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# Nearly Ricci recurrent manifolds admitting Schouten tensor

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## Abstract

The object of the present paper is to study nearly Ricci recurrent manifolds admitting Schouten tensor and investigated some geometrical properties of this manifold.

**Keywords and phrases:** Nearly Ricci recurrent manifolds, constant scalar curvature tensor, Schouten tensor, Codazzi type Ricci tensor and divergence free tensor.

## 1.Introduction

Let $(M^n, g)$ be an dimensional Riemannian manifold with metric $g$. A tensor $\mathcal{T}$ of the type $(0, q)$ is said to be recurrent Roter (1974) if the relation

$$(D_X\mathcal{T})(Z_1, Z_2, \dots, Z_q).\mathcal{T}(Z_1, Z_2, \dots, Z_q) - \mathcal{T}(Y_1, Y_2, \dots, Y_q).(D_X\mathcal{T})(Z_1, Z_2, \dots, Z_q) = 0,$$

hold on $(M^n, g)$. From definition it follows that if at a point $x \in M$; $\mathcal{T}(X) \neq 0$, then on some neighbourhood of $x$, there exists a unique 1-form $A$ satisfying

$$(D_X\mathcal{T})(Y_1, Y_2, \dots, Y_q) = A(X)\mathcal{T}(Y_1, Y_2, \dots, Y_q).$$

In 1995 Patterson introduced a Ricci recurrent manifold. According to him, a manifold $(M^n, g)$ of dim n, was called Ricci recurrent if

$$(D_X Ric)(Y, Z) = A(X)Ric(Y, Z), \tag{1.1}$$

for some 1-form $A$. He denoted such a manifold by $R_n$. Ricci recurrent manifolds have been studied by several authors (Roter (1974); Chaki (1956); Praksha (1962) and Yamaguchi and Matsumoto (1968) ) and many others. In a recent paper, De, Guha and Kamilya (1995) introduced the notion of generalized Ricci recurrent manifold as follows:

A non-flat Riemannian manifold$(M^n, g)n > 2$, is called generalized Ricci recurrent if the Ricci tensor $Ric$ is non-zero and satisfies the condition:

$$(D_X Ric)(Y,Z) = A(X)Ric(Y,Z) + B(X)g(Y,Z), \quad (1.2)$$

where$A$ and $B$ non-zero 1-forms. Such a manifold denoted by them as $GR_n$. If the associated 1-form $B$ becomes zero. Then the manifold $GR_n$ reduces to a Ricci recurrent manifold $R_n$. This justifies the generalized Ricci recurrent manifold and the symbol $GR_n$ for it.

In a paper, De and Guha (1991) introduced a non-flat Riemannian manifold $(M^n, g) n > 2$, is called generalized recurrent manifolds if its curvature tensor $R(X,Y)Z$ of type (1,3) satisfies the condition:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y],$$

where$A$ and $B$ non-zero 1-forms and $D$ denotes the operator of differentiation with respect to metric $g$. Such a manifold has been denoted by $GK_n$ . If the associated 1-form $B$ becomes zero, then the manifold $GK_n$ reduces to recurrent manifold introduced by Ruse (1951) and Walker (1950) which has denoted by $K_n$. In recent paper Arslan et al (2009), Shaikh and Patra (2010), Malik, De and De (2013), Khairnar (2014), Shaikh, Prakasha and Ahmad (2015), Kumar, Singh and Chowdhary (2015), Hui (2017), Singh and Mayanglambam (2017), Singh and Kishor (2020) etc. explored various geometrical properties by using generalized recurrent and Ricci recurrent manifold on Riemannian manifold, LorentzianTrans-Sasakian manifolds, LP-Sasakian manifolds and $(k, \mu)$ $-$contact manifolds.

Further, the authors Prasad and Yadav (2023) considered a non-flat Riemannian manifold $(M^n, g) n > 2$, whose curvature tensor $R$ satisfies the condition:

$$(D_U R)(X,Y)Z = [A(U) + B(U)]R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y],$$

where$A$ and $B$ non-zero 1-forms and $D$ has the meaning already mention. Such a manifold were called by them as a nearly recurrent Riemannian manifold and denoted by $(NR)_n$.Nearly recurrent have been studied by Yadav and Prasad (2023), Yadav and Prasad (2022), Prasad and Yadav (2024).

Recently, Prasad and Yadav (2021) introduced the notion of nearly Ricci recurrent manifolds as follows:

$$(D_X Ric)(Y,Z) = [A(X) + B(X)]Ric(Y,Z) + B(X)g(Y,Z), \quad (1.3)$$

where$A$ and $B$ are two non-zero 1-forms, $\rho_1$ and $\rho_2$ are two vector fields such that

$$A(X) = g(X, \rho_1) \text{and } B(X) = g(X, \rho_2). \quad (1.4)$$

Such a manifold denoted by them as $N\{R(R_n)\}$. The name nearly Ricci recurrent manifold was chosen because if $B = 0$ in (1.3) then the manifold reduces to a Ricci recurrent which is very close to Ricci recurrent space. This justifies the name "Nearly Ricci recurrent manifold" for the manifold defined by (1.1) and the use of the symbol $N\{R(R_n)\}$ for it. Authors proved that in conformally flat $N\{R(R_n)\}$ with constant scalar curvature if the 1-form $A$ is closed then $R(X,Y).Ric = 0$ if and only if $\{A(X) + B(X)\}A(QZ) = \{A(Z) + B(Z)\}A(QX)$.The nearly Ricci recurrent have been studied by Yadav and Prasad (2022) and Prasad and Yadav (2023).

Gray (1978) introduced two groups of Riemannian manifolds based on the covariant differentiation of the Ricci tensor. The first group contains all Riemannian manifolds whose Ricci tensor $Ric$ is a Codazzi type tensor, that is,

$$(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z). \quad (1.5)$$

The second group contains all Riemannian manifolds whose Ricci tensor $Ric$ is cyclic parallel, that is

$$(D_X Ric)(Y,Z) + (D_Y Ric)(X,Z) + (D_Z Ric)(X,Y) = 0. \tag{1.6}$$

The Schouten tensor in a Riemannian manifold is given by Chen and Yano (1972)

$$P(Y,Z) = \frac{1}{n-2}\left[Ric(Y,Z) - \frac{r}{2(n-1)} g(Y,Z)\right]. \tag{1.7}$$

There is a decomposition formula in which the Riemannian curvature tensor decomposes into mom-conformally invariant part, the Schouten tensor and a conformally invariant part, the Conformal curvature tensor $K = P \odot g + C$, where $C$ is the Conformal curvature tensor of $g$ and $\odot$ denotes the Kuekarni-Nomizu product, $K$ is the Riemannian curvature tensor and $P$ is the Schouten tensor Chen and Yano (1972).

## 2. Schouten flat $N\{R(R_n)\}$

In this section, we assume that the manifold is Schouten flat, then from (1.7), we get

$$Ric(Y,Z) = \frac{r}{2(n-1)} g(Y,Z). \tag{2.1}$$

We also assume that manifold is $N\{R(R_n)\}$ admitting Schouten tensor.

Now, taking the covariant derivative of (2.1), we get

$$(D_X Ric)(Y,Z) = \frac{1}{2(n-1)} g(Y,Z).(D_X r). \tag{2.2}$$

From (1.3) and (2.1), we get

$$(D_X Ric)(Y,Z) = [A(X) + B(X)]\frac{1}{2(n-1)} g(Y,Z).r + B(X)g(Y,Z). \tag{2.3}$$

Contracting (2.3) over $Y$ and $Z$ we get

$$(D_X r) = [A(X) + B(X)]\frac{n}{2(n-1)}.r + nB(X). \tag{2.4}$$

Again contracting (2.3) over $X$ and $Y$ we get

$$(D_Z r) = [A(Z) + B(Z)]\frac{n}{2(n-1)}.r + nB(Z). \tag{2.5}$$

Replacing $Z$ be $X$ in (2.5), we get

$$(D_X r) = [A(X) + B(X)]\frac{1}{2(n-1)}.r + B(X). \tag{2.6}$$

Hence from (2.4) and (2.7), we get

$$r = \frac{-2(n-1)n}{[A(X)+B(X)]}, \quad \text{provided } [A(X) + B(X)] \neq 0. \tag{2.7}$$

Hence, we have the following theorem:

**Theorem (2.1):** For Schouten flat $N\{R(R_n)\}$, relation (2.6) shows that $r$ is not necessarily a constant, provided $[A(X) + B(X)] \neq 0$.

## 3. $N\{R(R_n)\}$ with Schouten tensor as conservative

In this section, we assume that Schouten tensor is conservative. That is,

$$div\ \mathrm{P} = 0. \tag{3.1}$$

Taking covariant derivative of (1.7), we get

$$(D_X P)(Y, Z) = \frac{1}{n-2}\Big[(D_X Ric)(Y, Z) - \frac{1}{2(n-1)} g(Y, Z)(D_X r)\Big]. \tag{3.2}$$

Contracting (1.3) with respect to $Y$ and $Z$ we get

$$(D_X r) = [A(X) + B(X)]r + B(X).n. \tag{3.3}$$

From (1.3), (3.2) and (3.3), we get

$$(D_X P)(Y, Z) = \frac{1}{n-2}[\{A(X) + B(X)\}Ric(Y, Z) + B(X)g(Y, Z)$$

$$-\frac{1}{2(n-1)}\{(A(X) + B(X)).rg(Y, Z) + nB(X)g(Y, Z)\}]. \tag{3.4}$$

Contracting (3.4) with respect to $X$ and $Y$ we get

$$(divP)(Z) = \frac{1}{n-2}[\{A(e_i) + B(e_i)\}Ric(e_i, Z) + B(e_i)g(e_i, Z)$$

$$-\frac{1}{2(n-1)}\{(A(e_i) + B(e_i)).rg(e_i, Z) + nB(e_i)g(e_i, Z)\}]. \tag{3.5}$$

Equation (3.5) can be put as

$$(divP)(Z) = \frac{1}{n-2}[\{g(e_i, \rho_1) + g(e_i, \rho_2)\}g(Qe_i, Z) + g(e_i, \rho_2)g(e_i, Z) -$$

$$\frac{1}{2(n-1)}\{(g(e_i, \rho_1) + g(e_i, \rho_2)).rg(e_i, Z) + ng(e_i, \rho_2)g(e_i, Z)\}]. \tag{3.6}$$

Solving (3.6), we get

$$(divP)(Z) = \frac{1}{n-2}\Big[\{A(QZ) + B(QZ)\} + B(Z) - \frac{1}{2(n-1)}\{(A(Z) + B(Z)).r + nB(Z)\}\Big]$$

$$\Rightarrow (div\ \mathrm{P})(Z) = \Big[\frac{1}{n-2}A(QZ) - \frac{1}{2(n-1)(n-2)}A(Z).r\Big] + \Big[\frac{1}{n-2}B(QZ) - \frac{1}{2(n-1)(n-2)}B(Z).r\Big] +$$

$$\Big[\frac{1}{n-2}B(Z) - \frac{1}{2(n-1)(n-2)}B(Z)\Big]. \tag{3.7}$$

Hence from (3.7) and (3.1), we have the following theorem:

**Theorem (3.1):** In $N\{R(R_n)\}$, Schouten tensor is divergence free if and only if

$$\Big[A(QZ) - \frac{1}{2(n-1)}A(Z).r\Big] + \Big[B(QZ) - \frac{1}{2(n-1)}B(Z).r\Big] + \frac{2n-3}{2(n-1)}.B(Z) = 0.$$

**3. $N\{R(R_n)\}$with Schouten tensor admitting Codazzi type Ricci tensor**

Let us suppose that the scalar curvature $r$ is non-zeero constant, then from (3.2), we get

$$(D_X P)(Y, Z) = \frac{1}{n-2}(D_X Ric)(Y, Z). \tag{4.1}$$

From (1.3) and (4.1), we get

$$(D_XP)(Y,Z) = \frac{1}{n-2}[\{A(X) + B(X)\}Ric(Y,Z) + B(X)g(Y,Z)]. \tag{4.2}$$

Interchanging $X$ and $Y$ in (4.2), we get

$$(D_YP)(X,Z) = \frac{1}{n-2}[\{A(Y) + B(Y)\}Ric(X,Z) + B(Y)g(X,Z)]. \tag{4.3}$$

From (4.2) and (4.3), we get

$$(D_XP)(Y,Z) - (D_YP)(X,Z) = \frac{1}{n-2}[\{A(X) + B(X)\}Ric(Y,Z) + B(X)g(Y,Z)]$$

$$-\frac{1}{n-2}[\{A(Y) + B(Y)\}Ric(X,Z) + B(Y)g(X,Z)]. \tag{4.4}$$

Contracting (4.4) over $Y$ and $Z$ we get

$$\big(A(X) + B(X)\big).r + nB(Z) = A(QX) + B(QX) + B(X). \tag{4.5}$$

Hence, in view of (4.5), we have the following theorem:

**Theorem (4.1):** For $N\{R(R_n)\}$ of non-zero constant scalar curvature, if Schouten tensor is of Codazzi type then scalar curvature is given by $r = \frac{A(QX)+B(QX)-(n-1)B(X)}{A(X)+B(X)}$.

Let $r$ is non-zero constant and Schouten tensor $P$ is covariantly constant. Then from (4.1), we get

$$(D_XRic)(Y,Z) = 0. \tag{4.6}$$

Hence from (1.3) we get

$$\{A(X) + B(X)\}Ric(Y,Z) + B(X)g(Y,Z) = 0. \tag{4.7}$$

Contraction of (4.7), we get

$$\{A(X) + B(X)\}.r + n.B(X) = 0.$$

$$\Rightarrow r = \frac{-nB(X)}{A(X)+B(X)},\ A(X) + B(X) \neq 0. \tag{4.8}$$

Hence in view of (4.8), we have the following theorem:

**Theorem (4.2):** In $N\{R(R_n)\}$ of non-zero vanishing constant scalar curvature and Schouten tensor is covariantly constant, then the scalar curvature is given by (4.8), provided $A(X) + B(X) \neq 0$.

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