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On nearly W_2 −symmetric manifold

***Bhagwat Prasad and **JayPrakash Singh**

*Department of Mathematics, Shree M.M. Town PG College, Ballia-277001, U.P., India
**Department of Mathematics, School of Mathematics, Statistics and Computer Science,
Central University of South Bihar, Gaya, India

Email:*bhagwatprasad2010@rediffmail.comand**jpsmaths@gmail.com

Abstract

The present paper deals with nearly W_2 −symmetric manifold and find some geometrical properties.

Keywords and phrases-Nearly W_2- symmetric manifold, Codazzi type tensor, symmetric $\mathbb{Z}$ −tensor,Q −curvature tensor, semi-projective curvature tensor $\mathbb{P}$

1.Introduction

Riemannian symmetric spaces have been an important role in differential geometry. They were first classified by Carten (1926) in the late twenties and he also gave a classification of Riemannian symmetric spaces. In 1926, Carten studied the certain class of Riemannian spaces and introduced the notion of a symmetric spaces. According to him an n-dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where " , " represents the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M. This condition of locally symmetric is equivalent to the fact that the local geodesic symmetry $F(p)$ is an isometric Neill (1986) at every point $p \in M$.

After Carten the notion of locally symmetric manifolds has been reduced by many authors in several ways to a different extent such as pseudo-symmetric manifolds introduced by Chaki (1987), recurrent manifolds introduced by Walker (1957), conformally symmetric manifold introduced by Chaki and Gupta (1965), Conformally recurrent manifolds, introduced by Adati and Miyazawa (1967), weakly symmetric manifolds introduced by Tamassy and Binh (1989) etc.

A non-flat pseudo-Riemannian manifold (M^n, g) $(n > 2)$ is said to be a pseudo symmetric Chaki (1987) if its curvature tensor R of the type (0,4) satisfies the condition:

$$(D_X R)(Y,Z,U,V) = 2A(X)R(Y,Z,U,V) + A(Y)R(X,Z,U,V) + A(Z)R(Y,X,U,V) + A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X), \tag{1.1}$$

where A is a non-zero 1-form, ρ is a vector field by $g(X,\rho) = A(X)$ for all X, D denotes the operator of covariant differentiation with respect to the metric g and $R(Y,Z,U,V) = g(\mathbb{R}(Y,Z)\,U,V)$ where $\mathbb{R}$ is the curvature tensor of the type (1,3). The 1-form A is called the associated 1-form of the manifold. If $A = 0$ then the manifold reduces to a locally symmetry manifold in the sense of Carten.

An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. It should be taken into account that the notation of pseudo symmetric manifold studied in particular by Deszez (1987, 1992, 2002 and 2008) differ from that of Chaki (1987). Gray (1978) introduced two groups of Riemannian manifolds based on the covariant differentiation of the Ricci tensor. The first group contains all Riemannian manifold whose Ricci tensor Ric is a Codazzi type tensor, that is

$$(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z). \tag{1.2}$$

The second group contains all Riemannian manifolds whose Ricci tensor Ric is cyclic parallel, that is

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0. \tag{1.3}$$

In 1970, Pokhariyal and Mishra were introduced a new tensor field, called W_2 −curvature tensor in a Riemannian manifold and studied their properties. According to them a W_2 − curvature tensor in a Riemannian manifold (M^n, g) $(n > 2)$ defined by Pokhariyal and Mishra (1970)

$$W_2(X,Y)Z = \mathbb{R}(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(Y,Z)QX], \tag{1.4}$$

where$\mathbb{R}$ is the Riemannian curvature tensor of the type (1,3) and Ric is the Ricci tensor of the type (0,2). The W_2 −curvature tensor defined by Pokhariayal and Mishra (1970) has been widely studied in differential geometry as well as in the spacetime of general Relativity. Matsumoto and Mihai (1986) and Pokhariyal (1982) studied W_2 −curvature tensor for P-Sasakian manifold and Sasakian manifold and many others. Shaikh, Jana and Eyasmin (2007) have introduced the notion of weakly W_2 −symmetric manifold and obtained their properties. Moreover Yildiz and De (2010) have studied this tensor in Kenmotsu manifolds while Taleshian, Hosseinz-deh (2010) considered $N(K)$ −quasi Einstein manifolds satisfying the condition $\mathbb{R}(\xi, X)W_2 = 0$. Further Venkatesha, Bagewadi, and Kumar (2011) have studied LP-Sasakian manifold satisfying some condition on W_2 −curvature tensor. Ahsan and Ali (2017) have studied spacetime satisfying Einstein field equations with vanishing of W_2 −curvature as well as existence of killing and conformal killing vector fields. Further, the vanishing and divergence of W_2 −curvature have also been studied by in perfect fluid spacetime. The P −curvature have been defined by breaking W_2 −curvature tensor is skew-symmetric part and some of its properties have been studied Pokhariyal and Mishra (1970) and Pokhariyal (2000). Further, W_2 −curvature tensor was shown to extend Pirani formulation of gravitational waves to Einstein space Pokhariyal (1982).

In 2012, Mantica and Molinari defined a generalized (0,2) symmetric $\mathbb{Z}$ −tensor as

$$\mathbb{Z}(X,Y) = Ric(X,Y) + \phi g(X,Y), \tag{1.5}$$

whereϕ is an arbitrary scalar function. In Mantica and Suh (2012, 2014) various properties of the $\mathbb{Z}$ −tensor were pointed out.

Subsequently, Mantica and Suh (2013) introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ denoted by Q and defined by

$$Q(X,Y)W = \mathbb{R}(X,Y)W - \frac{\psi}{(n-1)}[g(Y,W)X - g(X,W)Y], \tag{1.6}$$

whereψ is an arbitrary scalar function. Such a tensor Q is known as Q −curvature tensor. The notion of Q −tensor is also suitable to excavated some-differential structures on a Riemannian manifold.

Recently, De and Majhi published a paper intitled "On semi-pseudo-projective symmetric manifolds" in which they introduced the semi-projective curvature tensor $\mathbb{P}$ of type (1,3) as follows

$$\mathbb{P}(X,Y)U = \mathbb{R}(X,Y)U - \frac{\phi}{(n-1)}[Ric(Y,U)X - Ric(X,U)Y], \tag{1.7}$$

whereϕ is an arbitrary scalar function. In particular if $\phi = 1$, then semi-projective curvature tensor reduced to projective curvature tensor $\mathbb{W}(X,Y)Z$ Neil (1983). This justify the name semi-projective curvature tensor. If $\phi = 0$, then semi-projective curvature tensor and curvature tensor are equivalent. Motivation of above studies in the present paper we define nearly W_2 −curvature tensor of the type (1,3) as follows:

$$\mathbb{W}_2(X,Y)U = \mathbb{R}(X,Y)U - \frac{\phi}{(n-1)}[g(Y,U)QX - g(X,U)QY], \tag{1.8}$$

whereϕ is an arbitrary scalar function. We prefer the name "Nearly W_2 −curvature tensor "; since it is clear that if $\phi = 1$, nearly W_2 −curvature tensor reduces to W_2 −curvature tensor Pokhariyal and Mishra (1970).

We can express (1.8) as follows:

$$'\mathbb{W}_2(X,Y,U,V) = {}'\mathbb{R}(X,Y,U,V) - \frac{\phi}{(n-1)}[g(Y,U)Ric(X,V) - g(X,U)Ric(Y,V)], \tag{1.9}$$

where $'\mathbb{W}_2(X,Y,U,V) = g(\mathbb{W}_2(X,Y)U,V)$ and $'\mathbb{R}(X,Y,U,V) = g(\mathbb{R}(X,Y)U,V)$.

A non-flat pseudo-Riemannian manifold (M^n, g) $(n > 2)$ is said to be pseudo nearly W_2 symmetric manifold if the nearly W_2 −curvature tensor of type (0,4) satisfies the condition:

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V) = 2A(X){}'\mathbb{W}_2(Y,Z,U,V) + A(Y){}'\mathbb{W}_2(X,Z,U,V) + A(Z){}'\mathbb{W}_2(Y,X,U,V) + A(U){}'\mathbb{W}_2(Y,Z,X,V) + A(V)\,{}'\mathbb{W}_2(Y,Z,U,X), \tag{1.10}$$

where$A$ is a non-zero 1-form, ρ is a vector field by $g(X,\rho) = A(X)$.

An n-dimensional pseudo nearly W_2 −symmetric manifold is denoted by $(PN\mathbb{W}_2S)_n$ where P stands for pseudo, $N\mathbb{W}_2$ stands for nearly W_2 and S stands for symmetric. If $\phi = 0$, then pseudo nearly W_2 symmetric manifold reduces to pseudo-symmetric manifolds introduced by Chaki (1987). Moreover, if $\phi = 1$, then pseudo nearly W_2 −symmetric manifold includes pseudo W_2 −symmetric manifold $(PW_2S)_n$ introduced by De and Ghosh (1994). The present paper organized as follows:

After introduction in section 2, we study some basic geometric properties of nearly W_2 −curvature. Section 3 is devoted to study of curvature property of $(PN\mathbb{W}_2S)_n$. In section 4, we study $(PN\mathbb{W}_2S)_n$ admitting Codazzi type Ricci tensor. Section 5 and 6 deals with Einstein $(PN\mathbb{W}_2S)_n$ and $(PN\mathbb{W}_2S)_n$ with $div\mathbb{W}_2 = 0$ respectively. Section 7 is devoted to study of $(PN\mathbb{W}_2S)_n$ admitting parallel vector fieldρ. Among others we prove that in $(PN\mathbb{W}_2S)_n$if the associated vector field ρ is unit vector field, then either the manifold reduces to a pseudo symmetric manifold or pseudo W_2 −symmetric manifold.

2. Preliminaries

Let Ric and r denote the Ricci tensor of the type (0,2) and the scalar curvature respectively and Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric, that is

$$g(QX,Y) = Ric(X,Y). \tag{2.1}$$

In this section, some basic formulae are derived, which will be useful to the study of $(PN\mathbb{W}_2S)_n$. In $(PN\mathbb{W}_2S)_n$, let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$. The Ricci tensor Ric is defined by $Ric(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)e_i, Y)$.

From (1.8) we can easily verify that the tensor $\mathbb{W}_2$ satisfies the following properties:

$$\left.\begin{array}{c}\mathbb{W}_2(X,Y)U + \mathbb{W}_2(Y,X)U = 0, \\ and \\ \mathbb{W}_2(X,Y)U + \mathbb{W}_2(Y,U)X + \mathbb{W}_2(U,X)Y = 0.\end{array}\right\} \tag{2.2}$$

From (1.8), we obtain

$$\sum_{i=1}^{n} {}'\mathbb{W}_2(X,Y,e_i,e_i) = \sum_{i=1}^{n} {}'\mathbb{W}_2(e_i,e_i,U,V) = 0, \tag{2.3}$$

$$\sum_{i=1}^{n} {}'\mathbb{W}_2(e_i,Y,U,e_i) = \left(1 + \frac{\phi}{n-1}\right) Ric(Y,U) - \frac{\phi r}{n-1} g(Y,U) = \mathbb{W}_3(Y,U), \tag{2.4}$$

$$\sum_{i=1}^{n} {}'\mathbb{W}_2(X,e_i,e_i,V) = (1-\phi)Ric(X,V) = \mathbb{W}_4(X,V), \tag{2.5}$$

where$r = \sum_{i=1}^{n} Ric(e_i,e_i)$, is the scalar curvature tensor.

$${}'\mathbb{W}_2(X,Y,U,V) + {}'\mathbb{W}_2(Y,X,U,V) = 0,$$

$${}'\mathbb{W}_2(X,Y,U,V) + {}'\mathbb{W}_2(X,Y,V,U) \neq 0,$$

$${}'\mathbb{W}_2(X,Y,U,V) - {}'\mathbb{W}_2(U,V,X,Y) \neq 0,$$

and

$${}'\mathbb{W}_2(X,Y,U,V) + {}'\mathbb{W}_2(Y,U,X,V) + {}'\mathbb{W}_2(U,X,Y,V) = 0. \tag{2.6}$$

Proposition (2.1): A Riemannian manifold is nearly W_2 −flat if and only if it is of constant curvature, provided the scalar curvature is non-zero.

Proof: Nearly W_2 −curvature tensor of the type (0,4) is given by

$${}'\mathbb{W}_2(X,Y,U,V) = {}'\mathbb{R}(X,Y,U,V) - \frac{\phi}{(n-1)}[g(Y,U)Ric(X,V) - g(X,U)Ric(Y,V)], \tag{2.7}$$

whereϕ is an arbitrary scalar function. If nearly W_2 −curvature vanishes then

$${}'\mathbb{R}(X,Y,U,V) = \frac{\phi}{(n-1)}[g(Y,U)Ric(X,V) - g(X,U)Ric(Y,V)]. \tag{2.8}$$

Contracting (2.8), we get

$$Ric(Y,U) = \frac{\phi r}{n-1+\phi} g(Y,U). \tag{2.9}$$

Again contraction (2.9), we get

$$r.(n-1).(\phi-1) = 0. \tag{2.10}$$

Hence, from (2.10), we get either $r = 0$or $\phi = 1$. For $r = 0$, we get

$$'\mathbb{W}_2(X,Y,U,V) = {}'\mathbb{R}(X,Y,U,V),$$

that is the nearly W_2 −curvature tensor is equal to the curvature tensor $'\mathbb{R}$. Also for $\phi = 1$, the nearly W_2 −curvature tensor is equal to W_2 −curvature tensor. Thus we can say that nearly W_2 −flatness and W_2 −flatness are equivalent. It is known Mishra and Pokhariyal (1970) that in a Riemainnian manifold is $'W_2$ −flat if and only if it is space of constant curvature. Therefore a Riemannian manifold is nearly W_2 −flat if and only if it is a manifold of constant curvature, provided the scalar curvature is non-zero. This completes the proof.

Proposition (2.2): If the nearly W_2 −curvature tensor is symmetric in the sense of Carten, then the manifold reduces to Ricci symmetric.

Proof: From (1.8), we get

$$\mathbb{W}_2(Y,Z)U = \mathbb{R}(Y,Z)U - \frac{\phi}{(n-1)}[g(Z,U)QY - g(Y,U)QZ], \tag{2.11}$$

whereϕ is an arbitrary scalar function.

Differentiating covariantly (2.11), we get

$$(D_X\mathbb{W}_2)(Y,Z)U = (D_X\mathbb{R})(Y,Z)U - \frac{d\phi(X)}{(n-1)}[g(Z,U)QY - g(Y,U)QZ] - \frac{\phi}{(n-1)}[g(Z,U)(D_XQ)Y - g(Y,U)(D_XQ)Z]. \tag{2.12}$$

Here we assume that nearly W_2 −curvature tensor is symmetric, hence from (2.12), we get

$$(D_X\mathbb{R})(Y,Z)U = \frac{d\phi(X)}{(n-1)}[g(Z,U)QY - g(Y,U)QZ] + \frac{\phi}{(n-1)}[g(Z,U)(D_XQ)Y - g(Y,U)(D_XQ)Z]. \tag{2.13}$$

Contraction (2.13), with respect to U, we get

$$(D_XRic)(Y,Z) = 0. \tag{2.14}$$

This proves the theorem.

Proposition (2.3): For a nearly W_2 −curvature tensor $(div\mathbb{W}_2)(Y,Z)U = (divW_2)(Y,Z)U$ if and only if r is constant, provided ϕ is constant.

Proof: From (1.8), we get

$$\mathbb{W}_2(Y,Z)U = \mathbb{R}(Y,Z)U - \frac{\phi}{(n-1)}[g(Z,U)QY - g(Y,U)QZ]. \tag{2.15}$$

Differentiating (2.15) covariantly with respect to X, we get

$$(D_X\mathbb{W}_2)(Y,Z)U = (D_X\mathbb{R})(Y,Z)U - \frac{d\phi(X)}{(n-1)}[g(Z,U)QY - g(Y,U)QZ] - \frac{\phi}{(n-1)}[g(Z,U)(D_XQ)Y - g(Y,U)(D_XQ)Z]. \tag{2.16}$$

Contracting (2.16) with respect to X, we get

$$(div\mathbb{W}_2)(Y,Z)U = (div\mathbb{R})(Y,Z)U - \frac{1}{(n-1)}[g(Z,U)\phi(QY) - g(Y,U)\phi(QZ)]$$

$$-\frac{\phi}{(n-1)}[g(Z,U)(divQ)(Y) - g(Y,U)(divQ)(Z)]. \quad (2.17)$$

Making the use of $(divQ)(Y) = \frac{1}{2}dr(Y)$ and ϕ is constant in (2.17), we get

$$(div\mathbb{W}_2)(Y,Z)U = (div\mathbb{R})(Y,Z)U - \frac{\phi}{2(n-1)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \quad (2.18)$$

Equation (2.18) can be written as

$$(div\mathbb{W}_2)(Y,Z)U = (divW_2)(Y,Z)U + \frac{(1-\phi)}{2(n-1)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \quad (2.19)$$

From (2.19), we see that if

$$(div\mathbb{W}_2)(Y,Z)U = (divW_2)(Y,Z)U, \quad (2.20)$$

then

$$\frac{(1-\phi)}{2(n-1)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)] = 0. \quad (2.21)$$

Hence from (2.21), we see that r is constant. Conversely, if r is constant then from (2.19), we get (2.20). this proves the preposition (2.3).

3. Bianchi's 2nd identity of $(PN\mathbb{W}_2S)_n$, $(n > 2)$

In this section we prove that in a $(PN\mathbb{W}_2S)_n$, the nearly W_2 −curvature $\mathbb{W}_2(Y,Z,U,V)$ satisfies Bianchi's 2nd identity, that is

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V) + (D_Y{}'\mathbb{W}_2)(Z,X,U,V) + (D_Z{}'\mathbb{W}_2)(X,Y,U,V) = 0.$$

From (1.10), we get

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V) + (D_Y{}'\mathbb{W}_2)(Z,X,U,V) + (D_Z{}'\mathbb{W}_2)(X,Y,U,V) =$$

$$A(U)[\,'\mathbb{W}_2(Y,Z,X,V) + \,'\mathbb{W}_2(Z,X,Y,V) + \,'\mathbb{W}_2(X,Y,Z,V)] +$$

$$A(V)[\,'\mathbb{W}_2(Y,Z,U,X) + \,'\mathbb{W}_2(Z,X,U,Y) + \,'\mathbb{W}_2(X,Y,U,Z)]. \quad (3.1)$$

Using (2.6) in (3.1), we get

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V) + (D_Y{}'\mathbb{W}_2)(Z,X,U,V) + (D_Z{}'\mathbb{W}_2)(X,Y,U,V) =$$

$$A(V)[\,'\mathbb{W}_2(Y,Z,U,X) + \,'\mathbb{W}_2(Z,X,U,Y) + \,'\mathbb{W}_2(X,Y,U,Z)]. \quad (3.2)$$

Again using (1.9) in (3.2), we get

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V) + (D_Y{}'\mathbb{W}_2)(Z,X,U,V) + (D_Z{}'\mathbb{W}_2)(X,Y,U,V) = 0. \quad (3.3)$$

Hence from (3.3), we can state the following:

Theorem (3.1): A nearly W_2 −curvature $\mathbb{W}_2(Y,Z,U,V)$, $(n > 2)$ satisfies Bianchi's 2nd identity.

4. $(PN\mathbb{W}_2S)_n,\ (n>2)$ with Codazzi type Ricci tensor

From (1.9), we get

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V)+(D_Y{}'\mathbb{W}_2)(Z,X,U,V)+(D_Z{}'\mathbb{W}_2)(X,Y,U,V)=$$
$$-\frac{\phi}{n-1}[(D_XRic)(Y,V)g(Z,U)-(D_XRic)(Z,V)g(Y,U)+(D_YRic)(Z,V)g(X,U)$$
$$-(D_YRic)(X,V)g(Z,U)+(D_ZRic)(X,V)g(Y,U)-(D_ZRic)(Y,V)g(X,U)]$$
$$-\frac{(X\phi)}{n-1}[Ric(Y,V)g(Z,U)-Ric(Z,V)g(Y,U)]-\frac{(Y\phi)}{n-1}[Ric(Z,V)g(X,U)$$
$$-Ric(X,V)g(Z,U)]-\frac{(Z\phi)}{n-1}[Ric(X,V)g(Y,U)-Ric(Y,V)g(X,U)]. \tag{4.1}$$

We assume that $(PN\mathbb{W}_2S)_n$ admits Codazzi type Ricci tensor, then we have from (1.2) and (4.1), we get

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V)+(D_Y{}'\mathbb{W}_2)(Z,X,U,V)+(D_Z{}'\mathbb{W}_2)(X,Y,U,V)=$$
$$-\frac{(X\phi)}{n-1}[Ric(Y,V)g(Z,U)-Ric(Z,V)g(Y,U)]-\frac{(Y\phi)}{n-1}[Ric(Z,V)g(X,U)$$
$$-Ric(X,V)g(Z,U)]-\frac{(Z\phi)}{n-1}[Ric(X,V)g(Y,U)-Ric(Y,V)g(X,U)]. \tag{4.2}$$

From (3.3) and (4.2), we get

$$-\frac{(X\phi)}{n-1}[Ric(Y,V)g(Z,U)-Ric(Z,V)g(Y,U)]-\frac{(Y\phi)}{n-1}[Ric(Z,V)g(X,U)$$
$$-Ric(X,V)g(Z,U)]-\frac{(Z\phi)}{n-1}[Ric(X,V)g(Y,U)-Ric(Y,V)g(X,U)]=0. \tag{4.3}$$

Contracting (4.3) with respect to $Y$and$U$, we get

$$(X\phi)Ric(V,Z)=(Z\phi)Ric(X,V). \tag{4.4}$$

Again contracting (4.4) with respect to $Z$and$V$, we get

$$(X\phi)r=g(grad\phi,QX). \tag{4.5}$$

From (4.5), we get

$$Ric(grad\phi,X)=r.g(grad\phi,X). \tag{4.6}$$

Equation (4.6) says that r is an eigenvalue of Ric corresponding to the eigenvector for $grad\phi$.Thus we conclude the following theorem:

Theorem (4.1): For a $(PN\mathbb{W}_2S)_n,\ (n>2)$ with Codazzi type Ricci tensor, r is an eigenvalue of Ric corresponding to the eigenvector for $grad\phi$.

If ϕ is constant, then from (4.1) and (3.3), we get

$$(D_XRic)(Y,V)g(Z,U)-(D_XRic)(Z,V)g(Y,U)+(D_YRic)(Z,V)g(X,U)$$
$$-(D_YRic)(X,V)g(Z,U)+(D_ZRic)(X,V)g(Y,U)-(D_ZRic)(Y,V)g(X,U)=0. \tag{4.7}$$

Contracting (4.7), we get

$$(D_X Ric)(Z,V) = (D_Z Ric)(X,V). \quad (4.8)$$

Hence we can state the following Corollary:

Corollary (4.1): In a $(PN\mathbb{W}_2S)_n, (n > 2)$ the Ricci tensor is of Codazzi type, provided ϕ is constant.

Now from (2.5), we get

$$\mathbb{W}_4(X,V) = (1-\phi)Ric(X,V). \quad (4.9)$$

Contracting (4.9), we get

$$\mathbb{W}_4 = (1-\phi)r. \quad (4.10)$$

In $(PN\mathbb{W}_2S)_n,\ n > 2)$ the nearly W_2 −curvature tensor satisfies the following relation:

$$(D_X{'}\mathbb{W}_2)(Y,Z,U,V) = 2A(X){'}\mathbb{W}_2(Y,Z,U,V) + A(Y){'}\mathbb{W}_2(X,Z,U,V) + A(Z){'}\mathbb{W}_2(Y,X,U,V) + A(U){'}\mathbb{W}_2(Y,Z,X,V) + A(V)\,{'}\mathbb{W}_2(Y,Z,U,X). \quad (4.11)$$

where$A$ is a non-zero 1-form, ρ is a vector field by $g(X,\rho) = A(X)$.

Contraction (4.11), we get

$$(D_X{'}\mathbb{W}_4)(Y,V) = 2A(X){'}\mathbb{W}_4(Y,V) + A(Y){'}\mathbb{W}_4(X,V) + {'}\mathbb{W}_4(Y,X,\rho,V) + {'}\mathbb{W}_4(Y,\rho,X,V) + A(V)\,{'}\mathbb{W}_4(Y,Z,U,X). \quad (4.12)$$

Again contraction (4.12), we get

$$D_X{'}\mathbb{W}_4 = 2A(X){'}\mathbb{W}_4(Y,V) + 2{'}\mathbb{W}_4(X,\rho) + 2{'}\mathbb{W}_3(X,\rho)\,. \quad (4.13)$$

From (4.9), (4.10), (4.13) and (2.4), we get

$$(1-\phi)dr(X) - d\phi(X)r = 2A(X)(1-\phi)r + 2(1-\phi)Ric(X,\rho) + 2\left[\left(1+\frac{\phi}{n-1}\right)Ric(X,\rho) - \frac{\phi}{n-1}rA(X)\right]. \quad (4.14)$$

Simplification of (4.14) yields

$$(1-\phi)dr(X) - d\phi(X)r = 2\left[(1-\phi)r - \frac{r\phi}{n-1}\right]A(X) + 2\left[(2-\phi) + \frac{\phi}{n-1}\right]Ric(X,\rho). \quad (4.15)$$

Hence, we have the following theorem:

Theorem (4.2): In $(PN\mathbb{W}_2S)_n,\ n > 2$ the following identity hold:

$$(1-\phi)dr(X) - d\phi(X)r = 2\left[(1-\phi)r - \frac{r\phi}{n-1}\right]A(X) + 2\left[(2-\phi) + \frac{\phi}{n-1}\right]A(QX).$$

5. Einstein $(PN\mathbb{W}_2S)_n,\ (n > 2)$

In this section we consider Einstein $(PN\mathbb{W}_2S)_n,\ (n > 2)$. Since for every Einstein manifold the scalar curvature r is constant, hence for Einstein $(PN\mathbb{W}_2S)_n$: we have $dr(X) = 0$. Therefore, from (4.15), we have

$$-d\phi(X)r = 2\left[(1-\phi)r - \frac{r\phi}{n-1}\right]A(X) + 2\left[(2-\phi) + \frac{\phi}{n-1}\right]Ric(X,\rho). \tag{5.1}$$

For Einstein manifold, we have $Ric(X,Y) = \frac{r}{n}g(X,Y)$,

$$-d\phi(X)r = 2(1-\phi)\left(1+\frac{2}{n}\right)A(X)r. \tag{5.2}$$

Hence ϕ is constant, then$r = 0$, provided $A(X) \neq 0$.

Hence we can state the following theorem:

Theorem (5.1): An Einstein $(PN\mathbb{W}_2S)_n,\ (n > 2)$ is of zero scalar curvature provided ϕ is constant.

If possible, let $(PN\mathbb{W}_2S)_n,\ (n > 2)$ be a space of constant curvature. Then we have

$$\mathbb{R}(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]. \tag{5.3}$$

where$k$ is a constant. Contracting X in (5.3), we get

$$Ric(Y,Z) = k(n-1)g(Y,Z). \tag{5.4}$$

Again, contraction (5.4), we get

$$r = kn(n-1). \tag{5.5}$$

Using (5.5) in (5.3), we get

$$\mathbb{R}(X,Y)Z = \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.6}$$

Since every space of constant curvature is Einstein manifold, then from Theorem (5.1), we get $r = 0$. Hence from (5.6) it follows that $\mathbb{R}(X,Y)Z = 0$, which is in admissible by definition. This gives the following theorem:

Theorem (5.2): A $(PN\mathbb{W}_2S)_n,\ (n > 2)$ cannot be of the constant curvature provided ϕ is constant.

6. $(PN\mathbb{W}_2S)_n,\ (n > 2)$with$div\mathbb{W}_2 = 0$

For $(PN\mathbb{W}_2S)_n,\ (n > 2)$, we have from (1.10), we get

$$(D_X{'}\mathbb{W}_2)(Y,Z,U,V) = 2A(X){'}\mathbb{W}_2(Y,Z,U,V) + A(Y){'}\mathbb{W}_2(X,Z,U,V) + A(Z){'}\mathbb{W}_2(Y,X,U,V) + A(U){'}\mathbb{W}_2(Y,Z,X,V) + A(V){'}\mathbb{W}_2(Y,Z,U,X), \tag{6.1}$$

where$A$ is a non-zero 1-form, ρ is a vector field by $g(X,\rho) = A(X)$.

Contracting of (6.1) gives

$$(div\mathbb{W}_2)(Y,Z,U) = 3A\big(\mathbb{W}_2(Y,Z,U)\big) + A(Y)\mathbb{W}_3(Z,U) - A(Z)\mathbb{W}_3(Y,U). \tag{6.2}$$

According to our assumption$div\mathbb{W}_2 = 0$, we get

$$3{'}\mathbb{W}_2(Y,Z,U,V) + A(Y)\mathbb{W}_3(Z,U) - A(Z)\mathbb{W}_3(Y,U) = 0. \tag{6.3}$$

Contracting (6.3) with respect to $Z$and U, we get

$$3\mathbb{W}_4(Y,V)+A(Y)\mathbb{W}_3(e_i,e_i)-A(e_i)\mathbb{W}_3(Y,e_i)=0. \tag{6.4}$$

From (6.4), (2.4) and (2.5), we get

$$Ric(Y,\rho)=\frac{[(n-2)\phi-(n-1)]}{(2-3\phi)(n-1)-\phi}.g(Y,\rho).r. \tag{6.5}$$

Equation (6.5) can be written as

$$Ric(Y,\rho)=\lambda g(Y,\rho), \tag{6.6}$$

where$\lambda=\frac{[(n-2)\phi-(n-1)]r}{(2-3\phi)(n-1)-\phi}$. Hence in view of (6.6), we have the following theorem:

Theorem (6.1): For a $(PN\mathbb{W}_2S)_n,\ (n>2)$ with $div\mathbb{W}_2=0$, λ is eigenvalue of the Ricci tensor Ric corresponding to the eigenvector ρ.

7. $(PN\mathbb{W}_2S)_n,\ (n>2)$ Admitting a parallel vector field

In this section, we obtain condition for a $(PN\mathbb{W}_2S)_n$ to be a $(PS)_n$or $(PW_2S)_n$. For this we require a notion of parallel vector field defined as follows:

A vector field V^* is said to parallel Ficken (1939) if

$$D_XV^*=0. \tag{7.1}$$

We now suppose that a $(PN\mathbb{W}_2S)_n,\ (n>2)$ admitting a unit parallel vector field ρ such that

$$D_X\rho=0. \tag{7.2}$$

Applying Ricci identity to (7.2), we have

$$\mathbb{R}(X,Y)\rho=0. \tag{7.3}$$

Contracting Y in (7.3), we get

$$Ric(X,\rho)=0. \tag{7.4}$$

From (2.5) and (7.4), we get

$$\mathbb{W}_4(X,\rho)=(1-\phi)Ric(X,\rho)=0. \tag{7.5}$$

Again definition of $(PN\mathbb{W}_2S)_n$, we have

$$(D_X{}'\mathbb{W}_2)(Y,Z,U,V)=2A(X)'\mathbb{W}_2(Y,Z,U,V)+A(Y)'\mathbb{W}_2(X,Z,U,V)+A(Z)'\mathbb{W}_2(Y,X,U,V)+A(U)'\mathbb{W}_2(Y,Z,X,V)+A(V)'\mathbb{W}_2(Y,Z,U,X), \tag{7.6}$$

where$A$ is a non-zero 1-form, ρ is a vector field by $g(X,\rho)=A(X)$.

Hence in view of (7.6), we get

$$(D_X\mathbb{W}_4)(Y,V)=\sum_{i=1}^{n}(D_X{}'\mathbb{W}_2)(Y,e_i,e_i,V)=\sum_{i=1}^{n}\{2A(X)'\mathbb{W}_2(Y,e_i,e_i,V)+$$

$$A(Y)'\mathbb{W}_2(X, e_i, e_i, V) + A(e_i)'\mathbb{W}_2(Y, X, e_i, V)$$
$$+ A(e_i)'\mathbb{W}_2(Y, e_i, X, V) + A(V)\,'\mathbb{W}_2(Y, e_i, e_i, X),$$

which gives

$$(D_X\mathbb{W}_4)(Y, V) = 2A(X)\mathbb{W}_4(Y, V) + A(Y)\mathbb{W}_4(X, V)$$
$$+'\mathbb{W}_2(\rho, X, U, V) + A(V)\mathbb{W}_4(Y, X). \quad (7.7)$$

Putting ρ for V in (7.7), we get

$$(D_X\mathbb{W}_4)(Y, \rho) = 2A(X)\mathbb{W}_4(Y, \rho) + A(Y)\mathbb{W}_4(X, \rho)$$
$$+'\mathbb{W}_2(\rho, X, U, \rho) + A(\rho)\mathbb{W}_4(Y, X). \quad (7.8)$$

From (7.7) and (7.8), we get

$$(D_X{}'\mathbb{W}_4)(Y, \rho) = A(\rho)\,'\mathbb{W}_4(Y, X). \quad (7.9)$$

Again from (7.5) and (7.9), we get

$$\mathbb{W}_4(Y, X). \quad (7.10)$$

Thus we have

$$(1 - \phi)Ric(X, Y) = 0. \quad (7.11)$$

Therefore either $\phi = 1$ or $Ric(X, Y) = 0$. For $\phi = 1$, $(PN\mathbb{W}_2S)_n$, $(n > 2)$ reduces to pseudo W_2 − symmetric manifold, that is $(PW_2S)_n$. Also, for $Ric(X, Y) = 0, (PN\mathbb{W}_2S)_n$, $(n > 2)$ reduces to pseudo symmetric manifolds that is, $(PS)_n$, $(n > 2)$. Therefore, we can state the following theorem:

Theorem (7.1): In a $(PN\mathbb{W}_2S)_n$, $(n > 2)$, if the associated vector field ρ is a unit parallel vector field, then either the manifold reduces to a pseudo symmetric manifold or pseudo W_2 −symmetric manifold.

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