0$.

Hence, we can state the following theorem **:**

Theorem 3.2

If the vector L is a parallel vector field in an Einstein $G(PCBS)_n$, then $G(PCBS)_n$ reduces to $G(PBS)_n$ provided the vector field L corresponding to the 2-forms, A and B are not co-directional .

4. Conformal transformation of G(PCBS) $_n$

Definition

Let M_n be an Riemannian manifold metric tensor < ,> A transformation ϕ of M is said to be conformal is $\phi \langle ^* \rangle = \sigma^2 \langle , \rangle$, where σ is a positive function on M_n. If σ is a constant function, ϕ is called homothetic transformation (Kobayashi and Nomizu, 1963).

Let $\overset{*}{D}_W \overset{*}{D}_X$ be the operator of bi- covariant differentiation with respect to $\langle ^*,\rangle$, we have

$$\overset{*}{D}_W \overset{*}{D}_X Y - D_w D_x Y = \omega(W,X)\, Y + \omega(W,Y)X - \langle (W,X)Y\rangle U , \quad (4.1)$$

for any vector field $W,X,Y \in \mathcal{X}(M_n)$, where ω is 2-form defined by

$\omega = d \log \sigma$ (4.2)

and U is a vector field defined by

$\langle U, (W,X)\rangle = \omega(W,X)$. (4.3)

By the conformal transformation, it is well - known that

$$\overset{*}{C}(Y,Z,V) = C(Y,Z,V) , \quad (4.4)$$

where the symbol * denote the quantities of $\overset{*}{M}_n$.

Taking bi-covariant differention of equation (4.4) and making use of the relation (4.1) , we get

$$(\overset{*}{D}_W \overset{*}{D}_X C)(Y,Z,V) = (D_w D_x C)(Y,Z,V) - 2\,\omega(W,X)\, C(Y,Z,V) \quad (4.5)$$
$$- [\omega(W,Y)C(X,Z,V) + \omega(W,X)C(Y,Z,V) + \omega(W,V)(Y,Z,X) + \langle C(Y,Z,V)(W,X)\rangle U] + \omega(C(Y,Z,V),(W,X)) + \langle (W,X),Y\rangle\, C(U,Z,V)$$

$+ <(W,X),Z>C(Y,U,V) + <(W,X),V>C(Y,Z,U)$.

Now, we assume that both M_n and $\overset{*}{M}_n$ are $\overset{*}{G}(PCBS)_n$, then

$$(D_W D_X C)(Y,Z,V) = 2A(W,X)C(Y,Z,V) + B(W,Y)C(X,Z)V + D(W,Z)(Y,X,V) + E(W,V)C(Y,Z,X) + <C(Y,Z,V),(W,X)>L \quad (4.6)$$

and

$$(\overset{*}{D}_W \overset{*}{D}_X \overset{*}{C})(Y,Z,V) = 2\overset{*}{A}(W,X)\overset{*}{C}(Y,Z,V) + \overset{*}{B}(W,Y)\overset{*}{C}(X,Z,V) + \overset{*}{D}(W,Z)\overset{*}{C}(Y,X,V) + \overset{*}{E}(W,V)\overset{*}{C}(Y,Z,X) + <\overset{*}{C}(Y,Z,V),(W,X)>\overset{*}{L} , \quad (4.7)$$

for a non-zero 2- forms A, B, D, E and $\overset{*}{A}, \overset{*}{B}, \overset{*}{D}, \overset{*}{E}$ and

$<(\overset{*}{W},X), \overset{*}{L}> = \overset{*}{A}(W,X)$.

Substituting equations (4.6) and (4.7) in (4.5) and using equation (4.4) , we have

$$2\{\overset{*}{A}(W,X) - \overset{*}{A}(W,X)\}C(Y,Z,V) + \{\overset{*}{B}(W,Y) - B(W,Y)\}C(X,Z,W) + \{\overset{*}{D}(W,Z) - D(W,Z)\}C(Y,X)V + \{\overset{*}{E}(W,V) - E(W,V)\}C(Y,Z,X) + <C(Y,Z,V),(W,X)>(\overset{*}{L} - L) \quad - 2\,\omega(W,X)C(Y,Z,V) - \omega(W,Y)C(X,Z,V) \quad \omega(W,Z)C(Y,V,X) + \omega(W,V)C(Y,Z,X) \quad <C(Y,Z,V),(W,X)>U + \omega\{C(Y,Z,V),(W,X)\} \quad <(W,X),Y>C(U,Z,V) + <(W,X),Z>C(Y,U,V) \quad <(W,X),V>C(Y,Z,U). \quad (4.8)$$

Now, we consider two cases **:**

CASE- I

$\overset{*}{A}(W,X) = A(W,X)$, $\overset{*}{B}(W,X) = B(W,X)$, $\overset{*}{D}(W,X) = D(W,X)$,

$\overset{*}{E}(W,X) = E(W,X)$.

In this case, equation (4.8) reduces to

$$2\,\omega(W,X)C(Y,Z,V) + \omega(W,Y)C(X,Z,V) + \omega(W,Z)C(Y,X,V) + \omega(W,V)C(Y,Z,X) + <C(Y,Z,V),(W,X)>U - \omega\{(Y,Z,V),(W,X)\} + <(W,X),Y> C(U,Z,V) - <(W,X),Z>C(Y,Z,V) + <(W,X),V>C(Y,Z,V) = 0 . \quad (4.9)$$

Contracting equation (4.9) over X, we get

$$(n-3)\,\omega(W, C(Y,Z,V)) = 0 . \quad (4.10)$$

Since $n > 3$, equation (4.10) implies that

$$\omega(W, C(Y,Z,V)) = 0 , \quad (4.11)$$

from which it follows that

$$C(U,Y,Z) = 0 \; ; \; C(Y,U,Z) = 0 \; \text{ and } \; C(Y,Z,U) = 0 . \quad (4.12)$$

Applying equations (4.11) and (4.12) in equation (4.9), it follows that

$$2\,\omega(W,X)C(Y,Z,V) + \omega(W,Y)C(X,Z,V) + \omega(W,Z)C(Y,X,V) + \omega(W,V)C(Y,Z,X) + <C(Y,Z,V),(W,X)>U = 0 . \quad (4.13)$$

Putting X = U in equation (4.13) & using equations (4.11) and (4.12) , we get

$$\omega(W,U)C(Y,Z,V) = 0 .$$

Hence , either $C(Y,Z,V) = 0$. (4.14)

or

$\omega(W,V) = 0$. (4.15)

From equations (4.2) and (4.15), we get σ = constant .
Hence , we can state the following theorem:

Theorem 4.1

If a $G(PCBS)_n$ is transformed into a $G(PCBS)_n$ with same associate 2- forms by a conformal transformation, then either the smooth Riemannian manifold is conformally flat or the transformation is homothetic .

Now, if L = constant and smooth Riemannian manifold is conformally flat , then equation (4.5), can be written as

$$(\overset{*}{D}_W D_X C)(Y,Z,V) = (D_W D_X C)(Y,Z,W).$$

Consequently , a $G(PCBS)_n$ may be transformed into $G(PCBS)_n$ by a conformal transformation of equation .

Thus considering from the theorem 4.1 , we have

Theorem 4.2

In order that $G(PCBS)_n$ which is not conformally flat is transformed into another $G(PCBS)_n$ with the same associated 2- forms by a conformal transformation, it is necessary and sufficient that L is constant .

CASE II

In this case we assume that ,

$$\overset{*}{A}(W,X) \neq A(W,X) , \ \overset{*}{B}(W,X) \neq B(W,X) ,.$$

$$\overset{*}{D}(W,X) \neq D(W,X) , \ \overset{*}{E}(W,X) \neq E(W,X).$$

Contracting equation (4.8) over X , we get

$$3[\overset{*}{A}\{C(Y,Z,V),(W,X)\}] - A\{C(Y,Z,V),(W,X)\} = (n-3)\,\omega\ \{C(Y,Z,V),(W,X)\} . \quad (4.16)$$

Let us consider

$$\overset{*}{A}(W,X) - A(W,X) = \omega(W,X) . \quad (4.17)$$

Then from (4.16), we get

$$\omega\{C(Y,Z,V),(W,X)\} = 0 . \quad (4.18)$$

Putting X= U in equation (4.8) and using equation (4.18) , we get

$$[2\{\overset{*}{A}W,U) - A(W,U)\} + 2\,\omega(W,U)]\,C(Y,Z,V) = 0 . \quad (4.19)$$

By equation (4.17), it follows from equation (4.19), either $C(Y,Z,V) = 0$.
or

$$\omega(W,V) = 0 . \quad (4.20)$$

But equation (4.20) is impossible , since in this case $\overset{*}{A}(W,X) \neq A(W,X)$, $\overset{*}{B}(W,X) \neq B(W,X)$, $\overset{*}{D}(W,X) \neq D(W,X)$, $\overset{*}{E}(W,X) \neq E(W,X)$ **.**

Thus , we have the following theorem **:**

Theorem 4.3

If $G(PCBS)_n$ is transformed into another $G(PCBS)_n$ with difference associated 2- forms by a conformal transformation , satisfying the condition (4.17), then the manifold is conformally flat.

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