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Certain modular equations from second note book of Ramanujan

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Abstract

An attempt has been made to give simple proofs of certain modular equations of chapter XVIII of Ramanujan's second notebook.

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Introduction, Notations and Definitions : Here and in the sequel we employ the customary q-product notation. For $| q | < |$, let $[a ;q]_o = 1$,

$$[a ;q]_n = \prod_{r=0}^{n-1} (1-aq^r), \text{ for } n \geq,1 \text{ and,}$$

$$[a ;q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r), r \tag{1.1}$$

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ | ab | < | \tag{1.2}$$

Applying Jacobi Triple product identity we have,

$$f(a,b) = [-a ; ab]_\infty [-b ; ab]_\infty [ab ; ab]_\infty \tag{1.3}$$

Most important special cases of (1.2) are

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q ;-q]_\infty}{[q ;-q]_\infty} = \frac{[-q ;q^2]_\infty [q^2 ;q^2]_\infty}{[-q^2 ;q^2]_\infty [q ;q^2]_\infty} \tag{1.4}$$

$$\Psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2;q^2]_\infty}{[q ;q^2]_\infty} \tag{1.5}$$

and $$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n-1)/2} = [q ;q]_\infty . \tag{1.6}$$

In chapter 16, entry 25, Ramanujan established following beautiful relations:

(i) $$\phi^2(x) + \phi^2(-x) = 2\phi^2(x^2), \tag{1.12}$$

(ii) $$\phi^4(x) - \phi^4(-x) = 16x\Psi^4(x^4). \tag{1.13}$$

(iii)In entry 10 of chapter 17, Ramanujan has mentioned following $\phi(x) + \phi(-x) = 2\phi(x^4),$ (1.7)

(iv) $$\phi(x) - \phi(-x) = 4x(x^8), \tag{1.8}$$

(v) $\phi(x)\phi(-x) = \phi^2(-x^2)$ and $\Psi(x)\Psi(-x^2) = \Psi(x^2)\phi(-x^2),$ (1.9)

(vi) $$\phi(x)\Psi(x^2) = \Psi^2(x), \tag{1.10}$$

(vii) $$\phi^2(x) - \phi^2(-x) = 8x\Psi^2(x^4), \tag{1.11}$$

results which hold for modular equations of signature 2,

(i) $$\phi(q) = \sqrt{z} \tag{1.14}$$

(ii) $$\phi(-q) = \sqrt{z}\ 4\sqrt{(1-x)} \tag{1.15}$$

where,

$$q = e^{-y} = e^{-\pi}\ \frac{{}_2F_1[1/2, 1/2\ ; 1\ ; 1-x]}{{}_2F_1[1/2, 1/2\ ; 1\ ; x]} \tag{1.16}$$

and $\quad z = {}_2F_1[1/2\ ; 1/2\ ; 1\ ; x]$

$$m = \frac{{}_2F_1[1/2, 1/2, 1, \alpha]}{{}_2F_1[1/2, 1/2, 1, \beta]} = \frac{Z_1}{Z_n}. \tag{1.17}$$

In entry 24 of chapter 18, Ramanujan has given following modular equations of signature 2,

(a)If β is of degree 2 over α, then

(i) $$m\sqrt{1-\alpha} + \sqrt{\beta} = 1, \tag{1.18}$$

(ii) $$m^2\sqrt{1-\alpha} + \beta = 1, \tag{1.19}$$

(iii) $$\frac{m^2}{2} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = \frac{1+\beta}{1+(1-\alpha)}. \tag{1.20}$$

(b)If β is of degree 4 over α, then

(i) $$\sqrt{m}\ \sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1, \tag{1.21}$$

(ii) $$m\sqrt[4]{1-\alpha}+\sqrt{\beta}=1, \tag{1.22}$$

(iii) $$\frac{m}{2}=\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}=\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}. \tag{1.23}$$

(c)If β is of degree 8 over α, then

$$\sqrt{m}\sqrt[8]{1-\alpha}+\sqrt[4]{\beta}=1 \tag{1.24}$$

(A) If β is of degree 16 over α, then

$$\frac{\sqrt{m}}{2}=\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}. \tag{1.25}$$

In this paper, making use of (1.7)-(1.15) an attempt has been made to give simple proofs of (1.18)-(1.25). In the entry 24(V) of chapter 18, Ramanujan has mentioned:

In any equation α may be changed to 1-β, β to 1-α and m to n/m where n is the degree of β." Thus we see that

2nd degree: $\frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha}=1$ and $\left(1-\sqrt{1-\alpha}\right)\left(1-\sqrt{\beta}\right)=\sqrt[2]{\beta(1-\alpha)}$. (1.26)

4th degree: $\frac{2}{\sqrt{m}}\sqrt[4]{\beta}+\sqrt[4]{1-\alpha}=1$ and $\left(1-\sqrt[4]{1-\alpha}\right)\left(1-\sqrt[4]{\beta}\right)=2\sqrt[4]{\beta(1-\alpha)}$. (1.27)

8th degree : $\frac{2\sqrt{2}}{\sqrt{m}}\sqrt[8]{\beta}+\sqrt[4]{1-\alpha}=1$ and $\left(1-\sqrt[4]{1-\alpha}\right)\left(1-\sqrt[4]{\beta}\right)=2\sqrt{2}\sqrt[8]{\beta(1-\alpha)}$.

(1.28)

We shall also deduce (1.26) - (1.28) from (1.18)- (1.25).

Proofs of (1.18) – (1.25): (a) (i) Proof of (1.18)- Since β is of degree 2 over α so from (1.14) and (1.15) we have,

$$\phi(q)=\sqrt{z_1},\phi(-q)=\sqrt{z_1}(1-\alpha)^{1/4}\, and\, \phi(q^2)=\sqrt{z_2},\phi(-q^2)=\sqrt{z_2}\,(1-\beta)^{1/4}. \tag{2.1}$$

From (2.1) we find

$$\frac{\phi(-q)}{\phi(q^2)}=\sqrt{m}\,(1-\alpha)^{1/4} \text{ and } \frac{\phi(-q^2)}{\phi(q^2)}=(1-\beta)^{1/4}. \tag{2.2}$$

$$m\sqrt{1-\alpha}+\sqrt{\beta} = \frac{\phi^2(-q)+\sqrt{\phi^4(q^2)-\phi^4(-q^2)}}{\phi^2(q^2)}$$

So,

$$= \frac{\phi^2(-q)+\phi^2(-q)+8q\Psi^2(q^4)}{2\phi^2(q^2)} \quad [\text{from } (1.13)]$$

$$= \frac{\phi^2(-q)+\phi^2(q)}{2\phi^2(q^2)} \quad [\text{from } (1.11)]$$

$$= 1 \quad [\text{from } (1.12)]$$

(ii)Proof of (1.19) - From (2.1) we have $\frac{\phi(-q)}{\phi(q^2)}\frac{\phi(q)}{\phi(q^2)} = m(1-\alpha)^{1/4}$

$$m^2\sqrt{1-\alpha}+\beta = \frac{\phi^2(-q)}{\phi^4(q^2)}\phi^2(q)+\left\{1-\frac{\phi^4(-q^2)}{\phi^4(q^2)}\right\}$$

Thus we have

$$= \frac{\phi^4(-q^2)+\phi^4(q^2)-\phi^4(-q^2)}{\phi^4(q^2)} = 1 \quad [\text{from } (1.9)]$$

Replacing α by 1- β, β by 1-α and m by 2/m in (1.18) we get: $\frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha}=1$ (2.3)

From (1.18) and (2.3) we have

$$\left.\begin{aligned} m\sqrt{1-\alpha}+\sqrt{\beta} &= 1 \\ \frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} m\sqrt{1-\alpha} &= 1-\sqrt{\beta} \\ \frac{2}{m}\sqrt{\beta} &= 1-\sqrt{1-\alpha} \end{aligned}$$

Multiplying these relation we get $(1-\sqrt{\beta})(1-\sqrt{1-\alpha}) = 2\sqrt{\beta(1-\alpha)}$ (2.4)

(2.3) and (2.6) are same as given in (1.26).

(iii)Proof of (1.20) - From (1.18) and (1.19) we may deduce two more modular equations respectively.

$$\frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha} = 1 \quad (2.5)$$

$$\frac{4}{m^2}\sqrt{\beta}+(1-\alpha) = 1 \quad (2.6)$$

From (1.18) and (2.5) we have $\frac{m^2}{2}\sqrt{\frac{1-\alpha}{\beta}} = \frac{1-\sqrt{\beta}}{1-\sqrt{1-\alpha}}$ (2.7)

Also from (1.19) and (2.6) we have $\frac{m^4}{4}\sqrt{\frac{1-\alpha}{\beta}} = \frac{1-\beta}{1-(1-\alpha)}$ (2.8)

Dividing (2.8) by (2.7) we find $\frac{m^2}{2} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}$. (2.9)

Again, (1.18) and (2.5) can be written as:

$$\left.\begin{array}{l} m\sqrt{1-\alpha} + \sqrt{\beta} = 1 \\ 2\sqrt{\beta} + m\sqrt{1-\alpha} = m \end{array}\right\} \Rightarrow \quad 1+\sqrt{\beta} = m. \qquad (2.10)$$

From (2.6) we get $4\sqrt{\beta} + m^2(1-\alpha) = m^2$

$$= (1+\sqrt{\beta})^2 \quad = 1+\beta+2\sqrt{\beta} \qquad (2.10\text{ a})$$

or $m^2(1-\alpha) + 2\sqrt{\beta} = 1+\beta.$ (2.11)

From (2.10 a) we get $2\sqrt{\beta} = \frac{m^2}{2} - \frac{m^2}{2}(1-\alpha)$ (2.12)

From (2.11) and (2.12) we find $\frac{m^2}{2}\{1+(1-\alpha)\} = 1+\beta$ or $\frac{m^2}{2} = \frac{1+\beta}{1+(1-\alpha)}$. (2.13)

From (2.9) and (2.13) we get (1.20).

(b) (i) Proof of (1.21) --:

since β is of degree four over α, so from (1.14) and (1.15)

$$\phi(q) = \sqrt{z_1}, \phi(-q_1) = z_1(1-\alpha)^{1/4}, \phi(q^4) = \sqrt{z_4}, \phi(-q^4) = \sqrt{z_4}(1-\beta)^{1/4}. \qquad (2.14)$$

From (2.14) we have,

$$\left.\begin{array}{ll} \frac{\phi(q)}{\phi(q^4)} = \sqrt{m}, & \frac{\phi(-q)}{\phi(-q^4)} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}, \\ \frac{\phi(-q)}{\phi(q)} = (1-\alpha)^{1/4}, & \frac{\phi(-q)}{\phi(q^4)} = \sqrt{m}(1-\alpha)^{1/4}, \\ \frac{\phi(-q^4)}{\phi(q^4)} = (1-\beta)^{1/4}. & \end{array}\right\} \qquad (2.15)$$

Now, making use of relations of (2.15) we get:

$$\sqrt{m}\;\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = \frac{\phi(-q)}{\phi(q^4)} + \left\{1 - \frac{\phi^4(-q^4)}{\phi^4(q^4)}\right\}^{1/4}$$ which is (1.21)

(ii)Proof of (1.22): Making use of (2.15) we have: $m\sqrt[4]{(1-\alpha)}+\sqrt{\beta}$

$$= \frac{\phi(q)}{\phi(q^4)}\frac{\phi(-q)}{\phi(q^4)} + \left\{1 - \frac{\phi^4(-q^4)}{\phi^4(q^4)}\right\}^{1/2} = \frac{\phi^2(-q^2)}{\phi^2(q^4)} + \frac{\sqrt{\phi^4(q^4)-\phi^4(-q^4)}}{\phi^2(q^4)} \quad \text{(from 1.9)}$$

$$= \frac{\phi^2(-q^2)+4q^2\psi^2(q^8)}{\phi^2(q^4)} \quad \text{(from 1.13)}$$

$$= \frac{2\phi^2(-q^2)+\phi^2(q^2)-\phi^2(-q^2)}{2\phi^2(q^4)} \quad \text{(from 1.11)}$$

$$= \frac{\phi^2(q^2)+\phi^2(-q^2)}{2\phi^2(q^4)} = 1, \quad \text{(from 1.12)}$$

which is (1.22).

(iii)Proof of (1.23) - From (1.21) and (1.23) we have two more modular equations by replacing α by 1-β, β by 1-α and m by n/m :

$$\frac{2}{\sqrt{m}}\sqrt[4]{\beta}+\sqrt[4]{1-\alpha}=1, \tag{2.16}$$

$$\frac{4}{m}\sqrt[4]{\beta}+\sqrt{1-\alpha}=1. \tag{2.17}$$

Now, from (1.21) and (2.16) we have, $\frac{m}{2}\sqrt[4]{\frac{1-\alpha}{\beta}} = \frac{1-\sqrt[4]{\beta}}{1-\sqrt[4]{1-\alpha}}.$ (2.18)

Also, from (1.22) and (2.7) we have, $\frac{m^2}{4}\sqrt[4]{\frac{1-\alpha}{\beta}} = \frac{1-\sqrt{\beta}}{1-\sqrt{1-\alpha}}.$ (2.19)

Divide (2.19) by (2.18) we get : $\frac{m}{2} = \frac{1-\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$ (2.20)

Again, we can write (1.21) and (2.16) as :

$$\left.\begin{aligned}\sqrt{m}\sqrt[4]{1-\alpha}+\sqrt[4]{\beta}&=1\\ \sqrt[4]{\beta}+\sqrt{m}\sqrt[4]{1-\alpha}&=\sqrt{m}\end{aligned}\right\}\Rightarrow\quad 1+\sqrt[4]{\beta}=m\quad. \tag{2.21}$$

Also, from (1.22) and (2.17) we have $2\sqrt[4]{\beta}+m\sqrt{1-\alpha}=1+\sqrt{\beta},$ (2.22)

(2.22) can be written as $\frac{m}{2}=\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}.$ (2.23)

Thus from (2.20) and (2.23) we get (1.23).

Also, from (1.21) and (2.16) we have

$$2\sqrt[4]{\beta(1-\alpha)} = (1-\sqrt[4]{\beta})\,(1-\sqrt[4]{1-\alpha}),\qquad \text{which is precisely (1.27).}$$

(c) Proof of (1.24) -: Since β is of degree 8 over α, so from (1.14) and (1.15) we have :

$$\phi(q) = \sqrt{z_1}, \phi(-q) = \sqrt{z_1}(1-\alpha)^{1/4}, \phi(q^8) = \sqrt{z_8}, \phi(-q^8) = \sqrt{z_8}(1-\beta)^{1/4} \quad . \tag{2.25}$$

From (2.25) we have $\dfrac{\phi(-q^8)}{\phi(q^8)} = (1-\beta)^{1/4}$.

Therefore

$$\begin{aligned}
&\sqrt{m}(1-\alpha)^{1/8} + (\beta)^{1/4} \\
&= \frac{\sqrt{\phi(q)\phi(-q)}}{\phi(q^8)} + \frac{\sqrt[4]{\phi^4(q^8)-\phi^4(-q^8)}}{\phi(q^8)} \\
&= \frac{\sqrt{\phi(-q^2)+2q^2\psi(q^{16})}}{\phi(q^8)} && \textit{from } (1.9) \textit{ and } (1.13) \\
&= \frac{\phi(-q^2)+\phi(-q^2)+4q^2\psi(q^{16})}{2\phi(q^8)} \\
&= \frac{\phi(-q^2)+\phi(q^2)}{2\phi(q^8)} && \textit{from } (1.8) \\
&= 1, && \textit{from } (1.12)
\end{aligned}$$

which is (1.24).

From (1.24) we have another modular equation

$$2\sqrt{2/m}\ \sqrt[8]{\beta} + \sqrt[4]{1-\alpha} = 1$$

So,
$$2\sqrt{2}\ \sqrt[8]{\beta(1-\alpha)} = (1-\sqrt[4]{\beta})\ (1-\sqrt[4]{1-\alpha}), \tag{2.26}$$

which is (1.28) .

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