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Study of G −projective curvature tensor on a Riemannian manifold

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Abstract

The object of the present paper is to study some properties of ‘ G −projective’curvature tensor and ‘G^* −projective’ curvature tensor in a Riemannian manifold which have been defined as

$$'G(Y,Z,U,T) = {}'R(Y,Z,U,T) - \frac{1}{2(n-1)}[g(Y,U)Ric(Z,T) - g(Y,T)Ric(Z,U)$$
$$-g(Z,U)Ric(Y,T) + g(Z,T)Ric(Y,U)]$$

and

$$G^*(Y,Z,U,T) = {}'R(Y,\mathrm{Z},U,T) - \frac{1}{2(n-1)}[g(Y,U)Z(\mathbb{Z},T) - g(Y,T)Z(\mathbb{Z},U)$$
$$-g(\mathbb{Z},U)Z(Y,T) + g(\mathbb{Z},T)Z(Y,U)],$$

where$'R$ is the curvature tensor,Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor Ric i.e $g(QY,Z) = Ric(Y,Z)$ and $\mathbb{Z}$ −tensor of type (0,2).

Keywords and phrases-W_2 −curvature tensor, G −projective curvature tensor, constant curvature and $'G^*$ −projective curvature tensor.

1.Introduction

In 1970, Pokhariyal and Mishra were introduced a new tensor field, called W_2 −curvature tensor in a Riemannian manifold and studied their properties. According to them a W_2 − curvature tensor in a Riemannian manifold (M^n, g) $(n > 2)$, is defined by the following expression:

$$W_2(Y,Z)U = R(Y,Z)U + \frac{1}{(n-1)}[g(Y,U)QZ - g(Z,U)QY]. \tag{1.1}$$

Equation (1.1) can be put as

$$'W_2(Y,Z,U,T) = {}'R(Y,Z,U,T) + \frac{1}{(n-1)}[g(Y,U)Ric(Z,T) - g(Z,U)Ric(Y,T)],$$

whereQY is the Ricci operator of type (1,1) defined by $g(QY,Z)=Ric(Y,Z)$, and

$$'W_2(Y,Z,U,T)=g(W_2(Y,Z)U,T) \text{ and } 'R(Y,Z,U,T)=g(R(Y,Z)U,T).$$

In this connection it may be mentioned that Pokhariyal and Mishra (1970 and (1971)) and Pokhariyal (1972) introduced some new curvatures defined on the line of Weyl projective curvature tensor. The geometrical and physical properties of W_2 −curvature tensor have been fairly widely studied by authors in different structures as Prasad (1997), Prakasha (2010), Malik and De (2014), Hui (2012), Zengin et al (2019), Ahsan (2017), Shenaw and Unal (2016) and many others.

On breaking of $'W_2(Y,Z,U,T)$ in two part viz.:

$$'M(Y,Z,U,T)=\frac{1}{2}['W_2(Y,Z,U,T)-'W_2(Y,Z,T,U)], \tag{1.2}$$

and

$$'N(Y,Z,U,T)=\frac{1}{2}['W_2(Y,Z,U,T)+'W_2(Y,Z,T,U)],$$

which are respectively skew-symmetric and symmetric in U and T.

From (1.1) and (1.2), it follows that

$$'M(Y,Z,U,T)='R(Y,Z,U,T)+\frac{1}{(n-1)}[g(Y,U)Ric(Z,T)-g(Z,U)Ric(Y,T)$$
$$-g(Y,T)Ric(Z,U)+g(Z,T)Ric(Y,U)]. \tag{1.3}$$

Further, the curvature tensor (1,3) satisfies the skew-symmetric and symmetric, as well as cyclic properties that are satisfied by the Riemannian curvature tensor. Some geometrical properties of this curvature tensor initiated by Ojha (1986) and called it as M −projective curvature tensor. In a recent papers Ghosh and De (1994), Prasad and Verma (2004), Singh (2009), Prakash (2010), Singh (2012), Chuabey and Ojha (2010), Pokhariyal (2020) and many workers.

In 1971, Pokhariyal and Mishra investigated following curvature tensor of type (0,4) as follows:

$$'W^*(Y,Z,U,T)='R(Y,Z,U,T)-\frac{1}{(n-1)}[g(Y,U)Ric(Z,T)-g(Z,U)Ric(Y,T)]. \tag{1.4}$$

From (1.4), we notice that (1.4) is skew-symmetric in U and T and it also satisfies cyclic property.

Now, breaking (1.4) in two parts, viz.:

$$'G(Y,Z,U,T)=\frac{1}{2}['W^*(Y,Z,U,T)-'W^*(Z,Y,U,T)], \tag{1.5}$$

and

$$'H(Y,Z,U,T)=\frac{1}{2}['W^*(Y,Z,U,T)+'W^*(Z,Y,U,T)],$$

which are respectively skew-symmetric and symmetric in Y and Z.

From (1.1) and (1.2), it follows that

$$'G(Y,Z,U,T)='R(Y,Z,U,T)-\frac{1}{2(n-1)}[g(Y,U)Ric(Z,T)-g(Y,T)Ric(Z,U)$$

$$-g(Z,U)Ric(Y,T) + g(Z,T)Ric(Y,U)]. \quad (1.6)$$

It can be easily seen that $'G(Y,Z,U,T)$ possesses all the skew symmetric and symmetric properties of $'R(Y,Z,U,T)$ as well as cyclic property. That is,

$$'G(Y,Z,U,T) + 'G(Y,Z,T,U) = 0,$$

$$'G(Y,Z,U,T) + 'G(Z,Y,U,T) = 0,$$

$$'G(Y,Z,U,T) - 'G(\mathrm{U},T,Y,Z) = 0,$$

$$'G(Y,Z,U,T) + 'G(Z,U,Y,T) + 'G(U,Y,Z,T) = 0.$$

Since the curvature tenors define in (1.6) is very close to M −projective curvature tensor, hence we call it as $'G$projective curvature tensor Kumar (2012).

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \le i \le n$. In a Riemannian manifold the Ricci tensor Ric is defined by $Ric(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)e_i,Y)$ and $r = \sum_{i=1}^{n} Ric(e_i,e_i)$, where r is the scalar curvature tensor.

From (1.6), we get

$$\sum_{i=1}^{n} 'G(Y,Z,e_i,e_i) = \sum_{i=1}^{n} 'G(e_i,e_i,U,T) = 0,$$

$$\tilde{G}(Z,U) = \sum_{i=1}^{n} 'G(e_i,Z,U,e_i) = \frac{3n-4}{2(n-1)}\left[Ric(Z,U) + \frac{r}{(3n-4)}g(Z,U)\right], \quad (1.7)$$

$$\tilde{G}(Y,T) = \sum_{i=1}^{n} 'G(Y,e_i,e_i,T) = \frac{3n-4}{2(n-1)}\left[Ric(Y,T) + \frac{r}{(3n-4)}g(Y,T)\right].$$

Definition (1.1). TheBianchi second differential identity is given by

$$(D_X{}'R)(Y,Z,T,U) + (D_Y{}'R)(Z,X,T,U) + (D_Z{}'R)(X,Y,T,U) = 0. \quad (1.8)$$

Definition (1.2). If Ricci tensor $Ric(X,Y)$ is of Codazzi type, then we have

$$(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z) = (D_Z Ric)(X,Y). \quad (1.9)$$

The geometrical and topological consequences of the existence of a non-trival Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen (1983).

Definition (1.3). A Riemannian manifold (M^n,g) is said to be manifold of constant curvature Chen and Yano (1972) if the Riemannian curvature tensor $'R(Y,Z,U,T)$ of type (0,4) satisfies the condition:

$$'R(Y,Z,U,T) = k[g(U,T)\mathrm{g}(Y,T) - g(Y,U)g(Z,T)], \quad (1.10)$$

where$k$ is scalar function.

Definition (1.4). A Riemannian manifold (M^n,g) are said to be $'G-$flat and $'G-$conservative if $'G(Y,Z,U,T) = 0$ and $divG = 0$ respectively Chaki and Ghosh (1997).

2. G −projectively flat (M^n, g) $(n > 2)$

We have

$$'G(Y,Z,U,T) = 0. \tag{2.1}$$

Then from (1.6), we get

$$'R(Y,Z,U,T) = \frac{1}{(n-1)}[g(Y,U)Ric(Z,T) - g(Y,T)Ric(Z,U)$$
$$-g(Z,U)Ric(Y,T) + g(Z,T)Ric(Y,U)]. \tag{2.2}$$

Contracting (2.2), we get

$$\frac{(3n-4)}{2(n-1)}\left[Ric(Z,U) + \frac{r}{(3n-4)}g(Z,U)\right] = 0.$$

This gives

$$Ric(Z,U) = -\frac{r}{(3n-4)}g(Z,U)\,. \tag{2.3}$$

Hence, in view of (2.2) and (2.3), we get

$$'R(Y,Z,U,T) = \frac{r}{(n-1)(3n-4)}[g(Z,U)g(Y,T) - g(Y,U)g(Z,T)]. \tag{2.4}$$

From (1.10) and (2.4), it follows that the manifold $(M^n, g)$is of constant curvature.

Thus, we have the following:

Theorem (2.1): A $G-$projectively flat (M^n, g) $(n > 2)$ in which $\frac{(3n-4)}{2(n-1)} \neq 0$ is a manifold of constant curvature.

3. G −projectively conservative (M^n, g) $(n > 2)$

In this, we assume that

$$divG = 0. \tag{3.1}$$

Now, differentiating (1.6) covariantly, we get

$$(D_X G)(Y,Z)U = (D_X R)(Y,Z)U - \frac{1}{2(n-1)}[(D_X Ric)(Y,U)Z - (D_X Ric)(Z,U)Y$$
$$+g(Y,Z)(D_X Q)(U) - g(Z,U)(D_X Q)(Y)]. \tag{3.2}$$

Contracting (3.2), we get

$$(divG)(Y,Z)U = (divR)(Y,Z)U - \frac{1}{2(n-1)}[(D_Z Ric)(Y,U) - (D_Y Ric)(Z,U)$$
$$+g(Y,Z)(divQ)(U) - g(Z,U)(divQ)(Y)].$$

(3.3)

But

$$\left.\begin{aligned} (divR)(Y,Z)U &= (D_Y Ric)(Z,U) - (D_Z Ric)(Y,U), \\ &\text{and} \\ (divQ)(Y) &= \frac{1}{2}dr(Y). \end{aligned}\right\} \tag{3.4}$$

Hence, from (3.3) and (3.4), we get

$$(divG)(Y,Z)U = \left[\frac{2n-1}{2(n-1)}\right][(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U)] + \frac{1}{4(n-1)}[g(Y,Z)dr(U) - g(Z,U)dr(Y)]. \quad (3.5)$$

From (1.8) and (3.5), we get

$$(divG)(Y,Z)U = \frac{1}{4(n-1)}[g(Y,Z)dr(U) - g(Z,U)dr(Y)]. \quad (3.6)$$

Again, from (3.1) and (3.6), we get

$$g(Y,Z)dr(U) - g(Z,U)dr(Y) = 0. \quad (3.7)$$

Consequently r is constant. Again if r is constant then from (3.6), we get

$$(divG)(Y,Z)U = 0. \quad (3.8)$$

Thus, we have the following theorem:

Theorem (3.1): If the manifold admits Codazzi type Ricci tensor, then $G-$projectively manifold is conservative if and only if the manifold is of constant scalar curvature.

If $(D_X G)(Y,Z)U = 0$, then from (3.2), we get

$$(D_X R)(Y,Z)U = \frac{1}{2(n-1)}[(D_X Ric)(Y,U)Z - (D_X Ric)(Z,U)Y + g(Y,Z)(D_X Q)(U) - g(Z,U)(D_X Q)(Y)]. \quad (3.9)$$

Contracting with respect to Y in (3.9), we get

$$(D_X Ric)(Z,U) = -\frac{(n-1)}{2(n-1)}(D_X Ric)(Z,U) + \frac{1}{2(n-1)}[g(e_i,Z)(D_X Q)(U) - g(Z,U)(D_X Q)(e_i)],$$

$$\Rightarrow (D_X Ric)(Z,U) = \frac{1}{3(n-1)}[g(e_i,Z)(D_X Q)(U) - g(Z,U)(D_X Q)(e_i)]. \quad (3.10)$$

Again contraction (3.10) with respect to $Z$and U, we get

$$dr(X) = 0. \quad (3.11)$$

Hence, we have the following theorem:

Theorem (3.2): If $G-$projective curvature tensor is symmetric in the sense of Carten, then the scalar curvature tensor is constant.

Using r =constant and $(divG)(Y,Z)U = 0$ in equation (3.5), then we get

$$(D_X Ric)(Y,U)Z - (D_X Ric)(Z,U)Y = 0.$$

Thus, we in position to the state the following:

Corollary (3.1): If the manifold (M^n, g) $(n > 2)$ possesss $(divG) = 0$ and r is constant, then the Ricci tenor Ric of M^n is of Coddazi type.

4. G −Projective curvature tensor with cyclic Ricci tensor

A Riemannian manifold is said to be cyclic Ricci tensor if

$$(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0. \tag{4.1}$$

From (1.9), we get

$$\left(D_X\tilde{G}\right)(Y,Z) + \left(D_Y\tilde{G}\right)(Z,X) + \left(D_Z\tilde{G}\right)(X,Y) = \frac{3n-4}{2(n-1)}[(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y)] + \frac{1}{2(n-1)}[(D_X r)g(Y,Z) + (D_Y r)g(Z,X) + (D_Z r)g(X,Y)]. \tag{4.2}$$

From (4.1) and (4.2), we get

$$\left(D_X\tilde{G}\right)(Y,Z) + \left(D_Y\tilde{G}\right)(Z,X) + \left(D_Z\tilde{G}\right)(X,Y) = \frac{1}{2(n-1)}[(D_X r)g(Y,Z) + (D_Y r)g(Z,X) + (D_Z r)g(X,Y)]. \tag{4.3}$$

Here, we assume that $\tilde{G}(X,Y)$ be cyclic Ricci tenosr, then from (4.3), we get

$$(D_X r)g(Y,Z) + (D_Y r)g(Z,X) + (D_Z r)g(X,Y) = 0. \tag{4.4}$$

Walker's Lemma (1970) states that if $\tilde{a}(X,Y)$ and $\tilde{b}(X)$ are such that $\tilde{a}(X,Y) = \tilde{a}(Y,X)$ and

$$\tilde{a}(X,Y)\tilde{b}(Z) + \tilde{a}(Y,Z)\tilde{b}(X) + \tilde{a}(Z,X)\tilde{b}(Y) = 0, \tag{4.5}$$

for all X, Y, Z, then either $\tilde{a}(X,Y) = 0$ or all $\tilde{b}(X)$ is zero. Hence by Walker Lemma, we get (4.4) and (4.5), we get

either $g(X,Y) = 0$ or $(D_X r) = 0$. But $g(X,Y) \neq 0$ and we get $D_X r = 0 \Rightarrow r$ is constant.

Therefore from (4.3), we get $\tilde{G}(X,Y)$ is cyclic Ricci tensor. Hence, we have the following theorem:

Theorem (4.1): If the Ricci tensor of M^n admitting a G −projective curvature tensor be a cyclic Ricci tensor then a necessary and sufficient condition for $\tilde{G}(X,Y)$ to be cyclic Ricci tensor is that scalar curvature is constant.

5. Bianchi differential identity for the G −projective curvature tensor

The Bianchi differential identity given by (1.7). Thus, we have from (1.6)

$$(D_X{}'G)(Y,Z,T,U) = (D_X{}'R)(Y,Z,T,U) - \frac{1}{2(n-1)}[g(Y,T)(D_X Ric)(Z,U) - g(Y,U)(D_X Ric)(Z,T) - g(Z,T)(D_X \mathrm{Ric})(Y,U) + g(Z,U)(D_X Ric)(Y,T)]. \tag{5.1}$$

Writing two more equation by cyclic permutation of $X, Y$and Z, we get

$$(D_Y{}'G)(Z,X,T,U) = (D_Y{}'R)(Z,X,T,U) - \frac{1}{2(n-1)}[g(Z,T)(D_Y Ric)(X,U) - g(Z,U)(D_Y Ric)(X,T) - g(X,T)(D_Y Ric)(Z,U) + g(X,U)(D_Y Ric)(Z,T)], \tag{5.2}$$

$$(D_Z{}'G)(X,Y,T,U) = (D_Z{}'R)(X,Y,T,U) - \frac{1}{2(n-1)}[g(X,T)(D_Z Ric)(Y,U) - g(X,U)(D_Z Ric)(Y,T) - g(Y,T)(D_Z Ric)(X,U) + g(Y,U)(D_Z Ric)(X,T)]. \tag{5.3}$$

Adding equation (5.1), (5.2) and (5.3), with the fact of the equation (1.7), we get

$$(D_X{}'G)(Y,Z,T,U) + (D_Y{}'G)(Z,X,T,U) + (D_Z{}'G)(X,Y,T,U) = \frac{1}{2(n-1)}[g(X,T)\{(D_Z Ric)(Y,U) - (D_Y Ric)(Z,U)\} + g(Y,T)\{(D_X Ric)(Z,U) - (D_Z Ric)(X,U)\} + g(Z,T)\{(D_Y Ric)(X,U) - (D_X Ric)(Y,\mathrm{U})\} + g(Y,U)\{(D_Z Ric)(X,T) - (D_X Ric)(Z,T)\} + g(Z,U)\{(D_X Ric)(Y,T) - (D_Y Ric)(X,T)\} + g(X,U)\{(D_Y Ric)(Z,T) - (D_Z Ric)(Y,T)\}]. \tag{5.4}$$

In view of (1.8) and (5.4), we get

$$(D_X{}'G)(Y,Z,T,U) + (D_Y{}'G)(Z,X,T,U) + (D_Z{}'G)(X,Y,T,U) = 0, \tag{5.5}$$

which shows that G −projective curvature tensor satisfied Bianchi second identity.

Thus, we have the following theorem:

Theorem (5.1): In (M^n, g) $(n > 2)$ the $G-$projective curvature tensor satisfies Bianchi differential identity if the Ricci tensor is Codazzi type.

6. G −Projective curvature tensor admitting $\mathbb{Z}$ −tensor

In 2012, Montica and Suh introduced a new generalized (0,2) symmetric tensor $\mathbb{Z}$ and studied various geometric properties of it on a Riemannian manifold. A new tensor $\mathbb{Z}$ is defined as

$$\mathbb{Z}(X,Y) = Ric(X,Y) + \phi g(X,Y), \tag{6.1}$$

whereϕ is an arbitrary scalar function named as generalized $\mathbb{Z}$ −tensor.

In view of (6.1) and (1.6), we get

$${}'G(Y,Z,U,\mathrm{T}) = {}'R(Y,Z,U,T) - \frac{1}{2(n-1)}[g(Y,U)\mathbb{Z}(Z,T) - g(Y,T)\mathbb{Z}(Z,U) - g(Z,U)\mathbb{Z}(Y,T) + g(Z,T)\mathbb{Z}(Y,U)] + \frac{\phi}{(n-1)}[g(Y,U)g(Z,T) - g(Y,T)g(Z,U)]. \tag{6.2}$$

Now, we define

$$'G^*(Y,Z,U,T) = 'R(Y,Z,U,T) - \frac{1}{2(n-1)}[g(Y,U)\mathbb{Z}(Z,T) - g(Y,T)\mathbb{Z}(Z,U)$$

$$-g(Z,U)\mathbb{Z}(Y,T) + g(Z,T)\mathbb{Z}(Y,U)]. \quad (6.3)$$

In view of (6.2) and (6.3), we get

$$'G(Y,Z,U,T) = 'G^*(Y,Z,U,T) + \frac{\phi}{(n-1)}[g(Y,U)g(Z,T) - g(Y,T)g(Z,U)], \quad (6.4)$$

where$'G^*(Y,Z,U,T)$ is called $'G^*$ −projective curvature tensor. Equation (6.4) can be written as

$$'G^*(Y,Z,U,T) = 'G(Y,Z,U,T) - \frac{\phi}{(n-1)}[g(Y,U)g(Z,T) - g(Y,T)g(Z,U)]. \quad (6.5)$$

If ϕ (scalar function) vanishes then (6.4) becomes

$$'G(Y,Z,U,T) = 'G^*(Y,Z,U,T). \quad (6.6)$$

Hence, we can say that $'G(Y,Z,U,T) = 'G^*(Y,Z,U,T)$ if and only if $\phi = 0$.

From (6.5), we can say that $'G^*(Y,Z,U,T)$ possesses all the skew-symmetric and symmetric properties of $'R(Y,Z,U,T)$ as well as cyclic properties. That is

$$'G^*(Y,Z,U,T) + 'G^*(Y,Z,T,U) = 0, \quad (6.6a)$$

$$'G^*(Y,Z,U,T) + 'G^*(Z,Y,U,T) = 0, \quad (6.6b)$$

$$'G(Y,Z,U,T) - 'G(U,T,Y,Z) = 0, \quad (6.6c)$$

and

$$'G^*(Y,Z,U,T) + 'G^*(Z,U,Y,T) + 'G^*(U,Y,Z,T) = 0. \quad (6.6d)$$

Now, differentiating covariantly (6.3) with respect to X we get

$$(D_X{}'G^*)(Y,Z,T,U) = (D_X{}'R)(Y,Z,T,U) - \frac{1}{2(n-1)}[g(Y,U)(D_X\mathbb{Z})(Z,T) -$$

$$g(Y,T)(D_X\mathbb{Z})(Z,U) - g(Z,U)(D_X\mathbb{Z})(Y,T) + g(Z,T)(D_X\mathbb{Z})(Y,U)]. \quad (6.7)$$

From (6.7), we get

$$(D_Y{}'G^*)(Z,X,T,U) = (D_Y{}'R)(Z,X,T,U) - \frac{1}{2(n-1)}[g(Z,U)(D_Y\mathbb{Z})(X,T) -$$

$$g(Z,T)(D_Y\mathbb{Z})(X,U) - g(X,U)(D_Y\mathbb{Z})(Z,T)$$

$$+g(X,T)(D_Y\mathbb{Z})(Z,U)], \quad (6.8)$$

and

$$(D_Z{}'G^*)(X,Y,T,U) = (D_Z{}'R)(X,Y,T,U) - \frac{1}{2(n-1)}[g(X,U)(D_Z\mathbb{Z})(Y,T) -$$

$$g(X,T)(D_Z\mathbb{Z})(Y,U) - g(Y,U)(D_Z\mathbb{Z})(X,T) +$$

$$g(Y,T)(D_Z\mathbb{Z})(X,U)]. \tag{6.9}$$

Adding (6.7), (6.8) and (6.9) using (1.7) in resulting equation, we get

$$(D_X{}'G^*)(Y,Z,T,U) + (D_Y{}'G^*)(Z,X,T,U) + (D_Z{}'G^*)(X,Y,T,U) = \frac{1}{2(n-1)}[g(Y,U)\{(D_X\mathbb{Z})(Z,T) - (D_Z\mathbb{Z})(X,T)\} + g(Z,U)\{(D_Y\mathbb{Z})(X,T) - (D_X\mathbb{Z})(Y,T)\} + g(X,U)\{(D_Z\mathbb{Z})(Y,T) - (D_Y\mathbb{Z})(Z,T)\} + g(X,T)\{(D_Y\mathbb{Z})(Z,U) - (D_Z\mathbb{Z})(Y,U)\} + g(Y,T)\{(D_Z\mathbb{Z})(X,U) - (D_X\mathbb{Z})(Z,U)\} + g(Z,T)\{(D_X\mathbb{Z})(Y,U) - (D_Y\mathbb{Z})(X,U)\}]. \tag{6.10}$$

Here, assume that $\mathbb{Z}$ −tensor is of Codazzi type tensor. That is

$$(D_X\mathbb{Z})(Y,U) = (D_Y\mathbb{Z})(X,U). \tag{6.11}$$

Therefore, in view of (6.10) and (6.11) we get

$$(D_X{}'G^*)(Y,Z,T,U) + (D_Y{}'G^*)(Z,X,T,U) + (D_Z{}'G^*)(X,Y,T,U) = 0. \tag{6.12}$$

Equation (6.12) show that $'G^*$ −projective curvature tensor satisfied Bianchi's second identity.

In view of (6.6a), (6.6b), (6.6c), (6.6d), (6.10), (6.11), and (6.12), we can state the following theorem:

Theorem (6.1): A $'G^*$ −projective curvature tensor on (M^n, g) is

I. skew-symmetric with respect to last two pair of slots,
II. skew-symmetric with respect to first two pair of slots,
III. symmetric in pair of slots,
IV. satisfies Bianchi's first identity,
V. satisfies Bianchi's second identity if $\mathbb{Z}$ −tensor is Codazzi tensor.

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