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Riemannian manifolds admitting a new type of semi-symmetric non-metric connection

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Abstract

In this paper, we define a semi-symmetric non-metric connection $\overline{D}$ on a Riemannian manifold M and proved its existence. In particular cases, this connection reduces to several symmetric, semi-symmetric and semi-symmetric non-metric connections; even some of them are not introduced so for. We also find the expression for curvature tensor, Ricci tensor and scalar curvature of this new connection $\overline{D}$. Further, it is also proved that such connection on a Riemannian manifold is conformally and projectively invariant under certain conditions. Also we study a hypersurface of a Riemannian manifold admitting such a semi-symmetric non-metric connection. To validate our findings, we construct a non-trivial example of 3–dimensional Riemannian manifold equipped with a semi-symmetric non-metric connection.

Keywords and phrases- Riemannian manifold, semi-symmetric non-metric connection, curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and conformal curvature tensor.

1.Introduction

Let M^n be an n-dimensional Riemannian manifold and D denote the Levi-Civita connection corresponding to the Riemannian metric gon M^n. A linear connection $\overline{D}$ defined on M^n is said to be symmetric if the torsion tensor $\overline{T}$ of $\overline{D}$ defined by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y], \tag{1.1}$$

is zero for all X and Y on M^n; otherwise it is non-symmetric.

It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In 1924, Friedmann and Schouten considered a differentiable manifold and introduced the idea of a semi-symmetric linear connection on it. Moreover in 1932, Hayden gave the idea of a metric connection $\overline{D}$ on a Riemannian manifold and later named such connection a Hyden connection. After a long gap, Pak (1963) considered the Hyden connection $\overline{D}$ equipped with the torsion tensor $\overline{T}$ with non-zero and proved that it is a semi-symmetric metric connection. A linear connection $\overline{D}$ is said to be metric on M^n if $\overline{D}_X g = 0$; otherwise it is non-metric. A systematic study of semi-symmetric metric connection $\overline{D}$ on a Riemannian manifold was initiated by Yano (1970). He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection $\overline{D}$ where curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection $\overline{D}$ for which the manifold is a group manifold Eisenhart (1946). Various properties of Yano’s connection since then have been studied by many authors. In 1992, Agashe and Chafle introduced a new class of the semi-symmetric connection on a Riemannian manifold and studied some of its geometrical properties. This was further developed by Agashe and Chafle (1994), Prasad (1994), De and Kamily (1995), Ojha and Chaubey (2012) and by several geometers. In the earlier paper De and Sengupta (2000), Sengupta, De and Binh (2000) and Prasad and Verma (2004) defined a new type of semi-symmetric non-metric connection on a Riemannian manifold which generalizes the notion of semi-symmetric metric connection Yano (1970) and semi-symmetric non-metric connection Agashe and Chafle (1992). In 2008, Prasad, Verma and De and Tripathi (2008) defined the most general form of semi-symmetric metric and semi-symmetric non-metric connection and called by them as generalized semi-symmetric connection on a Riemannian manifold. In continuation of this study Prasad, Dubey and Yadav (2011), Prasad and Doulo (2015) and Prasad, Kumar and Singh (2021) defined again semi-symmetric non-metric connection on a Riemannian manifold and obtained many geometrical properties.

Motivation of the above studies in this paper authors defined another type of semi-symmetric non-metric connection $\overline{D}$ which includes the known semi-symmetric, semi-symmetric metric and semi-symmetric non-metric connections. At first we prove the existence of such a connection and then we obtained curvature properties and Ricci properties with respect to $\overline{D}$. We also defined projective curvature tensor and conformal curvature tensor with respect to this connection $\overline{D}$ and obtain relation connecting it with the projective curvature tensor and conformal curvature to the Levi-Civita connection D. Also we study a hypersurface of a Riemannian manifold admitting such a semi-symmetric non-metric connection $\overline{D}$ and in the last section; we construct a non-trivial example of 3-dimensional Riemannian manifold endowed with a semi-symmetric non-metric connection $\overline{D}$.

2. Semi-symmetric non-metric connection

Let (M^n, g) be a Riemannian manifold with Levi-Civita connection D. We define a linear connection $\overline{D}$ on M^n by the expression

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X + \eta_2(X)Y + \eta_3(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4 \tag{2.1}$$

where$\eta_1, \eta_2, \eta_3$and η_4 are 1-forms associate with the vector fields ξ_1, ξ_2, ξ_3 and ξ_4 on M^n by

$$\eta_1(X) = g(X, \xi_1), \tag{2.2}$$

$$\eta_2(X) = g(X, \xi_2), \tag{2.3}$$

$$\eta_3(X) = g(X, \xi_3), \tag{2.4}$$

and

$$\eta_4(X) = g(X, \xi_4). \tag{2.5}$$

Using (2.1) and (1.1), the torsion tensor T of M^n with respect to the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = [\eta_1(Y)X - \eta_1(X)Y] + [\eta_2(X)Y - \eta_2(Y)X] + [\eta_3(Y)X - \eta_3(X)Y]. \tag{2.6}$$

A linear connection satisfying (2.6) is called semi-symmetric connection.

Further, using (2.1), we have

$$\begin{aligned}(\overline{D}_X g)(Y,Z) &= Xg(Y,Z) - g(\overline{D}_X Y, Z) - g(Y, \overline{D}_X Z)\\ &= -2\eta_2(X)g(Y,Z) - [\eta_3(Y)g(X,Z) + \eta_3(Z)g(X,Y)] - \\ &\quad [\eta_4(Z)g(X,Y) + \eta_4(Y)g(X,Z)].\end{aligned} \tag{2.7}$$

A linear connection $\overline{D}$ defined by (2.1) satisfies (2.6) and (2.7) and hence we call $\overline{D}$, a semi-symmetric non-metric connection.

Conversely, we show that a linear connection $\overline{D}$ defined on M^n satisfying (2.6) and (2.7) is given by (2.1).

Let H be a tensor field of the type (1, 2) and

$$\overline{D}_X Y = D_X Y + H(X,Y), \tag{2.8}$$

whereD is the Levi-Civita connection on (M^n, g).

Then, we have

$$\overline{T}(X,Y) = H(X,Y) - H(Y,X). \tag{2.9}$$

Also, we have

$$(\overline{D}_X g)(Y,Z) = Xg(Y,Z) - g(\overline{D}_X Y, Z) - g(Y, \overline{D}_X Z). \tag{2.10}$$

In view of (2.8) and (2.10), we get

$$g(H(X,Y),Z) + g(H(X,Z),Y) = 2\eta_2(X)g(Y,Z) + [\eta_3(Y)g(X,Z) + \eta_3(Z)g(X,Y)] + [\eta_4(Z)g(X,Y) + \eta_4(Y)g(X,Z)]. \tag{2.11}$$

Also, we have

$$g(\overline{T}(X,Y),Z) + g(\overline{T}(Z,X),Y) + g(\overline{T}(Z,Y),X) = 2g(H(X,Y),Z) - 2\eta_2(X)g(Y,Z) - 2\eta_2(Y)g(X,Z) - 2\eta_3(Y)g(X,Y) - 2\eta_4(Z)g(X,Y) - 2\eta_2(Z)g(X,Y). \tag{2.12}$$

Equation (2.12) can be written as

$$H(X,Y) = \frac{1}{2}[\overline{T}(X,Y) + \overline{T}'(X,Y) + \overline{T}'(Y,X)] + \eta_2(X)Y + +\eta_2(Y)X + g(X,Y)\xi_2 + g(X,Y)\xi_3 + g(X,Y)\xi_4. \tag{2.13}$$

where $g(T'(X,Y),Z) = g(T(Z,X),Y)$.

Then

$$g(T'(X,Y),Z) = \eta_1(X)g(Y,Z) - \eta_1(Z)g(X,Y) + 2\eta_2(Z)g(X,Y) - 2\eta_2(X)g(Y,Z) + 2\eta_3(X)g(Y,Z) - 2\eta_3(Z)g(X,Y). \tag{2.14}$$

From (2.13) and (2.14), we get

$$H(X,Y) = \eta_1(Y)X + \eta_2(X)Y + \eta_3(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4. \tag{2.15}$$

Hence, from (2.8) and (2.15), we get

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X + \eta_2(X)Y + \eta_3(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4. \tag{2.16}$$

Further, we define

$$'\overline{T}(X,Y,Z) = g(\overline{T}(X,Y),Z). \tag{2.17}$$

Thus, we have from (2.6) and (2.17)

$$\left.\begin{aligned} '\overline{T}(X,Y,Z) + '\overline{T}(Y,X,Z) &= 0, \\ '\overline{T}(X,Y,Z) + '\overline{T}(X,Z,Y) &\neq 0, \\ '\overline{T}(X,Y,Z) + '\overline{T}(Y,Z,X) + '\overline{T}(Z,X,Y) &= 0. \end{aligned}\right\} \tag{2.18}$$

Next, let πbe any form defined on manifold with associated vector field ρ i.e.

$$g(X,\rho) = \pi(X). \tag{2.19}$$

Thus, we get

$$(\bar{D}_X\pi)(Y) = X\pi(Y) - \pi(\bar{D}_X Y)$$
$$= X\pi(Y) - \pi(D_X Y + \eta_1(Y)X + \eta_2(Y)X + \eta_3(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4)$$
$$= (D_X\pi)(Y) - \eta_1(Y)\pi(X) - \eta_2(Y)\pi(X) - \eta_3(Y)\pi(X)$$
$$+g(X,Y)\eta_1(\rho) - g(X,Y)\eta_4(\rho), \tag{2.20}$$

for any vector field X and Y on M^n.

From (2.20), we have

$$(\bar{D}_X\pi)(Y) - (\bar{D}_Y\pi)(X) = (D_X\pi)(Y) - (D_Y\pi)(X) - [\eta_1(Y) - \eta_2(Y) + \eta_3(Y)]\pi(X)$$
$$+[\eta_1(X) - \eta_2(X) + \eta_3(X)]\pi(Y). \tag{2.21}$$

In view of (2.18) and (2.21), we have the following theorem:

Theorem (2.1): On an n-dimensional Riemannian manifold (M^n, g) endowed with a semi-symmetric non-metric connection $\bar{D}$, we get

i. $'\bar{T}(X,Y,Z) + '\bar{T}(Y,X,Z) = 0$,
ii. $'\bar{T}(X,Y,Z) + '\bar{T}(X,Z,Y) \neq 0$,
iii. $'\bar{T}(X,Y,Z) + '\bar{T}(Y,Z,X) + '\bar{T}(Z,X,Y) = 0$.
iv. $\bar{d}\pi(X) = d\pi(X)$if and only if
$[\eta_1(Y) - \eta_2(Y) + \eta_3(Y)]\pi(X) = [\eta_1(X) - \eta_2(X) + \eta_3(X)]\pi(Y)$.

3. Curvature tensor of M^n with respect to semi-symmetric non-metric connection $\bar{D}$

Analogous to the definition of curvature tensor of a Riemannian manifold M^n with respect to the Riemannian connection D, we define the curvature tensor of M^n with respect to semi-symmetric non-metric connection $\bar{D}$ by

$$\bar{R}(X,Y)Z = \bar{D}_X\bar{D}_Y Z - \bar{D}_Y\bar{D}_X Z - \bar{D}_{[X,Y]}Z. \tag{3.1}$$

From (2.1) and (3.1), we get

$$\bar{R}(X,Y)Z = \bar{D}_X[\eta_1(Z)Y + \eta_2(Y)Z + \eta_3(Z)Y - g(Y,Z)\xi_1 + g(Y,Z)\xi_4] -$$
$$\bar{D}_Y[\eta_1(Z)X + \eta_2(X)Z + \eta_3(Z)X - g(X,Z)\xi_1 + g(X,Z)\xi_4] -$$
$$[D_{[X,Y]}Z + \eta_1(Z)[X,Y] + \eta_2([X,Y])Z + \eta_3(Z)[X,Y] -$$
$$g([X,Y],Z)\xi_1 + g([X,Y],Z)\xi_4] .$$

After a long calculation, we get

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + \beta(X,Z)Y - \beta(Y,Z)X$$
$$-g(Y,Z)AX + g(X,Z)AY + g(Y,Z)EX - g(X,Z)EY$$
$$+\gamma(Y,Z)X - \gamma(X,Z)Y + d\eta_2(X,Y)Z. \tag{3.2}$$

where

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

is the curvature tensor of a Riemannian manifold M with respect to Riemannian connection D and

$$\alpha(X,Z) = g(AX,Z) = (D_X\eta_1)(Z) - \eta_1(X)\eta_1(Z) + \tfrac{1}{2}g(X,Z)\eta_1(\xi_1) - \tfrac{1}{2}g(X,Z)\eta_1(\xi_4), \tag{3.3}$$

$$\beta(X,Z) = g(BX,Z) = (D_X\eta_3)(Z) - \eta_3(X)\eta_3(Z) + \tfrac{1}{2}g(X,Z)\eta_3(\xi_1) - \tfrac{1}{2}g(X,Z)\eta_3(\xi_4), \tag{3.4}$$

$$\gamma(X,Z) = g(CX,Z) = \eta_1(X)\eta_3(Z) + \eta_3(X)\eta_1(Z) - \frac{1}{2}g(X,Z)\eta_3(\xi_1), \quad (3.5)$$

$$\delta(X,Z) = g(EX,Z) = (D_X\eta_4)(Z) - \eta_1(X)\eta_4(Z) + \eta_1(Z)\eta_4(X) + \eta_4(X)\eta_4(Z), \quad (3.6)$$

$$AX = D_X\xi_1 - \eta_1(X)\xi_1 + \frac{1}{2}\eta_1(\xi_1)X - \frac{1}{2}\eta_1(\xi_4)X, \quad (3.7)$$

$$BX = D_X\xi_3 - \eta_3(X)\xi_3 + \frac{1}{2}\eta_3(\xi_1)X - \frac{1}{2}\eta_3(\xi_4)X, \quad (3.8)$$

$$CX = \eta_1(X)\xi_3 + \eta_3(X)\xi_1 - \frac{1}{2}\eta_3(\xi_1)X, \quad (3.9)$$

and $$EX = D_X\xi_4 - \eta_1(X)\xi_4 + \eta_4(X)\xi_1 + \eta_4(X)\xi_4 - \frac{1}{2}\eta_1(\xi_4)X. \quad (3.10)$$

In view of (3.3), (3.4), (3.5) and (3.6), we get

$$\alpha(X,Z) - \alpha(Z,X) = d\eta_1(X,Z), \quad (3.11)$$

$$\beta(X,Z) - \beta(Z,X) = d\eta_3(X,Z), \quad (3.12)$$

$$\gamma(X,Z) - \gamma(Z,X) = 0, \quad (3.13)$$

and $$\delta(X,Z) - \delta(Z,X) = d\eta_4(X,Z). \quad (3.14)$$

From (3.11), (3.12) and (3.14), we can say that tensor α, β and δ are symmetric if and only if 1-form η_1, η_3 and η_4 are closed where as tensor γ is symmetric.

Let

$${}'R(X,Y,Z,W) = g(R(X,Y)Z,W) \text{ and } {}'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W), \quad (3.15)$$

for the arbitrary vector fields X, Y, Z and W on manifold.

From (3.2) and (3.15), we get

$$\begin{aligned}{}'\bar{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) + \alpha(X,Z)g(Y,W) - \alpha(Y,Z)g(X,W) + \\ \beta(X,Z)g(Y,W) - \beta(Y,Z)g(X,W) - g(Y,Z)\alpha(X,W) + \\ g(X,Z)\alpha(Y,W) + g(Y,Z)\delta(X,W) - g(X,Z)\delta(Y,W) + \\ \gamma(Y,Z)g(X,W - \gamma(X,Z)g(Y,W) + d\eta_2(X,Y)g(Z,W).\end{aligned} \quad (3.16)$$

From (3.16), we have

$${}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = [d\eta_2(X,Y) + d\eta_2(Y,X)]g(Z,W), \quad (3.17)$$

$$\begin{aligned}{}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) = [\beta(X,Z) + \delta(X,Z) - \gamma(X,Z)]g(Y,W) + \\ [\beta(X,W) + \delta(X,W) - \gamma(X,W)]g(Y,Z) - \\ [\beta(Y,Z) + \delta(Y,Z) - \gamma(Y,Z)]g(X,W) - \\ [\beta(Y,W) + \delta(Y,W) - \gamma(Y,W)]g(X,Z),\end{aligned} \quad (3.18)$$

$${}'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) = [d\eta_1(X,Z)g(Y,W) - d\eta_1(Y,Z)g(X,W)], \quad (3.19)$$

$$\begin{aligned}{}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,Z,X,W) + {}'\bar{R}(Z,X,Y,W) = [d\eta_1(\mathrm{X},Z)g(Y,W) + d\eta_1(Y,X)g(Z,W) \\ + d\eta_1(Z,Y)g(X,W)] + [d\eta_2(Z,X)g(Y,W) \\ + d\eta_2(X,Y)g(Z,W) + d\eta_2(Y,Z)g(X,W)] + \\ [d\eta_3(X,Z)g(Y,W) + d\eta_3(Y,X)g(Z,W) + \\ d\eta_3(Z,Y)g(X,W)\ .\end{aligned} \quad (3.20)$$

Analogous to the definition of Ricci tensor of Riemannian manifold M^n with respect to the Riemannian connection$D$, we define Ricci tensor $\bar{R}ic$ of M^n with respect to semi-symmetric non-metric connection $\bar{D}$by the expression

$$\bar{R}ic(Y,Z) = \sum_{i=1}^{n} {}'\bar{R}\,(E_i\,,Y,Z,E_i) \tag{3.21}$$

where$E_i's, 1 \leq\ i \leq\ n$, orthonormal vector fields on M^n. From (3.16) and (3.21), we get

$$\bar{R}ic(Y,Z) = Ric(Y,Z) - (n-2)\alpha(Y,Z) - (n-1)[\beta(Y,Z) - \gamma(Y,Z)] -$$
$$\delta(Y,Z) - (a-b)g(Y,Z) + d\eta_2(Y,Z), \tag{3.22}$$

where $a = traceA,\ b = traceE$ and $Ric(Y,Z) = \sum_{i=1}^{n} R\,(E_i\,,Y,Z,E_i)$ is the Ricci tensor of M^n with respect to the Riemannian connection. A relation between Ricci tensor $\bar{R}ic$ with respect to semi-symmetric non-metric connection $\bar{D}$ and the Riemannian connection D is given by (3.22).

From (3.22) it follows that the Ricci tensor $\bar{R}ic$ is symmetric if and only if

$$0 = \alpha(Y,Z) - (n-1)[\beta(Y,Z) - \gamma(Y,Z)] - \delta(Y,Z) - (a-b)g(Y,Z) + d\eta_2(Y,Z). \tag{3.23}$$

Again, from (3.22), we get

$$\bar{R}ic(Y,Z) + \bar{R}ic(Z,Y) = 2Ric(Y,Z) - (n-2)[\alpha(Y,Z) + \alpha(Z,Y)] +$$
$$(n-1)[\beta(Y,Z) + \beta(Z,Y)] - [\delta(Y,Z) + \delta(Z,Y)] -$$
$$2\gamma(Y,Z) - 2(a-b)g(Y,Z) + [d\eta_2(Y,Z) + d\eta_2(Z,Y)]. \tag{3.24}$$

Analogous to the definition of scalar curvature of Riemannian manifold M^n with respect to the Riemannian connection$D$, we define scalar curvature r of M^n with respect to semi-symmetric non-metric connection $\bar{D}$by

$$\bar{r} = \sum_{i=1}^{n} \bar{R}ic\,(E_i\,,E_i) \tag{3.25}$$

From (3.22) and (3.25), we get

$$\bar{r} = r - (n-1)[2a + b - c + d], \tag{3.24}$$

where$c = traceB, d = traceC$ and $r = \sum_{i=1}^{n} Ric\,(E_i\,,E_i)$ is the scalar curvature of Riemannian manifold M^n with respect to the Riemannian connection$D$.

In consequences of (3.16), (3.17), (3.18), (3.19), (3.20), (3.22), (3.23), (3.24) and (3.26), we can state the following theorem:

Theorem (3.1): For a Riemannian manifold (M^n, g) admitting semi-symmetric non-metric connection $\bar{D}$

- **(i)** The curvature tensor $\bar{R}$ of $\bar{D}$ is given by (3.2)
- **(ii)** ${}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,X,Z,W) = 0$, if and only if 1-form η_2 is closed,
- **(iii)** ${}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(X,Y,W,Z) \neq 0$,
- **(iv)** ${}'\bar{R}(X,Y,Z,W) - {}'\bar{R}(Z,W,X,Y) = 0$, if and only if 1-form η_1 is closed,
- **(v)** ${}'\bar{R}(X,Y,Z,W) + {}'\bar{R}(Y,Z,X,W) + {}'\bar{R}(Z,X,Y,W) = 0$, if and only if 1-form $\eta_1,\ \eta_2$ and η_3 are closed,
- **(vi)** The Ricci tensor $\bar{R}ic$ of $\bar{D}$ is given by (3.22),

(vii) The Ricci tensor $\bar{R}ic$ is symmetric if and only if

$$\alpha(Y,Z)-(n-1)[\beta(Y,Z)-\gamma(Y,Z)]-\delta(Y,Z)-(a-b)g(Y,Z)+d\eta_2(Y,Z)=0,$$

(viii) The Ricci tensor $\bar{R}ic$ is skew symmetric if and only if

(ix) $2Ric(Y,Z)=(n-2)[\alpha(Y,Z)+\alpha(Z,Y)]-(n-1)[\beta(Y,Z)+\beta(Z,Y)]+$ $[\delta(Y,Z)+\delta(Z,Y)]+2\gamma(Y,Z)+2(a-b)g(Y,Z)-[d\eta_2(Y,Z)+d\eta_2(Z,Y)],$

(x) The scalar curvature $\bar{r}$ of $\bar{D}$ is given by (3.26),

(xi) The necessary and sufficient condition for the scalar curvature $\bar{r}$ and r to coincide is that $2a+b-c+d=0$.

4. Projective curvature tensor of a Riemannian manifold with respect to semi-symmetric non-metric connection $\bar{D}$

The Projective curvature tensor of the manifold of the type (1,3) with respect to semi-symmetric non-metric connection $\bar{D}$ is given by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{n-1}\,[\bar{R}ic(Y,Z)X-\bar{R}ic(X,Z)Y\,]. \tag{4.1}$$

From (3.2), (3.22) and (4.1) it follows that

$$\bar{P}(X,Y)Z=P(X,Y)Z-2\,[\{\beta(Y,Z)X-\beta(X,Z)Y\}-\{\gamma(Y,Z)X-\gamma(X,Z)Y\}]-$$
$$g(Y,Z)AX+g(X,Z)AY+g(Y,Z)EX-g(X,Z)EY+$$
$$\frac{1}{n-1}[(n-1)d\eta_2(X,Y)Z-d\eta_2(Y,Z)X+d\eta_2(X,Z)Y+$$
$$\delta(Y,Z)X-\delta(X,Z)Y+(a-b)\{g(Y,Z)X-g(X,Z)Y\}], \tag{4.2}$$

where $P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}\,[Ric(Y,Z)X-Ric(X,Z)Y\,]$ is the projective curvature tensor with respect to D (Mishra,1984) .

If $\bar{P}(X,Y)Z=P(X,Y)Z$, then from (4.2), we get

$$2\,[\{\beta(Y,Z)X-\beta(X,Z)Y\}-\{\gamma(Y,Z)X-\gamma(X,Z)Y\}]+$$
$$g(Y,Z)AX-g(X,Z)AY-g(Y,Z)EX+g(X,Z)EY-$$
$$\frac{1}{n-1}[(n-1)d\eta_2(X,Y)Z-d\eta_2(Y,Z)X+d\eta_2(X,Z)Y+$$
$$\delta(Y,Z)X-\delta(X,Z)Y+(a-b)\{g(Y,Z)X-g(X,Z)Y\}]=0. \tag{4.3}$$

Contracting (4.3) with respect to X, we get

$$\alpha(Y,Z)+2(n-1)[\beta(Y,Z)-\gamma(Y,Z)]+(a-b-n+1)g(Y,Z)=0.$$

Theorem (4.1): A necessary condition for the projective curvature tensor P of the manifold (M^n,g) with respect to the Levi-Civita connection D and projective curvature tensor $\bar{P}$ of the manifold with respect to the semi-symmetric non-metric connection to be equal is that

$$\alpha(Y,Z)+2(n-1)[\beta(Y,Z)-\gamma(Y,Z)]+(a-b-n+1)g(Y,Z)=0.$$

From (4.2), we have

$$\bar{P}(X,Y)Z+\bar{P}(Y\,,X)Z=0, \tag{4.4}$$

and

$$\bar{P}(X,Y)Z + \bar{P}(Y,Z)X + \bar{P}(Z,X)Y = -2[\eta_3(X,Y)Z + \eta_3(Y,Z)X + \eta_3(Z,X)Y] + \eta_2(X,Y)Z + \eta_2(Y,Z)X + \eta_2(Z,X)Y + \frac{1}{n-1}[\eta_4(X,Y)Z + \eta_4(Y,Z)X + \eta_4(Z,X)Y]. \quad (4.5)$$

Hence, we have the following theorem:

Theorem (4.2): The projective curvature tensor $\bar{P}$ with respect to $\bar{D}$ satisfies the following algebraic properties:

$\bar{P}(X,Y)Z + \bar{P}(Y,X)Z = 0$, i.e. skew symmetric in first pair of slots

and

Cyclic sum of projective curvature tensor $\bar{P}$ is equal to zero if and only if 1-form η_2, η_3 and η_4 are closed.

Let

$$\bar{R}(X,Y)Z = 0, \quad (4.6)$$

$$\bar{R}ic(Y,Z) = 0. \quad (4.7)$$

Hence, in view of (4.1), (4.6) and (4.7), we get

$$\bar{P}(X,Y)Z = 0. \quad (4.8)$$

From (4.2) and (4.8), we get

$$P(X,Y)Z = 2\,[\{\beta(Y,Z)X - \beta(X,Z)Y\} - \{\gamma(Y,Z)X - \gamma(X,Z)Y\}] + [g(Y,Z)AX - g(X,Z)AY - g(Y,Z)EX + g(X,Z)EY] - \frac{1}{n-1}[(n-1)d\eta_2(X,Y)Z - d\eta_2(Y,Z)X + d\eta_2(X,Z)Y + \delta(Y,Z)X - \delta(X,Z)Y + (a-b)\{g(Y,Z)X - g(X,Z)Y\}]. \quad (4.9)$$

Let us consider

$$P(X,Y)Z = 0. \quad (4.10)$$

Hence in view of (4.9) and (4.10), we get

$$2\,[\{\beta(Y,Z)X - \beta(X,Z)Y\} - \{\gamma(Y,Z)X - \gamma(X,Z)Y\}] + [g(Y,Z)AX - g(X,Z)AY - g(Y,Z)EX + g(X,Z)EY] - \frac{1}{n-1}[(n-1)d\eta_2(X,Y)Z - d\eta_2(Y,Z)X + d\eta_2(X,Z)Y + \delta(Y,Z)X - \delta(X,Z)Y + (a-b)\{g(Y,Z)X - g(X,Z)Y\}] = 0. \quad (4.11)$$

Contraction (4.11), we get

$$\alpha(Y,Z) + 2(n-1)[\beta(Y,Z) - \gamma(Y,Z)] + (a-b-n+1)g(Y,Z) = 0. \quad (4.12)$$

Theorem (4.3): A necessary condition for the projective curvature tensor P of the manifold (M^n, g) with respect to the Levi-Civita connection D flat if

$$\alpha(Y,Z) + 2(n-1)[\beta(Y,Z) - \gamma(Y,Z)] + (a-b-n+1)g(Y,Z) = 0.$$

5. Conformal curvature tensor of a Riemannian manifold with respect to semi-symmetric non-metric connection $\bar{D}$

The Conformal curvature tensor of the manifold of the type (1,3) with respect to Levi-Civita connection D is given by (Mishra, 1984)

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

Equation (5.1) can be written as

$$'\tilde{C}(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-1}[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) + g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{5.2}$$

where $'\tilde{C}(X,Y,Z,W) = g(\tilde{C}(X,Y)Z,W)$ and $g(RX,Y) = Ric(X,Y)$

Similarly, we define Conformal curvature tensor of the manifold with respect to semi-symmetric non-metric connection $\bar{D}$ of type (0,4) by the expression:

$$'\bar{\tilde{C}}(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) - \frac{1}{n-1}[\bar{R}ic(Y,Z)g(X,W) - \bar{R}ic(X,Z)g(Y,W) + g(Y,Z)\bar{R}ic(X,W) - g(X,Z)\bar{R}ic(Y,W)] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{5.3}$$

where $'\bar{\tilde{C}}(X,Y,Z,W) = g(\bar{\tilde{C}}(X,Y)Z,W)$.

From (3.16), (3.22), (3.26), (5.2), and (5.3), we have

$$'\bar{\tilde{C}}(X,Y,Z,W) = {}'\tilde{C}(X,Y,Z,W) + \frac{1}{n-2}[\{\beta(Y,Z) - \gamma(Y,Z) - (n-1)\delta(Y,Z) + d\eta_2(Y,Z)\}g(X,W) + \{-\beta(X,Z) + \gamma(X,Z) + (n-1)\delta(X,Z) - d\eta_2(X,Z)\}g(Y,W)\} + \{(n-1)\beta(X,W) - (n-1)\gamma(X,W) + \delta(X,W) + d\eta_2(X,W)\}g(Y,Z)\} + \{-(n-1)\beta(Y,W) + (n-1)\gamma(Y,W) - \delta(Y,W) - d\eta_2(Y,W)\}g(X,Z)\}] + \frac{4a-b-c-d}{(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{5.4}$$

If $'\bar{\tilde{C}}(X,Y,Z,W) = {}'\tilde{C}(X,Y,Z,W)$, then from (5.4), we get

$$\frac{1}{n-2}[\{\beta(Y,Z) - \gamma(Y,Z) - (n-1)\delta(Y,Z) + d\eta_2(Y,Z)\}g(X,W) + \{-\beta(X,Z) + \gamma(X,Z) + (n-1)\delta(X,Z) - d\eta_2(X,Z)\}g(Y,W)\} +$$

$$\{(n-1)\beta(X,W)-(n-1)\gamma(X,W)+\delta(X,W)+d\eta_2(X,W)\}g(Y,Z)\}+$$
$$\{-(n-1)\beta(Y,W)+(n-1)\gamma(Y,W)-\delta(Y,W)-d\eta_2(Y,W)\}g(X,Z)\}]+$$
$$\frac{4a-b-c-d}{(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]=0. \tag{5.5}$$

Contraction (5.5), we get

$$\beta(Y,Z)-\gamma(Y,Z)-[(n-1)^2+1]\delta(Y,Z)+(n-2)d\eta_2(Y,Z)$$
$$+(4a-b-c-d)g(Y,Z)=0.$$

Hence, we can state the following theorem:

Theorem (5.1): A necessary condition for the Conformal curvature tensor $'\tilde{C}$ of the manifold (M^n,g) with respect to the Levi-Civita connection D flat if

$$\beta(Y,Z)-\gamma(Y,Z)-[(n-1)^2+1]\delta(Y,Z)+(n-2)d\eta_2(Y,Z)$$
$$+(4a-b-c-d)g(Y,Z)=0.$$

From (5.4), we have

$$\bar{\tilde{C}}(X,Y,Z,W)+\bar{\tilde{C}}(Y,X,Z,W)=0, \tag{5.6}$$

$$\bar{\tilde{C}}(X,Y,Z,W)+\bar{\tilde{C}}(X,Y,W,Z)=[-\beta(Y,Z)+\gamma(Y,Z)+\delta(Y,Z)]g(X,W)+$$
$$[\beta(X,Z)-\gamma(X,Z)-\delta(X,Z)]g(Y,W)+$$
$$[\beta(X,W)-\gamma(X,W)-\delta(X,W)]g(Y,Z)+$$
$$[-\beta(Y,W)+\gamma(Y,W)+\delta(Y,W)]g(X,Z), \tag{5.7}$$

and

$$\bar{\tilde{C}}(X,Y,Z,W)+\bar{\tilde{C}}(Y,Z,X,W)+\bar{\tilde{C}}(Z,X,Y,W)=\frac{1}{n-2}[\{d\eta_3(Y,Z)+(n-1)d\eta_4(Y,Z)\}g(X,W)$$
$$\{d\eta_3(Z,X)+(n-1)d\eta_4(Z,X)\}g(Y,W)+$$
$$\{d\eta_3(X,Y)+(n-1)d\eta_4(X,Y)\}g(Z,W)+$$
$$\{d\eta_4(X,W)+(n-1)d\eta_3(X,W)\}g(Y,Z)+$$
$$\{d\eta_4(Y,W)+(n-1)d\eta_3(Y,W)\}g(X,Z)+$$
$$\{d\eta_4(Z,W)+(n-1)d\eta_3(Z,W)\}g(X,Y)].(5.8)$$

Theorem (5.2): The Conformal curvature tensor $\bar{\tilde{C}}$ with respect to $\bar{D}$ satisfies the following algebraic properties:

(i) $\bar{\tilde{C}}(X,Y,Z,W)+\bar{\tilde{C}}(Y,X,Z,W)=0$, i.e. skew symmetric in first pair of slots,

(ii) $\bar{\tilde{C}}(X,Y,Z,W)+\bar{\tilde{C}}(X,Y,W,Z)=[-\beta(Y,Z)+\gamma(Y,Z)+\delta(Y,Z)]g(X,W)+$
$[\beta(X,Z)-\gamma(X,Z)-\delta(X,Z)]g(Y,W)+$
$[\beta(X,W)-\gamma(X,W)-\delta(X,W)]g(Y,Z)+$
$[-\beta(Y,W)+\gamma(Y,W)+\delta(Y,W)]g(X,Z)$,

and

(iii) Cyclic sum of Conformal curvature tensor $\bar{\bar{C}}$ is equal to zero if and only if 1-form $\eta_1,\ \eta_2, \eta_3$ and η_4 are closed.

6. Hypersurfaces of a Riemannian manifold with respect to semi-symmetric non-metric connection

Let M^{n+1} be an (n+1)-dimensional differentiable manifold of class C^{∞} and M^n be n-dimensional differentiable manifold immersed in M^{n+1} by a differentiable immersion $i: M^n \to M^{n+1}$. We identify the image $i(M^n)$ with M^n is called hypersurface of M^{n+1}. We denote the differential di of the immersion $i$by B, so that a vector field X in M^n there corresponds a vector BX is M^{n+1}. We assume that the metric tensor $\bar{g}$ defined by

$$g(X,Y) = \bar{g}(BX, BY), \tag{6.1}$$

$\forall\, X, Y \in \chi(M^n)$.

If the Riemannian manifold M^n and M^{n+1} are both orientable we can choose a unique vector field N defined in M^n so that

$$\bar{g}(BX, N) = 0, \tag{6.2}$$

And

$$\bar{g}(N, N) = 0, \tag{6.3}$$

this vector field N is called the unit normal vector field to the hypersurface M^n.

Let $(M^{n+1}, \bar{g})$ be an (n+1)-dimensional Riemannian manifold. A linear connection $\overset{0}{\bar{D}}$ is said to be a semi-symmetric non-metric connection if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X},\bar{Y}) = [\bar{\eta}_1(\bar{Y})X - \bar{\eta}_1(\bar{X})\bar{Y}] + [\bar{\eta}_2(\bar{X})\bar{Y} - \bar{\eta}_2(\bar{Y})\bar{X}] + [\bar{\eta}_3(\bar{Y})\bar{X} - \bar{\eta}_3(\bar{X})\bar{Y}], \tag{6.4}$$

and

$$\left(\overset{0}{\bar{D}}_{\bar{X}}\bar{g}\right)(\bar{Y},\bar{Z}) = -2\bar{\eta}_2(\bar{X})g(\bar{Y},\bar{Z}) - [\bar{\eta}_3(\bar{Y})g(\bar{X},\bar{Z}) + \bar{\eta}_3(\bar{Z})g(\bar{X},\bar{Y})] - [\bar{\eta}_4(\bar{Z})g(\bar{X},\bar{Y}) + \bar{\eta}_4(\bar{Y})g(\bar{X},\bar{Z})]. \tag{6.5}$$

for all $\bar{X},\bar{Y},\bar{Z} \in \chi(M^{n+1})$, where $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3$ and $\bar{\eta}_4$ are four 1-forms associate with the vector fields $\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3$ and $\bar{\xi}_4$ respectively by

$$g(\bar{X},\bar{\xi}_1) = \bar{\eta}_1(\bar{X}), \tag{6.6}$$

$$g(\bar{X},\bar{\xi}_2) = \bar{\eta}_2(\bar{X}), \tag{6.7}$$

$$g(\bar{X},\bar{\xi}_3) = \bar{\eta}_3(\bar{X}), \tag{6.8}$$

and

$$\bar{\eta}_4(\bar{X}) = g(\bar{X},\bar{\xi}_4). \tag{6.9}$$

It assumed that the Riemannian manifold $(M^{n+1}, \bar{g})$ admits a semi-symmetric non-metric connection given by

$$\overset{0}{\bar{D}}_{\bar{X}}\bar{Y} = \overset{0}{D}_{\bar{X}}\bar{Y} + \bar{\eta}_1(\bar{Y})\bar{X} + \bar{\eta}_2(\bar{X})\bar{Y} + \bar{\eta}_3(\bar{Y})\bar{X} - \bar{g}(\bar{X},\bar{Y})\bar{\xi}_1 + g(\bar{X},\bar{Y})\bar{\xi}_4, \tag{6.10}$$

where $\overset{0}{D}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\bar{g}$.

Now, we put

$$\bar{\xi}_1 = B\xi_1 + \lambda n, \tag{6.11}$$

$$\bar{\xi}_2 = B\xi_2 + \lambda n, \tag{6.12}$$

$$\bar{\xi}_3 = B\xi_3 + \lambda n, \tag{6.13}$$

$$\bar{\xi}_4 = B\xi_4 + \lambda n, \tag{6.14}$$

where ξ_1, ξ_2, ξ_3 and ξ_4 are vector fields and λ is a function in M^n. We denote by D and $\overline{D}$ connections induced on the hypersurface from $\overset{0}{D}$ and $\overset{0}{\overline{D}}$ respectively with respect to the unite normal N. Thus we have

$$\overset{0}{D}_{BX}BY = B(D_XY) + h(X,Y)N, \tag{6.15}$$

and

$$\overset{0}{\overline{D}}_{BX}BY = B(\overline{D}_XY) + m(X,Y)N. \tag{6.16}$$

Here m is a tensor field of the type (0,2) and h is second fundamental tensor. From (6.10), we obtain

$$\overset{0}{\overline{D}}_{BX}BY = \overset{0}{D}_{BX}BY + \bar{\eta}_1(BY)BX + \bar{\eta}_2(BX)BY + \bar{\eta}_3(BY)BX - \bar{g}(BX,BY)\bar{\xi}_1 + g(BX,BY)\bar{\xi}_4. \tag{6.17}$$

From (6.16) and (6.17), we get

$$B(\overline{D}_XY) + m(X,Y)N = \overset{0}{D}_{BX}BY + \bar{\eta}_1(BY)BX + \bar{\eta}_2(BX)BY + \bar{\eta}_3(BY)BX - \bar{g}(BX,BY)\bar{\xi}_1 + g(BX,BY)\bar{\xi}_4. \tag{6.18}$$

Again from (6.15) and (6.18), we get

$$B(\overline{D}_XY) + m(X,Y)N = B(D_XY) + h(X,Y)N + \bar{\eta}_1(BY)BX + \bar{\eta}_2(BX)BY + \bar{\eta}_3(BY)BX - \bar{g}(BX,BY)\bar{\xi}_1 + g(BX,BY)\bar{\xi}_4. \tag{6.19}$$

In view of (6.11), (6.12), (6.13), (6.14) and (6.19), we get

$$B(\overline{D}_XY) + m(X,Y)N = B(D_XY) + h(X,Y)N + \eta_1(Y)BX + \eta_2(X)BY + \eta_3(Y)BX - \bar{g}(BX,BY)B\xi_1 - \bar{g}(BX,BY)\lambda N + g(BX,BY)B\xi_4 + \bar{g}(BX,BY)\lambda N, \tag{6.20}$$

where we used $\bar{\eta}_1(BX) = \eta_1(X)$, $\bar{\eta}_2(BX) = \eta_2(X)$ and $\bar{\eta}_3(BX) = \eta_3(X)$.

Equating tangent and normal components from (6.20), we get

$$\overline{D}_XY = D_XY + \eta_1(Y)X + \eta_2(X)Y + \eta_3(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4, \tag{6.21}$$

and

$$m(X,Y) = h(X,Y) - \lambda g(X,Y). \tag{6.22}$$

In view of (6.21), we get

$$\overline{T}(X,Y) = [\eta_1(Y)X - \eta_1(X)Y] + [\eta_2(X)Y - \eta_2(Y)X] + [\eta_3(Y)X - \eta_3(X)Y], \tag{6.23}$$

and

$$(\overline{D}_X g)(Y,Z) = -2\eta_2(X)g(Y,Z) - [\eta_3(Y)g(X,Z) + \eta_3(Z)g(X,Y)] - [\eta_4(Z)g(X,Y) + \eta_4(Y)g(X,Z)]. \tag{6.24}$$

Hence, we can state the following theorem:

Theorem (6.1): The connection induced on a hypersurface of a Riemannian manifold with semi-symmetric non-metric connection with respect to the unit normal is also a semi-symmetric non-metric connection, provided that the associated vector fields are non-null on the hypersurface.

Moreover, let $e_1, e_2, e_3, \dots, e_n$ be n orthonormal local vector fields in M^n. Then the functions $\frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ and $\frac{1}{n}\sum_{i=1}^{n} m(e_i, e_i)$ are called the mean curvatures of M^n with respect to D and $\overline{D}$ respectively.

From (6.22) the following theorem follows:

Theorem (6.2): In order that the mean curvature if M^n with respect to D and $\overline{D}$ be equal, it is necessary and sufficient condition that the vector fields ξ_1, ξ_2, ξ_3 and ξ_4 are tangent to M^n.

If h is proportional to g, then M^n is totally umbilical with respect to D and if m is proportional to g, then M^n id said to be totally umbilical with respect to $\overline{D}$.

Hence, from (6.22), we have

Theorem (6.3): A hypersurface is totally umbilical with respect to the Levi-Civita connection D if and only if it is totally umbilical also with respect to the semi-symmetric non-metric connection $\overline{D}$.

7. Example

Let us consider the 4-dimensional manifold $M = \{(x,y,z,w)\,\epsilon\mathbb{R}^4, w \neq 0\}$, where (x,y,z,w) are standard co-ordinate of $\mathbb{R}^4$.

We choose the vector fields

$$e_1 = e^w\frac{\partial}{\partial x},\ \ e_2 = e^w\frac{\partial}{\partial y},\ \ e_3 = e^w\frac{\partial}{\partial z},\ \ e_4 = \frac{\partial}{\partial w}, \tag{7.1}$$

which is linearly independently at each point of M.

Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{7.2}$$

where$i, j = 1,2,3,4$. Let D be the Levi-Civita connection with respect to Riemannian metric g. Then from equation (7.1), we have

$$\left.\begin{aligned} &[e_1,e_2] = 0, [e_1,e_3] = 0, [e_1,e_4] = 0,\ [e_2,e_3] = 0, \\ &[e_2,e_3] = 0, [e_2,e_4] = -e_2, [e_3,e_4] = -e_3. \end{aligned}\right\} \tag{7.3}$$

The Riemannian connection Dif the metric g is given by

$$2g(D_XY,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \tag{7.4}$$

which is known as Koszul's formula. Using (7.2) and (7.3) in (7.4), we get

$$\left.\begin{array}{llll} D_{e_1}e_1 = 0, & D_{e_1}e_2 = 0, & D_{e_1}e_3 = 0, & D_{e_1}e_4 = 0, \\ D_{e_2}e_1 = 0, & D_{e_2}e_2 = e_4, & D_{e_2}e_3 = 0, & D_{e_2}e_4 = -e_2, \\ D_{e_3}e_1 = 0, & D_{e_3}e_2 = 0, & D_{e_3}e_3 = e_4, & D_{e_3}e_4 = -e_3, \\ D_{e_4}e_1 = 0, & D_{e_4}e_2 = 0, & D_{e_4}e_3 = 0, & D_{e_4}e_4 = 0. \end{array}\right\}$$

Also

$$\left.\begin{array}{llll} \overline{D}_{e_1}e_1 = 2e_4, & \overline{D}_{e_1}e_2 = 0, & \overline{D}_{e_1}e_3 = 0, & \overline{D}_{e_1}e_4 = -e_1, \\ \overline{D}_{e_2}e_1 = e_1 + e_2, & \overline{D}_{e_2}e_2 = -e_1 + e_2 + 2e_4, & \overline{D}_{e_2}e_3 = e_2 + e_3, & \overline{D}_{e_2}e_4 = -e_2, \\ \overline{D}_{e_3}e_1 = 0, & \overline{D}_{e_3}e_2 = 0, & \overline{D}_{e_3}e_3 = -e_1 + e_3 + 2e_4, & \overline{D}_{e_3}e_4 = -e_3, \\ \overline{D}_{e_4}e_1 = 0, & \overline{D}_{e_4}e_2 = 0, & \overline{D}_{e_4}e_3 = 0, & \overline{D}_{e_4}e_4 = -e_1 + e_4. \end{array}\right\} \quad (7.5)$$

From (2.6), (7.4) and (7.5) we get

$$\bar{T}(e_1, e_4) = [\eta_1(e_4)e_1 - \eta_1(e_1)e_4] + [\eta_2(e_1)e_4 - \eta_2(e_4)e_1] + [\eta_3(e_4)e_1 - \eta_3(e_1)e_4]$$

$\Rightarrow \bar{T}(e_1, e_4) = -e_4 \neq 0$. This shows that the linear connection $\overline{D}$defined as (2.1) is a semi-symmetric connection on (M^3, g). Also

$$(\overline{D}_{e_1}g)(e_2, e_4) = -1 \neq 0.$$

From above, we can say that the defined connection $\overline{D}$ is semi-symmetric non-metric connection.

8. Particular cases

In this section, we list of the following particular cases.

(i) If $\eta_2 = \eta_3 = \eta_4 = 0$. Then from (2.1), we obtain a semi-symmetric metric connection $\overline{D}$given by (Yano, 1970)

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X - g(X, Y)\xi_1.$$

(ii) If $\eta_1 = \eta_2$ and $\eta_3 = \eta_4 = 0$, then equation (2.1) becomes

$$\overline{D}_X Y = D_X Y + \eta_1(Y)X + \eta_1(X)Y - g(X, Y)\xi_1.$$

This connection is symmetric but non-metric connection due to (Yano, 1970). This connection $\overline{D}$ is conformally related to Levi-Civita connection.

(iii) If $\eta_2 = \eta_3$ and $\eta_1 = \eta_4 = 0$, the n equation (2.1) gives again symmetric but non-metric connection as (Yano, 1970; Smaranda, 1984)

$$\overline{D}_X Y = D_X Y + \eta_2(Y)X + \eta_2(X)Y.$$

(iv) If η_1 and η_2 are replaced by $-\frac{1}{2}\eta$, and $\eta_3 = \eta_4 = 0$. Then we obtain a Weyl connection constructed with η and ξ (Folland, 1970) given by

$$\overline{D}_X Y = D_X Y - \frac{1}{2}[\eta(Y)X + \eta(X)Y - g(X, Y)\xi].$$

This connection is a symmetric recurrent non-metric.

(v) If $\eta_1 = \eta_2 = \eta_4 = 0$. Then (2.1) becomes a semi-symmetric non-metric connection introduced by (Agashe and Chafle, 1992)

$$\overline{D}_X Y = D_X Y + \eta_3(Y)X.$$

(vi) If $\eta_1 = \eta_3 = \eta_4 = 0$ and η_2 is replaced by $-\eta$. Then (2.1) becomes

$$\overline{D}_X Y = D_X Y - \eta(X)Y.$$

Actually this is semi-symmetric non-metric connection but (Liang, 1994) called this is semi-symmetric recurrent metric connection.

(vii) If $\eta_1 = \eta_2 = 0$ and η_3 and η_4 are replaced by $-\eta$. Then (2.1) gives

$\bar{D}_X Y = D_X Y - \eta(Y)X - g(X,Y)\xi$. This connection investigated by Kumar and Chaubey in 2010 on a generalized co-symplectic manifold.

(viii) If $\eta_1 = \eta_3 = \eta_4 = 0$. Then (2.1) gives another semi-symmetric metric connection by the expression (Agashe and Chafle, 1992)

$$\bar{D}_X Y = D_X Y + \eta_2(X)Y,$$

whose metric and torsion are non-zero (Melhotra, 2012).

(ix) If $\eta_1 = \eta_2 = 0$and $\eta_3 = \eta_4$. Then equation (2.1) becomes

$\bar{D}_X Y = D_X Y + \eta_3(Y)X + g(X,Y)\xi_4$. This connection is also semi-symmetric non-metric connection and established by De and Biswas (1996/1997).

(x) If $\eta_1 = \eta_4 = 0$. Then equation (2.1) becomes

$$\bar{D}_X Y = D_X Y + \eta_2(X)Y + \eta_3(Y)X.$$

This semi-symmetric non-metric connection introduced by Prasad and Verma in (2004).

(xi) If $\eta_3 = \eta_4 = 0$. Then equation (2.1) gives

$$\bar{D}_X Y = D_X Y + \eta_1(Y)X + \eta_2(X)Y - g(X,Y)\xi_1.$$

Prasad, Dubey and Yadav (2011) proved the existence of such connection and obtained various properties.

(xii) If $\eta_2 = \eta_4 = 0$. Then equation (2.1) yields (Prasad and Doulo, 2015)

$$\bar{D}_X Y = D_X Y + \eta_1(Y)X + \eta_3(Y)X - g(X,Y)\xi_1.$$

(xiii) If $\eta_1 = \eta_4 = 0$ and η_2 and η_3 replaced by $a\eta$ and$b\eta$ respectively, a, $b \neq 0$. Then equation (2.1) gives another semi-symmetric non-metric connection and this was proved by De, Han and Zhao (2016)

$$\bar{D}_X Y = D_X Y + a\eta(X)Y + b\eta(Y)X.$$

(xiv) If $\eta_2 = \eta_3 = 0$. Then equation (2.1) becomes

$$\bar{D}_X Y = D_X Y + \eta_1(Y)X - g(X,Y)\xi_1 + g(X,Y)\xi_4.$$

This connection generalizes the notion of Agashe and Chafle (1992) connection and gives the connection of Sengupta, De and Binh (2000).

(xv) If $\eta_1 = \eta_4 = 0$, η_2 replaced by $-\frac{1}{2}\eta$ and η_3 replaced by $\frac{1}{2}\eta$. Then (2.1) becomes

$$\bar{D}_X Y = D_X Y + \frac{1}{2}[\eta(Y)X - \eta(Y)X].$$

This existence of such connection were established by Chaubey and Yieldz (2019).

(xvi) If $\eta_3 = \eta_4 = 0$, η_1 replaced by η and η_2 replaced by $-\eta$. Then (2.1) becomes

$$\bar{D}_X Y = D_X Y + \eta(Y)X - \eta(X)Y - g(X,Y)\xi.$$

This connection is known as "semi-symmetric recurrent non-metric connection" (communicated for publication).

Furthermore, we mention some connections that are not specific case of our connection but are of the same nature as identified by researchers:

(i) In 1992, Barua and Mukhopadhya defined

$$\bar{D}_X Y = D_X Y - \overset{*}{\eta}_1(X)Y + g(X,Y)\overset{*}{\xi}_1,$$

where $\bar{T}(X,Y) \neq 0$ and $(\bar{D}_X g)(Y,Z) \neq 0$ and they called "semi-symmetric non-metric connection"

(ii) Recently in (2021), Prasad, Kumar and Singh investigated another type of semi-symmetric non-metric connection which extended Agashe and Chafle's (1992) connection by another way

$\bar{D}_X Y = D_X Y + \overset{*}{\eta}_1(Y)X - \overset{*}{\eta}_2(Y)X$, whose torsion tensor and metric are

$\bar{T}(X,Y) = \left[\overset{*}{\eta}_1(Y)X - \overset{*}{\eta}_1(X)Y\right] - \left[\overset{*}{\eta}_2(Y)X - \overset{*}{\eta}_2(X)Y\right]$and

$$(\bar{D}_X g)(Y,Z) = -\overset{*}{\eta}_1(Y)g(X,Z) - \overset{*}{\eta}_1(Z)g(X,Y) + \overset{*}{\eta}_2(Y)g(X,Z) + \overset{*}{\eta}_2(Z)g(X,Y)$$

(iii) In 2000, Sengupta and De defined a semi-symmetric non-metric connection which generalized Yano's connection (1970)

$$\bar{D}_X Y = D_X Y + \overset{*}{\eta}_1(Y)X - g(X,Y)\overset{*}{\xi}_1 - \overset{*}{\eta}_2(X)Y - \overset{*}{\eta}_2(Y)X,$$

where$\overset{*}{\eta}_1$ and $\overset{*}{\eta}_2$ are 1-forms associated with the vector fields $\overset{*}{\xi}_1$ and $\overset{*}{\xi}_2$ by $g\left(X,\overset{*}{\xi}_1\right) = \overset{*}{\eta}_1(X)$ and $g\left(X,\overset{*}{\xi}_2\right) = \overset{*}{\eta}_2(\mathrm{X})$ with $\bar{T}(X,Y) = \overset{*}{\eta}_1(Y)X - \overset{*}{\eta}_1(X)Y$and

$$(\bar{D}_X g)(Y,Z) = 2\overset{*}{\eta}_2(X)g(Y,Z) + \overset{*}{\eta}_2(Y)g(X,Z) + \overset{*}{\eta}_2(Z)g(X,Y).$$

References

1. Agashe, N.S. and Chafle, M.R. (1992). A semi-symmetric non-metric connection in a Riemannian manifold, Indian J.Pure Appl. Math., 23: 399-409.
2. Agashe, N.S. and Chafle, M.R. (1994). On submanifold of a Riemannian manifold with a semi-symmetric non-metric connection, Tensor, N.S., 55:120-130.
3. Barua, B. and Mukhopahyay, S. (1992). A sequence of semi-symmetric connection of Riemannian manifold, Proceeding of seventh national seminar on Finshler Lagranges and Hamiltonian space, Brasav, Romania.
4. Chaubey, S.K. and Ojha, R.H. (2012). On a semi-symmetric non-metric connection,Filomat, 26: 63-69.
5. Chaubey, S.K. and Yieldiz, A. (2019). Riemannian manifolds admitting a new type of semi-symmetric non-metric connection, Turk. J. Math., 43:1887-1904.
6. De, U.C. and Kamilya, D. (1995). Hypersurfaces of Riemannian manifolds with semi-symmetric non-metric connection, J. Indian, Inst. Sci., 75:707-710.
7. De, U.C. and Biswas, S. (1996/1997). On a type semi-symmetric non-metric connection on a Riemannian manifold, Istanbul Univ. Math. Derge, 237-243.
8. De, U.C. and Sengupta, J. (2000). On a type of semi-symmetric non-metric connection on a Riemannian manifold,Bull. Cal. Math. Sock,92(5):375-384.
9. De, U.C., Yum, Y.L. and Zhao, P.B. (2016). A special type of semi-symmetric non-metric connection on a Riemannian manifold,FactUniversities (Nis), Ser. Math. Inform, 31(2): 529-541.
10. Eisenhert, L.P. (1961). Continuous group of transformation, Dover Publication, Inc. New York.
11. Friedmann, A. and Schouten, J. A. (1924). Uber die Geometric der halbsymmetrischen, Ubertragung. Math. Zeitschr, 21:211-223.
12. Folland, G.B. (1970). Weyl manifold,J. Diff. Geometry, 145-153.
13. Hayden, H. A. (1932). Subspaces of a space with torsion,Proc. London Math. Soc., 34: 27-50.
14. Han, Y., Yum, H.T. and Zhao, P. (2013). Some invariants of quarter-symmetric metric connection under the projective transformation, Filomat, 27(4): 679-691.

15. Kumar, A. and Chaubey, S.K. (2010). A semi-symmetric non-metric connection in a generalized co-symplectic manifold, Int J. of Math. Analysis, 17(4):809-817.

16. Liang, Y. (1994). On semi-symmetric recurrent metric connection, Tensor N.S., 55:107-112.

17. Melhotra, V. (2012). On a type of semi-symmetric non-metric connection on a Rimanniann manifold, Journal of Progressive science, 3(1):130-133.

18. Mishra, R. S. (1984). Structure on a differentiable manifold and their applications, Chandrama Prakashan, 50A. Balrampura House, Allahabad, India.

19. Pak, E. (1969). On the pseudo-Riemannain spaces, J. of Korean Math. Soc., 6:23-31.

20. Prasad, B., Dubey, D. D. and Yadav, R. P. S. Yadav (2011). On a semi-symmetric non-metric connection on a Riemannian manifold, J. Nat. Acad. Math., 25:53-68.

21. Prasad, B. (1994). On a semi-symmetric non-metric connection in SP-Sasakian manifold,Istanbul Uni. Fen. Fak. Mat. Der., 53:77-80.

22. Prasad, B. and Verma, R.K. (2004). On a type of semi-symmetric non-metric connection on a Riemannian manifold, Bull. Cal. Math. Soc., 96(5):483-488.

23. Prasad, B., Verma, R. K. and De, U. C. (2008). On a type of a semi-symmetric connection on a Riemannian manifold, Bull. Cal. Math. Soc., 100(3):227-240.

24. Prasad, B. and Doulo, K. (2015). On a semi-symmetric non-metric connection on a Riemannian manifold, Journal of Progressive Science, 6(1):36-48.

25. Prasad, B., Kumar, Raj and Singh, R. (2021). Semi-symmetric non-metric connection on a Riemannian manifold, JETIR, 8(3):100-108.

26. Prasad, B. and R. P. S. Yadav. Riemannian manifold admitting a new type of semi-symmetric non-metric connection, communicated for publication.

27. Sengupta, J., De, U.C. and Binh, T.Q. (2000). On a type of semi-symmetric non-metric connection on a Riemannian manifold. Indian J. Pure Appl. Math., 1(20):657-1670.

28. Smaranda, D. (1983). On projective transformations of symmetric connection with a recurrent projective tensor field, Proc. Nat. Call. on Geometry and Topology, 323-329.

29. Tripathi, M. M. (2008). A new connection in a Riemannian manifold, Int. Elec. J. of Geom., 1(1):15-24.

30. Yano, K. (1970). Integral formulas in a Riemannian geometry, Marcal Dekkar Ine.

31. Yano, K. (1970). On semi-symmetric connections, Revue Roumaine de Math. Pure Appl., 15:1579-1586.

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