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On conformal transformation in an almost Kaehlerian and Kaehlerian spaces

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Abstract

Yano (1952), studied on harmonic and killing vector fields. Lichnerowicz (1958) studied Geometric des groups de transformations. Tachibana (1959) studied and defined almost analytic vectors in almost Kaehlerian manifolds. Further, Goldberg (1960) studied conformal transformations of Kaehlerian manifolds. Rawat (2002) studied and defined Geometry of the locally product almost Tachibana space. Rawat and Silswal (2009) studied theory of Lie-derivatives and motions in Tachibana spaces. Rawat and Prasad (2009) studied on Lie- derivatives of scalars, vector and tensors. Rawat and Dobhal (2009) studied on bi- recurrent and bi- symmetric Kaehlerian manifolds. In the present paper, we have studied on Conformal Transformation in an almost Kaehlerian and Kaehlerian spaces; also several theorems have been established and proved within.

Key words- Conformal, transformation, almost Kaehlerian, kaehlerian

1. Introduction

Let X_{2n} be a $2n$- dimensional almost- complex space and F_j^i its almost- complex structure, then by definition, we have

$$F_j^s F_s^i = -\delta_j^i, \tag{1.1}$$

An almost- complex space with a positive definite Riemannian metric g_{ji} satisfying

$$g_{rs} F_j^r F_i^s = g_{ji} \tag{1.2}$$

is called an almost- Hermitian space. From (1.2), it follows that $F_{ji} \equiv g_{ri} F_j^r$ is skew- symmetric.
If an almost- Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0\,, \tag{1.3}$$

where ∇_j denotes the operator of covariant derivative with respect to Riemannian connection, then it is called an almost- Kaehlerian space and if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0\,. \tag{1.4}$$

then it is called a K- space.

In an almost- Hermitian space, if $\nabla_j F_{ih} = 0$, then it is called a Kaehlerian space.

Consider Conformal killing vector v^h in a $2n$- dimensional Kaehlerian space. Then the Lie- derivative of the fundamental tensor g_{ji} and that of Christoffel symbols $\left\{ {h \atop ji} \right\}$ with respect to v^h are respectively given by

$$\underset{v}{\mathcal{L}} g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\phi g_{ji} \tag{1.5}$$

and

$$\underset{v}{\mathcal{L}} \left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + R_{kji}^h v^k = A_j^h \phi_i + A_i^h \phi_j - \phi^h g_{ji} \, , \tag{1.6}$$

where R_{kji}^h is the curvature tensor, A_j^h the unit tensor and $\phi_i = \nabla_i \phi$, ϕ^h being contravariant components. For a skew- symmetric tensor $\omega_{i_p i_{p-1} \dots\dots i_1}$, we have in general Yano,(1957)

$$\underset{v}{\mathcal{L}} \nabla_j \omega_{i_p \dots\dots i_1} - \nabla_j \underset{v}{\mathcal{L}} \omega_{i_p \dots\dots i_1} = -\left(\underset{v}{\mathcal{L}} \left\{ {t \atop j \;\; i_p} \right\} \right) \omega t_{i_{p-1} \dots\dots i_1} \dots\dots\dots . - \left(\underset{v}{\mathcal{L}} \left\{ {t \atop j \;\; i_1} \right\} \right) \omega_{i_p \dots\dots i_1} t \tag{1.7}$$

Taking the skew- symmetric part with respect to $j, \ i_{p \dots\dots} i_1$, we have

$$\underset{v}{\mathcal{L}} \nabla_{[j} \omega_{ip \dots\dots i_1]} = \nabla_{[j} \underset{v}{\mathcal{L}} \omega_{ip \dots\dots i_1]} \, , \tag{1.8}$$

from which

THEOREM (1.1)- The Lie- derivative of a closed skew- symmetric tensor is closed.
Transvecting (1.7) with g^{ji_p} and taking account of (1.5) and (1.6), we get

$$\underset{v}{\mathcal{L}} g^{ji} \nabla_j \omega_{i i_{p-1} \dots\dots i_1} + 2\phi g^{ji} \nabla_j \omega_{i i_{p-1} \dots\dots i_1} - g^{ji} \nabla_j \underset{v}{\mathcal{L}} \omega_{i i_{p-1} \dots\dots i_1} = (n - 2p) \phi^t \omega_{t i_{p-1} \dots\dots i_1}, \tag{1.9}$$

from which

THEOREM (1.2)- The Lie- derivative of a coclosed skew- symmetric tensor of order p with respect to a Conformal killing vector is coclosed if and only if $p = n/2$, n being even, or

$$\nabla^t \left(\phi \omega_{t_{p-1} \dots\dots i_1} \right) = 0 \, ,$$

that is, $\phi \omega_{i_{p-1} \dots\dots i_1}$ is also coclosed, where ϕ is the function appearing in $\underset{v}{\mathcal{L}} g_{ji} = 2\phi g_{ji}$.

Combining Theorem (1.1) and (1.2), we have

THEOREM (1.3)- The Lie- derivative of a harmonic tensor ω of order p in an $n(= 2m)$-dimensional Kaehlerian space with respect to a Conformal killing vector is also harmonic if and only if $p = n/2$, n being even, or $\phi\omega$ is coclosed.

The most specific statement resulting is as follows.

THEOREM (1.4)- The Lie- derivative of a harmonic tensor ω of order p is an $n(= 2m)$-dimensional compact orientable Kaehlerian space with respect to a Conformal killing vector is zero if and only if $p = n/2$, $n$being even, or $\phi\omega$ is coclosed where ϕ is a function appearing in $\underset{v}{\mathcal{L}} g_{ji} = 2\phi g_{ji}$ Goldbergm (1960).

2. In an almost complex space, a contravariant almost analytic vector is defined as a vector v^h which satisfies

$$\underset{v}{\mathcal{L}} F_i^h = v^t \partial_t F_i^h - F_i^t \partial_t v^h + F_t^h \partial_i v^t = 0 \, , \tag{2.1}$$

In an almost Hermitian space, (2.1) may be written as

$$\underset{v}{\mathcal{L}}F_i^h = v^t\nabla_t F_i^h - F_i^t\nabla_t v^h + F_t^h\nabla_i v^t = 0\,, \tag{2.2}$$

from which, by a straight forward calculation, we have

$$\nabla^i\nabla_i v^h + R_i^h v^i - F_i^h\left(\underset{v}{\mathcal{L}}F^i\right) - \tfrac{1}{2}F_{ji}^h\left(\underset{v}{\mathcal{L}}F^{ji}\right) = 0\,, \tag{2.3}$$

where R_i^h is the Ricci- tensor and $F^i = \nabla^j F_j^i$ and $F_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}$.

If we put $S^{ji} = g^{jt}(\underset{v}{\mathcal{L}}F_t^i)$

and suppose that the space is compact, we have

$$\int\left[\left\{\nabla^i\nabla_i v^h + R_i^h v^i - F_i^h\left(\underset{v}{\mathcal{L}}F^i\right) - \tfrac{1}{2}F_{ji}^h(\underset{v}{\mathcal{L}}F^{ji})\right\}v_h + \tfrac{1}{2}S^{ji}S_{ji}\right]dv = 0\,, \tag{2.4}$$

dv being volume element of space. From (2.3) and (2.4), we have

THEOREM (2.1)- A necessary and sufficient condition for a vector v^h in a compact almost Hermitian space to be contravariant analytic is (2.3)

Suppose that a Conformal killing vector v^h satisfies

$$F_i^h\left(\underset{v}{\mathcal{L}}F^i\right) + \frac{1}{2}F_{ji}^h\left(\underset{v}{\mathcal{L}}F^{ji}\right) = 0\,,$$

Substituting

$$\nabla^i\nabla_i v^h + R_i^h v^i = -\tfrac{n-2}{n}\nabla^h\left(\nabla_i v^i\right)$$

obtained from (1.6) into (2.4), we find

$$\int[(n-2)/n\,(\nabla\_i\, v\^i\,)\^2 + 1/2\, S\^ji\, S\_ji\,]\ dv = 0\,, \tag{2.5}$$

from which, for $n > 2$

$$\nabla\_i\, v\^i = 0\,,\ S\_ji = 0$$

and consequently $v\^h$ is a Killing vector and at the same time a contravariant almost analytic vector, and for $n = 2$, we have $S\_ji = 0$.

Thus, we have

THEOREM (2.2)- If a Conformal killing vector $v\^h$ in an $n(= 2m)$-dimensional compact almost Hermitian space satisfies

$$F_i^h\left(\underset{v}{\mathcal{L}}F^i\right) + \tfrac{1}{2}F_{ji}^h\left(\underset{v}{\mathcal{L}}F^{ji}\right) = 0\,, \tag{2.6}$$

then, for $n > 2$, it defines an automorphism of the space, that is, the infinitesimal transformation $v\^h$ does not change both the metric and the almost complex structure of the space, and for $n = 2$, it is Contravariant almost analytic.

An almost Hermitian space in which $F\_i = 0$ is satisfied is called an almost semi- Kaehlerian space. In such a space, we have

$$F_{jih}\,F^{ji} = 2F_t\,F_h^t = 0\,,$$

Thus, from Theorem (2.2), we have

THEOREM (2.3)- If a Conformal killing vector v^h in an $n(> 2)$ dimensional compact almost semi-Kaehlerian space satisfies

$$F_{jih}\,[\![\left(\underset{v}{\mathcal{L}}\,F\right]\!]^{ji}\right) = 0 \qquad \text{or,} \qquad \left(\underset{v}{\mathcal{L}}\,F_{ji}\ \right)F^{ji} = 0\,,$$

then v^h defines an automorphism in the space.

An almost Hermitian space in which $F_{jih} = 0$ is satisfied, then it is called an almost Kaehlerian space. In such a space, we have

$$F_h = -1/2\, F_{jih}\; F^{ji}\, F_h^t = 0\,,$$

that is, F_{ji} is harmonic. Thus from Theorem (2.3), we have

THEOREM (2.4)- A Conformal Killing vector v^h in an $n(>2)$ dimensional Compact almost Kaehlerian space defines an automorphism of the space.

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