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On contact metric manifold with $\xi \in N(k)$

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Abstract

The object of the present paper is to study contact metric manifold with $\xi \in N(k)$. An irrotational pseudo projective curvature tensor in a contact metric manifold with $\xi \in N(k)$ is considered. Here it is proved that under this consideration an n-dimensional contact metric manifold with $\xi \in N(k)$ is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or is a space of constant curvature k. Finally, we study conharmonically flat contact metric manifold with $\xi \in N(k)$ satisfying R(X,Y).S=0 and here it is proved that the symmetric endomorphism Q of tangent space corresponding to S has two different non-zero eigen values (n − 1)k and −2(n − 1)k.

Key words- Contact metric manifold with $\xi \in N(k)$, Irrotational pseudo-projective curvature tensor, Conharmonic curvature tensor.

1. Introduction

In Tano (1988), introduced the class of contact metric manifolds M with contact metric structure (ϕ, ξ, η, g), which satisfy the equation

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad X,Y \in TM \qquad (1.1)$$

where R is the curvature tensor and k is a constant. A contact metric manifold belonging to this class is called a contact metric manifold with ξ belonging to the k-nullity condition or simply an N(k)−contact metric manifold Tripathi and Kim (2004).This class contains **Sasakian** manifolds for k = 1 (and h = 0, where 2h is the Lie derivative of ϕ in the direction ξ). In fact, for an N(k)-contact metric manifold, the conditions of being Sasakian manifold, K-contact manifold, k=1 and h=0 are all equivalent.

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A contact metric manifold with $\xi \in N(k)$ or N(k)-contact metric manifold was studied by D.E. Blair (1977), Blair, Koufogiorgos and Sharma (1990), Endo (1996), Papantoniou (1993), Perrone (1992), Tripathi and Kim (2004), Bagewadi, Prakasha and Venkatesha (2007 and 2008)and many others.
In 1997, Blair has studied contact metric manifolds with $R(X,Y)\xi = 0$. An irrotational quasi-conformal curvature on K-contact, Kenmotsu and trans-Sasakian manifolds is studied by Bagewadi and Gatti (2005 and 2003).
In this paper we study pseudo projective and conharmonic curvature tensors of an N(k)-contact metric manifold. In section 2, necessary details about contact metric manifolds, K-contact manifolds, Sasakian manifolds and N(k)-contact metric manifolds are given. In section 3, we study pseudo projectively flat contact metric manifold with $\xi \in N(k)$ and show that it is a space of constant curvature k. In section 4, using a result of D.E. Blair (1997), we prove that an n-dimensional N(k)-contact metric manifold with irrotational pseudo projective curvature tensor is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or is a space of constant curvature k. Further in section 5, we study conharmonically flat contact metric manifold with $\xi \in N(k)$ satisfying R(X, Y).S = 0 and it is proved that the symmetric endomorphism Q of tangent space corresponding to S has two different non-zero eigen values (n − 1)k and −2(n − 1)k. Also the dimensions of the manifold are in A.P., whose first term is 3 and common difference 6.

2. Preliminaries

An n-dimensional differentiable manifold M is called an almost contact manifold if either its structural group can be reduced to U(n) × 1 or equivalently, there is an almost
contact structure (ϕ,ξ,η) consisting of a (1, 1) tensor field ϕ, a vector field ξ, and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\circ\phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold M × R defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi,\ \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t the coordinate of R and λ a smooth function on M × R. The condition of normality is equivalent to vanishing of torsion tensor $[\phi,\phi] + 2d\eta\otimes\xi$, where $[\phi,\phi]$ is Nijenhuis tensor of ϕ. If g is a compatible Riemannian metric with (ϕ,ξ,η), such that

$$g(\phi X,\phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

or equivalently,

$$g(X,\phi Y) = -g(\phi X, Y) \quad \text{and} \quad g(X,\xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in TM$, then, M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g).
An almost contact metric structure becomes a contact metric structure if
$g(X,\phi Y) = d\eta(X,Y), \quad X, Y \in TM$
A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \quad X,Y \in TM$,

where ∇ is Levi-Civita connection; while a contact metric manifold M^n is Sasakian if and only if the curvature tensor R satisfies

$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$

holds for all vector fields X, Y on M^n. A contact metric manifold is called a K−contact manifold if the structure vector field ξ is a Killing vector field. An almost contact metric manifold is K−contact if and only if $\nabla\xi = -\phi$. A K−contact manifold is a contact metric manifold, while converse is true if h=0, where 2h is the Lie derivative of ϕ in the characteristic direction ξ. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold.

The nullity distribution Tano (1988) of a Riemannian manifold (M^n, g) for a real number k, is a distribution Tano (1988)

$$N(k): p \longrightarrow N_p(k) = [Z \in T_pM : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}] \tag{2.4}$$

where k is a constant. In a contact metric manifold M, if $\xi \in N(k)$ then M is an N(k)−contact metric manifold (2007). Consequently, a contact metric manifold with $\xi \in TM$ is a Sasakian manifold if and only if k = 1.

Next, suppose that $M^n(\phi,\xi,\eta,g)$ is a contact metric manifold with $\xi \in N(k)$, we have the following relations Baikoussis and Koufiorgos, (1993) and Blair (1976):

$$R(\xi,X)Y=k\{g(X,Y)\xi-\eta(Y)X\}, \tag{2.5}$$

$$\eta(R(X,Y)Z)=k\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \tag{2.6}$$

$$S(X,\xi) = (n-1)k\eta(X), \tag{2.7}$$

$$Q\xi = (n-1)k\xi \tag{2.8}$$

where S the Ricci tensor of type (0, 2) with Q is the Ricci operator i.e.,

$$g(QX, Y) = S(X, Y) \tag{2.9}$$

The pseudo projective curvature tensor $\overline{P}$ Prasad(2002) and the conharmonic curvature tensor C on a manifold M^n (n > 3) are defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y] \tag{2.10}$$

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y - g(Y,Z)QX - g(X,Z)QY] \tag{2.11}$$

where a and b are constants such that a, $b \neq 0$ and r is the scalar curvature of the manifold.

Lemma 1.Blair (1977): Let $M^n(\phi,\xi,\eta,g)$ be a contact metric manifold with $R(X,Y)\xi = 0$ for all vector fields X, Y. Then M^n is locally the Riemannian product of a flat (n+1)-dimensional manifold and an n-dimensional manifold of positive curvature 4, that is, $E^{n+1}(0) \times S^n(4)$.

The above result will be used in the next sections.

3. Pseudo projectively flat contact metric manifold with ξ∈N(k):

Let us consider a contact metric manifold M^n (n > 1) with $\xi \in N(k)$. Since the manifold is pseudo-projectively flat, we have $\tilde{P}(X,Y)Z$ for all X, Y and Z. Hence (1.9) yields

$$aR(X,Y)Z = b[S(X,Z)Y - S(Y,Z)X] \tag{3.1}$$

$$-\frac{r}{n}\left[\frac{a}{n-1}+b\right][g(X,Z)Y-g(Y,Z)X].$$

Taking the inner product on both sides of (3.1)by ξ and then using (2.6) and (2.7) we obtain

$$\left[ak-\frac{r}{n(n-1)}[a+(n-1)b]\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] +b[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]=0. \tag{3.2}$$

Substituting X by ξ in (3.2) we get by virtue of (2.7) that

$$-bS(Y,Z)=\left[ak-\frac{r}{n(n-1)}[a+(n-1)b]\right]g(Y,Z) -\left[k-\frac{r}{n(n-1)}\right][a+(n-1)b]\eta(Y)\eta(Z). \tag{3.3}$$

Let $\{e_i\}$, i = 1, 2, ..., n be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (3.3) and taking summation for $1 \le i \le n$ we get

$$r = n(n-1) \quad \text{if} \quad a+(n-1)b \neq 0. \tag{3.4}$$

In view of (3.3) and (3.4) we obtain (since $b \neq 0$)

$$S(Y, Z) = (n - 1)kg(Y, Z). \tag{3.5}$$

Using (3.4) and (3.5) in (3.1) we get (since $a \neq 0$)

R(X, Y)Z = k[g(Y, Z)X − g(X, Z)Y] for all X Y Z.

This leads to the following:

Theorem 2- A pseudo projectively flat contact metric manifold M^n (n > 1) with ξ belonging to the k-nullity distribution is a space of constant curvature k provided that $a+(n-1)b \neq 0$.

4. Irrotational pseudo projective curvature tensor in a contact metric manifold with $\xi \in N(k)$:

The rotation of pseudo projective curvature tensor $\text{curl}\,\tilde{P}=0$ on a Riemannian manifold is given by

$$\text{Rot}\,\tilde{P}=\left(\nabla_U\tilde{P}\right)(X,Y,Z)+\left(\nabla_X\tilde{P}\right)(U,Y,Z)+\left(\nabla_Y\tilde{P}\right)(U,X,Z)-\left(\nabla_Z\tilde{P}\right)(X,Y,U). \tag{4.1}$$

By virtue of second Bianchi identity

$$\left(\nabla_U\tilde{P}\right)(X,Y,Z)+\left(\nabla_X\tilde{P}\right)(U,Y,Z)+\left(\nabla_Y\tilde{P}\right)(U,X,Z)=0.$$

So (4.1) reduces to

$$\text{curl}\,\tilde{P}=-(\nabla_Z\tilde{P})(X,Y,U) \tag{4.2}$$

If the pseudo-projective curvature tensor is irrotational then $\text{curl}\,\tilde{P}=0$ and by (4.2) we have

$$(\nabla_Z\tilde{P})(X,Y,U)=0 \tag{4.3}$$

Thus the manifold is pseudo-projectively symmetric, which implies that

$$R(X,Y)\tilde{P}=0 \tag{4.4}$$

From (4.4), it follows that

$$g(R(X,Y)\tilde{P}(U,V)W,\xi)-g(\tilde{P}(R(X,Y)U,V)W,\xi)$$
$$-g(\tilde{P}(U,R(X,Y)V)W,\xi)-g(\tilde{P}(U,V)R(X,Y)W,\xi)=0 \quad (4.5)$$

From (2.10), it can be easily seen that

$$\eta(\tilde{P}(X,Y)\xi)=0 \quad \text{for all } X, Y. \quad (4.6)$$

By taking $Y=\xi$ in (4.5) we obtain by virtue of (2.5) and (4.6) that

$$k[\tilde{P}(U,V,W,X)-\eta(X)\eta(\tilde{P}(U,V)W)-g(X,U)\eta(\tilde{P}(\xi,V)W)$$
$$+\eta(U)\eta(\tilde{P}(X,V)W)-g(X,V)\eta(\tilde{P}(U,\xi)Y)$$
$$+\eta(V)\eta(\tilde{P}(U,X)W)+\eta(Y)\eta(\tilde{P}(U,V)X)]=0$$

where $\tilde{P}(U,V,W,X)=g(\tilde{P}(U,V)W,X)$.

The above relation implies that either k = 0, or

$$\tilde{P}(U,V,W,X)-\eta(X)\eta(\tilde{P}(U,V)W)-g(X,U)\eta(\tilde{P}(\xi,V)W)$$
$$+\eta(U)\eta(\tilde{P}(X,V)W)-g(X,V)\eta(\tilde{P}(U,\xi)Y)$$
$$+\eta(V)\eta(\tilde{P}(U,X)W)+\eta(W)\eta(\tilde{P}(U,V)X)=0 \quad (4.7)$$

If k = 0, then (1.1) yields $R(X,Y)\xi=0$ for all X, Y.

Now putting U = X = e_i in (4.7) and taking summation over $1\le i\le n$ we get

$$\sum_{i=1}^{n}\tilde{P}(e_i,V,W,e_i)-(n-1)\eta(\tilde{P}(\xi,V)W)+\sum_{i=1}^{n}\eta(\tilde{P}(e_i,V)e_i)\eta(W)=0 \quad (4.8)$$

From (2.10), it follows that

$$\sum_{i=1}^{n}\tilde{P}(e_i,V,W,e_i)=[a+(n-1)b]\left(S(V,W)-\frac{r}{n}g(V,W)\right) \quad (4.9)$$

$$\eta(\tilde{P}(\xi,V)W)=bS(V,W)+\left(ak-\frac{r}{n(n-1)[a+(n-1)b]}\right)g(V,W)$$
$$+\left[\frac{r}{n(n-1)}-k\right][a+n(n-1)b]\eta(V)\eta(W) \quad (4.10)$$

$$\sum_{i=1}^{n}\eta(\tilde{P}(e_i,V)e_i)\eta(W)=(a-b)\left(\frac{r}{n}-(n-1)k\right)\eta(V)\eta(W). \quad (4.11)$$

Using (4.9)-(4.11) in (4.8) we obtain

$$aS(V,W)-a(n-1)kg(V,W)-b[r-n(n-1)k]\eta(V)\eta(W)=0. \quad (4.12)$$

Putting V = W = e_i in (4.12) and taking summation for $1\le i\le n$, we get

$$r=n(n-1) \quad \text{if} \quad a-b\neq 0. \quad (4.13)$$

By virtue of (4.13) we obtain from (4.12) that

$$S(V, W) = (n-1)\, kg(V, W). \quad (4.14)$$

In view of (4.13), (4.14), (2.6) and (2.7), it can be easily seen from (2.10) that

$$\eta(\tilde{P}(X,Y)Z)=0 \quad \text{if for all } X, Y, Z. \quad (4.15)$$

The relation (4.7) implies that $\tilde{P}(U,V,W,X)=0$, for all U, V, W, X, i.e., $\tilde{P}(U,V)W=0$ for all U, V, Y, which means that the manifolds is pseudo-projectively flat. Hence by virtue of Lemma 1 and Theorem 2, we state the following:

Theorem 3. If in a contact metric manifold M^n (n > 2) with ξ belonging to the k-nullity distribution, the pseudo projective curvature tensor is irrotational, then the manifold M^n is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or is a space of constant curvature k, provided that $a-b\neq 0$.

5. Conharmonically flat contact metric manifold with $\xi \in N(k)$ satisfying R(X, Y)S = 0:

First we consider that the manifold is conharmonically flat. From(2.11) it follows that

$$R(X,Y)Z=\frac{1}{n-2}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]. \quad (5.1)$$

Next we consider in a contact metric manifold with ξ belonging to k-nullity distribution

$$R(X,Y).S=0. \quad (5.2)$$

Also (5.2) gives

$$S(R(X,Y)Z,W)+S(Z,R(X,Y)W)=0. \quad (5.3)$$

Using (5.1) in (5.3) and taking Y = Z we have,

$$g(Z,Z)S(QX,W)-g(X,Z)S(QZ,W)+g(Z,W)S(QX,Z)-g(X,W)S(QZ,Z)=0. \quad (5.4)$$

Taking $Z=\xi$ and using (2.1) and (2.3) we have

$$S(QX,W)-S(Q\xi,W)\eta(X)+\eta(W)S(QX,\xi)-g(X,W)S(Q\xi,\xi)=0. \quad (5.5)$$

Now by (2.7) and (2.8) we have

$$S(QX,W)-k^2(n-1)^2\eta(X)\eta(W)+\eta(W)S(QX,\xi)-k^2(n-1)^2g(X,W)=0. \quad (5.6)$$

Let λ be the eigen value of the endomorphism Q corresponding to an eigenvector X. Then

$$QX=\lambda X \quad (5.7)$$

Using (5.7) in (5.6) and then using (2.7) we have

$$\lambda S(X,W)-k^2(n-1)^2\eta(X)\eta(W)+k(n-1)\lambda\eta(X)\eta(W)-k^2(n-1)^2g(X,W)=0.$$

Again using (2.3) we obtain

$$\lambda^2g(X,W)-k^2(n-1)^2\eta(X)\eta(W)+k(n-1)\lambda\eta(X)\eta(W)-k^2(n-1)^2g(X,W)=0 \quad (5.8)$$

Putting $W=\xi$ in (5.8) and using (2.1) and (2.3) we get

$$[\lambda^2+k(n-1)\lambda-2k^2(n-1)^2]\eta(X)=0.$$

As $\eta(X)\neq 0$, we have

$$\lambda^2+k(n-1)\lambda-2k^2(n-1)^2=0. \quad (5.9)$$

From, (5.9), we get two non-null solutions

$$\lambda_1=(n-1)k, \quad \lambda_2=-2(n-1)k, \quad (5.10)$$

and

$$\lambda_1+\lambda_2=-(n-1)k. \quad (5.11)$$

Again, from (5.1) we have

$$(n-2)g(R(X,Y)Z,W)=g(Y,Z)g(QX,W)-g(X,Z)g(QY,W) \quad (5.12)$$

+S(Y, Z) g(X, W) − S(X, Z)g(Y, W).
Putting X = W, (5.12) reduces to
(n − 2)g(R(W, Y)Z, W) = g(Y, Z)g(QW, W) − g(W, Z)g(QY, W)
+S(Y, Z)g(W, W) − S(W, Z)g(Y, W).
The sum for $1 \le i \le n$ of the above expression for W = e_i yields

$$rg(Y, Z) = 0. \tag{5.13}$$

where r is the scalar curvature of the manifold and $\{e_i\}$ is an orthonormal basis of the tangent space of M. So

$$r = 0. \tag{5.14}$$

Since the scalar curvature is trace Q, we have

$$r = m\lambda_1 + (n - m)\lambda_2 , \tag{5.15}$$

where m is a positive integer which is the multiplicity of λ_1 and (n-m) is the multiplicity of λ_2 . By (5.11), (5.14) and (5.15) we get

$$n = \frac{3m}{2} . \tag{5.16}$$

Now if m is odd,

$$n = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}, ...$$

Here the dimension of manifold becomes a fraction, which cannot be so.
When m is even
n = 3, 6, 9, 12, ...
But, as the manifold is odd dimensional, so dimension of these type of contact metric manifold with $\xi \in N(k)$ will be 3,9,15,... etc. which is in airthematic progression (A.P) with first term 3 and common difference 6. So we state,

Theorem 4. In a contact metric manifold M^n (n > 2) with ξ belonging to the k-nullity distribution, which is conharmonically flat together with R(X, Y).S = 0, the symmetric endomorphism Q of tangent space corresponding to S has two different non-zero eigen values (n−1)k and −2(n−1)k. Also, the dimensions of these manifolds are in A.P. having first term 3 and common difference 6.

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