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Generalized semi-pseudo Ricci-symmetric admitting W_2 −Ricci tensor

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Abstract

The present paper deals with the study of generalized semi-pseudo Ricci-symmetric on W_2 −Ricci tensor and obtained some interesting results.

Keywords and phrases: Pseudo Ricci symmetric, semi-pseudo Ricci-symmetric, generalized semi-pseudo Ricci-symmetric, W_2 −Ricci tensor.

1.Introduction

The study of Riemann symmetric manifolds began with the work of Carten (1926). According to Carten, a Riemannian manifold is said to be the locally symmetric if its curvature tensor R satisfies the condition $DR = 0$, where D is the covariant differential operator with respect to the metric g. During the last six decades the notion of locally symmetric manifolds have been weekend by many investigators in several ways to a different extent such as recurrent manifolds Walker (1950), semi-symmetric manifolds Szabo (1982), Pseudo symmetric manifold Chaki (1987) and Deszcz (1992) etc. If the Ricci tensor Ric of type of (0,2) in a Riemannian manifold M satisfies the relation $D.Ric = 0$, then Ric is said to be Ricci symmetric. The notion of Ricci symmetric has been studied extensively by many authors in several ways to a different extent, viz., Ricci recurrent manifolds Patterson (1952), Ricci semi-symmetric manifold Szabo (1982), Pseudo Ricci-symmetric manifold Chaki (1988), Deszcz (1989) and Weakly Ricci symmetric manifold Tamassy and Binh (1992).

Let Q be the symmetric endomorphism corresponding to the Ricci tensor defined by $Ric(X,Y) = g(QX,Y)$ for all vector fields $X$and Y.

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensorRic of type (0,2) of the manifold is non-zero satisfies the condition Chaki (1988)

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(X,Y), \tag{1.1}$$

for all X, Y and Z, where A is nowhere vanishing. The pseudo Ricci symmetric manifolds have been studied by Chaki and Chakrabarti (1993), Chaki and Maity (1998), Arslan et.al. (2001), De *et.al*. (2010), Özen (2011) and many others.

In 1994, Chaki and Koley introduced the notion of generalized pseudo Ricci symmetric manifold which is defined as follows:

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + B(Y)Ric(X,Z) + D(Z)Ric(X,Y), \tag{1.2}$$

where A, B and D are non-zero 1-forms. Such a manifold was denoted by authors as $(GPRS)_n$.

In 1993, Tarafdar and Jawarneh introduced another type of non-flat Riemannian manifold whose Ricci tensor Ric of the type (0,2) satisfies the condition:

$$(D_X Ric)(Y,Z) = A(Y)Ric(X,Z) + A(Z)Ric(X,Y). \tag{1.3}$$

Such a manifold were called by them a semi-pseudo Ricci-symmetric manifold and n-dimensional of this kind was denoted by $(SPRS)_n$. Some contribution in this direction is due to Prasad (1998), Tarafdar et.al. (2011) and others discussed some properties on different structures. Pseudo Ricci symmetric manifolds, generalized pseudo Ricci symmetric manifolds and semi-pseudo Ricci symmetric manifolds have great importance in the general theory of Relativity. Considering this aspect Jawarneh and Tashtoush in 2012 generalized the notion of semi-pseudo Ricci symmetric manifold by the expression:

$$(D_X Ric)(Y,Z) = A(Y)Ric(X,Z) + B(Z)Ric(X,Y). \tag{1.4}$$

where A and B are two non-zero 1-forms defined by

$$A(X) = g(X,\rho_1) \text{and} B(X) = g(X,\rho_2) \tag{1.5}$$

Such a manifold was called by them a generalized semi-pseudo Ricci-symmetric manifold and denoted by $G(SPRS)_n$. In particular, if $A = B$ then equation (1.4) reduces to semi-pseudo Ricci-symmetric manifold Tarafdar and Jawarneh (1993). This justified the name generalized semi-pseudo Ricci-symmetric manifold defined by the equation (1.4) and the symbol for it. Recently Halder and Bhattacharyya (2017) studied this structure under the title “Some properties of generalized semi-pseudo Ricci symmetric manifold” and obtained various geometrical properties.

In a Riemannian manifold Ricci tensor is Ricci recurrent Pattersion (1952) if

$$(D_Z Ric)(X,Y) = A(Z)Ric(X,Y). \tag{1.6}$$

Moreover, in a Riemannian manifold the Ricci tensor are Codazzi type and Cyclic Ricci tensor if the following condition holds Hicks (1965):

$$(D_Z Ric)(X,Y) = (D_X Ric)(Y,Z) \tag{1.7}$$

and

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0. \tag{1.8}$$

In 1970 Pokhariyal and Mishra were introduced new tensor field called W_2 and E tensor fields in a Riemannian and studied their properties. According to them a W_2 −ccurvature tensor field on a manifold $(M^n, g), n > 2$, is defined as:

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{1.9}$$

In this connection it may be mentioned that Pokhariyal and Mishra (1970 and 1971) and Pokharyal (1972) introduced some new curvature tensors defined on the line of Weyl projective curvature tensor. Thus W_2 −ccurvature tensor possesses the properties almost similar to the Weyl projective curvature tensor. The W_2 −ccurvature tensor have also been studied by various authors on different structure such as Prasad (1997), Prakasha (2010), Malik and De (2014), Hui (2012) $\ddot{O}$zen et.al. (2019) and many others. In this paper it is shown that the scalar curvature of $G(SPRS)_n$ with $\overline{W}_2$ −flat Ricci tensor is not necessarily a constant. Further a necessarily and sufficient condition is obtained for $\overline{W}_2$ − Ricci tensor to be conservative. The question whether in $G(SPRS)_n$ the tensor $\overline{W}_2$ −Ricci tensor can be Codazzi type or a cyclic type tensor has been answered positive. Finally, it is shown that if $G(SPRS)_n$ admits a unit parallel vector field then V cannot be orthogonal to the basis vector fields $\rho_1$and ρ_2.

2. $G(SPRS)_n$with $\overline{W}_2$ −flat Ricci tensor

In this section, we denoted the contracted W_2 −curvature tensor which is type of (0,2) as $\overline{W}_2$ and we call it $\overline{W}_2$ −Ricci tensor. Now contracting equation (1.9), we get

$$\overline{W}_2(X,Y) = \frac{n}{n-1}\left[Ric(X,Y) - \frac{r}{n}g(X,Y)\right]. \tag{2.1}$$

Here, we assume that $\overline{W}_2$ −flat. Hence from (2.1), we get

$$Ric(X,Y) = \frac{r}{n}g(X,Y). \tag{2.2}$$

From (2.2), we get

$$(D_Z Ric)(X,Y) = \frac{1}{n}g(X,Y)(D_Z r). \tag{2.3}$$

Here we assume that the manifold is $G(SPRS)_n$. Hence we have from (1.4)

$$(D_Z Ric)(X,Y) = A(X)Ric(Z,Y) + B(Y)Ric(Z,X). \tag{2.4}$$

In view of (2.2), (2.3) and (2.4), we obtain

$$(D_Z r)g(X,Y) = r[A(X)g(Z,Y) + B(Y)g(Z,X)]. \tag{2.5}$$

Contracting equation (2.5) with respect to $X$and Y, we get

$$(D_Z r)n = r[A(Z) + B(Z)]. \tag{2.6}$$

Again contracting (2.4) with respect to $X$and Y, we get

$$(D_Y r) = A(QY) + rB(Y). \tag{2.7}$$

Now replacing Y by Z in (2.7), we get

$$(D_Z r) = A(QZ) + rB(Z). \tag{2.8}$$

From (2.6) and (2.8), we get

$$r = \frac{n.A(QZ)}{A(Z)-(n-1)B(Z)}, \quad A(Z) - (n-1)B(Z) \neq 0. \tag{2.9}$$

Thus, we have the following theorem:

Theorem (2.1): For $\overline{W}_2$ −flat $G(SPRS)_n$, relation (2.9) show that r is not necessarily a constant, provided $A(Z) - (n-1)B(Z) \neq 0$.

3. $G(SPRS)_n$with $\overline{W}_2$ −Ricci tensor as conservative

In this section, we denoted the contracted $\overline{W}_2$ is conservative Hicks (1969)

$$div\overline{W}_2 = 0. \tag{3.1}$$

Now differentiating covariantly (2.1), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}\left[(D_Z Ric)(X,Y) - \frac{1}{n}(D_Z r).g(X,Y)\right]. \tag{3.2}$$

Contracting (2.4) over $X$and Y, we get

$$(D_Z r) = A(QZ) + rB(Z). \quad (3.3)$$

From (2.4), (3.2) and (3.3), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}\Big[A(X)g(Z,Y) + B(Y)g(Z,X) - \frac{1}{n}A(QZ)g(X,Y)$$

$$-\frac{1}{n}B(QZ)g(X,Y)\Big]. \quad (3.4)$$

This gives on contraction

$$(div\overline{W}_2)(Y) = \frac{n}{n-1}\Big[\Big(\frac{n-1}{n}\Big)A(QY) + B(Y).r - \frac{1}{n}B(QY)\Big]. \quad (3.2)$$

Hence in view of (3.1) and (3.5), we can state the following theorem:

Theorem (3.1): For a $G(SPRS)_n$ contracted W_2 −curvature tensor $\overline{W}_2$ be divergence free if and only if $\left(\frac{n-1}{n}\right)A(QY) + B(Y).r - \frac{1}{n}B(QY) = 0$.

4. $G(SPRS)_n$ with $\overline{W}_2$ −Ricci tensor as Ricci -recurrent

In this section, we assume that $\overline{W}_2$ be Ricci recurrent, then from (1.6), we get

$$(D_Z\overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y), \quad (4.1)$$

whereα is 1-form.

Thus in view of (2.1), (3.4) and (4.1), we get

$$\alpha(Z)\Big[Ric(X,Y) - \frac{r}{n}g(X,Y)\Big] = A(X)Ric(Z,Y) + B(Y)Ric(Z,X) -$$

$$\frac{1}{n}A(QZ)g(X,Y) - \frac{1}{n}B(QZ)g(X,Y). \quad (4.2)$$

Contracting equation (4.2) with respect to $X$and Z, we get

$$\alpha(QY) - \frac{r}{n}\alpha(Y) = \Big(\frac{n-1}{n}\Big)A(QY) + B(Y).r - \frac{1}{n}B(QY).$$

Replacing above equation $Y$by Z, we get

$$\alpha(QZ) - \frac{r}{n}\alpha(Z) = \Big(\frac{n-1}{n}\Big)A(QZ) + B(Z).r - \frac{1}{n}B(QZ). \quad (4.3)$$

Let us take $\alpha(Z) = A(Z)$ in (4.3)

$$\alpha(QZ) - rA(Z) = \frac{r}{n}B(Z) - B(QZ). \quad (4.4)$$

Hence from (4.4), we have the following theorem:

Theorem (4.1): For a $G(SPRS)_n$, if $\overline{W}_2$ −Ricci tensor is recurrent with recurrence vector generated by the 1-form A. A necessary and sufficient condition $-\frac{r}{n}$ be an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector $\rho_1$where $g(X,\rho_1) = A(X)$.

and only if $\left(\frac{n-1}{n}\right)A(QY) + B(Y).r - \frac{1}{n}B(QY) = 0$.

5. $G(SPRS)_n$with$\overline{W}_2$ −Ricci tensor as Codazzi type and Cyclic type Ricci tensor

Here, we assume that r is constant scalar curvature then from (3.2), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}(D_Z Ric)(X,Y). \tag{5.1}$$

Using (1.4) in (5.1), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}[A(X)Ric(Z,Y) + B(Y)Ric(Z,X)]. \tag{5.2}$$

If $\overline{W}_2$ be Codazzi type Ricci tensor, then from (1.7) and (5.2), we have

$$A(X)Ric(Z,Y) + B(Y)Ric(Z,X) = A(Y)Ric(X,Z) + B(Z)Ric(X,Y). \tag{5.3}$$

Contracting (5.3) over $X$and Y, we get

$$B(QZ) = rB(Z). \tag{5.4}$$

Theorem (5.1): For an $G(SPRS)_n$ with constant scalar curvature, if $\overline{W}_2$ −Ricci tensor is Coddazi type then r is an eigenvalue if the Ricci tensor Ric corresponding to the eigenvector ρ_2 where $g(X,\rho_2) = B(X)$.

Again, suppose r is constant and $\overline{W}_2$ is covariantly constant, then

$$(D_Z S)(X,Y) = 0. \tag{5.5}$$

From (1.4) and (5.5), we get

$$A(QY) + rB(Y) = 0. \tag{5.6}$$

$Y$is replaced by Z in (5.6), we get

$$A(QZ) = -rB(Z). \tag{5.7}$$

Hence, we have the following theorem:

Theorem (5.2): For an $G(SPRS)_n$ with constant scalar curvature and covariantly constant $\overline{W}_2$ −Ricci tensor, the Ricci tensor is covariantly constant and $A(Z)$ and $B(Z)$ are related by $A(QZ) = -rB(Z)$.

Next, we assume that $\overline{W}_2$ −Ricci tensor is a cyclic type tensor. Hence in view of (1.8), we get

$$(D_X\overline{W}_2)(Y,Z) + (D_Y\overline{W}_2)(Z,X) + (D_Z\overline{W}_2)(X,Y) = 0. \tag{5.8}$$

Thus, from (5.2) and (5.8), we get

$$A(X)Ric(Z,Y) + A(Y)Ric(X,Z) + A(Z)Ric(Y,X) = B(X)Ric(Y,Z) + B(Y)Ric(Z,X) + B(Z)Ric(X,Y). \tag{5.9}$$

Contracting (5.9) over $X$and Y, we get

$$A(QZ) + \frac{r}{2}A(Z) = -B(QZ) - \frac{r}{2}B(Z). \tag{5.10}$$

Equation (5.10) can be put as

$$Ric(Z,\rho_1) + Ric(Z,\rho_2) = -\frac{r}{2}g(Z,\rho_1) - \frac{r}{2}g(Z,\rho_2),$$

or $\quad Ric(Z,\rho_1 + \rho_2) = -\frac{r}{2}g(Z,\rho_1 + \rho_2),$

or $\quad Ric(Z,\rho) = -\frac{r}{2}g(Z,\rho), \qquad$ where $\rho = \rho_1 + \rho_2$. $\quad$ (5.11)

Hence, in view of (5.11), we can state the following theorem:

Theorem (5.3): For an $G(SPRS)_n$ with constant scalar curvature, if $\overline{W}_2$ −Ricci tensor is cyclic type Ricci tensor then $-\frac{r}{2}$ is an eigenvalue if the Ricci tensor Ric corresponding to the eigenvector ρ.

References

1. Arslan, K., Ezertas, R. Murathan, C. and Özgür, C. (2001). On pseudo-Ricci symmetric warped product manifolds, Differ. Geom. Dyn. Syst, 3:1-5.
2. Carten, E. (1926). Surune classe remarquable d'espaces de Riemannian, Bull. Soc. Math. France, 54:214-264.
3. Chaki, M. C. (1987). On pseudo-symmetric manifolds, "AL. I. CUZA" Din Iasi., 33:53-58.
4. Chaki, M. C. (1988). On pseudo Ricci symmetric manifold, Bulg. J. Phys., 15:526-531.
5. Chaki, M.C. and Chakrabarti, P. (1993). On Conformally flat pseudo Ricci symmetric manifold, Tensor N.S., 52:217-222.
6. Chaki, M. C. and Koley, S. (1994). On generalized pseudo Ricci symmetric manifolds, Periodica Math. Hung., 28:123-129.
7. Chaki, M.C., and Maity, R.K. (1998). On Umbilically hypersurfaces of a Conformally flat pseudo Ricci symmetric manifold, Tensor N.S., 60: 254-257.
8. Deszcz, R. (1989). On Ricci-pseudo symmetric warped product manifolds. Demonstratio Math.. 22: 1053-1065.
9. Deszcz, R. (1992). On pseudo-symmetric space,Bull. Soc. Math. belg. Ser. A., 44(1):1-34.
10. De, U. C., Murathan, C. and Özgür, C. (2010). Pseudo-symmetric and pseudo Ricci symmetric warped product manifold, Commun. Korean Math. Soi., 25:615-626.
11. Hicks, N.J. (1965): Notes on differential Geometry. D. Van Nostrand company Inc. Princeton, New York.
12. Hui, S.K. (2012). On generalized quasi-Einstein manifold admitting W_2 −ccurvature tensor, Int. Jour. of Math. Analy., 6:1115-1121.
13. Halder, K. and Bhattacharyya, A. (2017). Some properties of generalized semi-pseudo Ricci-symmetric manifold, Palestine J. of Math., 6(Special issue II):348-354.
14. Jawarneh, Musa A. and Tashtoush, Mohommad A. (2012). Generalized semi-pseudo Ricci symmetric manifold. International Math. Forum. 7(6): 297-303.
15. Mallick, S., and De, U. C. (2014). Spacetimes admitting W_2 −ccurvature tensor, Int. Jour. Of geom. Math. In modern physics, 11(4): DOI: 10.1142/S0219887814500303.
16. $\ddot{O}$zen, F. (2011). On pseudo projective Ricci symmetric manifolds, Int. J. Pure Appl. Math., 72:249-258.
17. Patterson, E.M. (1952). Some theorem on Ricci-recurrent space, J. London. Math. Soc., 27:287-295.
18. Pokhariyal, G.P. and Mishra, R. S. (1970). Curvature tensor and their relativistic significance I, Yakohama, Math. J., 18:105-108.
19. Pokhariyal, G.P. and Mishra, R. S. (1971). Curvature tensor and their relativistic significance II, Yakohama, Math. J., 19:97-103.
20. Pokhariyal, G.P. and Mishra, R. S. (1972). Curvature tensor and their relativistic significance III, Yakohama, Math. J., 20:115-119
21. Prasad, B. (1997). W_2-curvature tensor Kenmotsu manifold, Indian Journal of Mathematics, 39(3):287-291.
22. Prasad, B. (1998). On semi-pseudo symmetric and semi-pseudo Ricci-symmetric Kenmotsu manifold, Indian Journal of Mathematics. 40(3):347-351.
23. Prakasha, D.G. (2010). On generalized W_2-recurrent $(LCS)_n$ manifolds, J.T.S., 4:33-40.

24. Szabo, Z.I. (1982). Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, J. diff. Geom., (17):531.

25. Tamassy, L. and Binh, T.Q. (1992). On weakly symmetric and weakly projective symmetric Riemannian manifold, Differential geometry and Appl., Collog. Math. Soc. Jnos Bolyai, 663-670.

26. Tarafdar, M. and J., A.A. Musa (1993). Semi-Pseudo Ricci Symmetric manifold, J.Indian Inst.Sci.,73(6):591-596.

27. Tarafdar, M.; Sengupta, J. and Chakraborty, S. (2011). On semi-pseudo symmetric manifolds admitting a type of quarter symmetric metric connection, Int. J. contemp Math. Sci., 6(4):169-175.

28. Walkar, A.G. (1950). On ruses spaces of recurrent curvature, Proc Lond Math Soc., 52(2):36-64.

29. Zengin, E. O. and Bektas, E. (2019). Almost pseudo Ricci symmetric manifold admitting W_2 −Ricci tensor, Balkan J. of Geom. and its Applications, 24:90-99.

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