ISSN:0976-4933
Journal of Progressive Science
Vol. 02, No.02, pp 147-159 (2011)

On generalized pseudo quasi-Einstein manifolds

Absos Ali Shaikh* **and Ananta Patra**
Department of Mathematics
University of Burdwan, Golapbag,
Burdwan-713104, W. B., India

Abstract

The object of the present paper is to introduce a type of non-flat Riemannian manifolds called generalized pseudo quasi-Einstein manifold and to study some geometric properties of such a manifold. It is shown that a generalized pseudo quasi-Einstein manifold can be expressed as a product manifold. Also the existence of such a manifold is ensured by a proper example.

2000 Mathematics Subject Classification: 53B05, 53B50, 53C15, 53C25.

Keywords- generalized pseudo quasi-Einstein manifold, generalized pseudo quasi-constant curvature, conformally flat manifold, Killing vector field, sectional curvature.

1. Introduction

Let $\left(M^n, g\right)$, $n \geq 3$, be a connected Riemannian or semi-Riemannian manifold. Let $U_{S_1} = \left\{x \in M : S \neq \frac{r}{n} g \text{ at } x\right\}$. Then the manifold $\left(M^n, g\right)$ is said to be a quasi-Einstein manifold (Chen *et al*., 1972, Deszes *et al*., 1998, Deszes *et al.,* 2001, Deszes *et al.* , 2001, Deszes *et al.,* 1996, Ferus, 1991,Gonzalez *et al*., 2001, Hicks,1969, Koufogiorgos *et al*.,2003, Koufogiorgos *et al*.,2003, Perrone,2004) if on $U_{S_1} \subset M$, we have

$$S - \alpha g = \beta A \otimes A, \tag{1.1}$$

where A is an 1-form on U_{S_1} and α, β are some smooth functions on U_{S_1}. It is clear that the function β and the 1-form A is non-zero at every point of U_{S_1}. The scalars α, β are known as the associated scalars of the manifold. Also the 1-form A is called the associated 1-form of the manifold defined by $g(X,U) = A(X)$ for any vector field X; U being a unit vector field called generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by QE_n.

Generalizing the notion of quasi-Einstein manifold, recently the first author (Shaikh, 2009) introduced the notion of pseudo quasi-Einstein manifold and studied its geometric properties with the existence of such notion by several non-trivial examples. Let $\left(M^n, g\right)$, $n \geq 3$, be a semi-Riemannian manifold. Let $U_{S_2} = \{x \in M : S - \alpha g \neq \beta A \otimes A \text{ at } x\}$. Then the manifold $\left(M^n, g\right)$ is said to be a pseudo quasi-Einstein manifold (Watenabe 1968) if on $U_{S_2} \subset M$, we have

$$S - \alpha g - \beta A \otimes A = \gamma D, \tag{1.2}$$

where A is an 1-form on U_{S_2} such that $g(.,U) = A(.)$ and α, β, γ are some smooth functions on U_{S_2} and D is a trace free symmetric tensor of type (0, 2) such that $D(X, U) = 0$ for any vector field X. Such an n-dimensional manifold is denoted by PQE_n. It follows that every QE_n is a PQE_n, but not conversely as follows by various examples given in (Watanabe, 1968).

It is known that the outer product of two covariant vectors is a tensor of type (0, 2) but the converse is not true, in general (De *et al*., 1981). Consequently, the tensor D can not be decomposed into product of two 1-forms. In particular, if $D = B \otimes B$, B being a non-zero 1-form, then a PQE_n reduces to generalized quasi-Einstein manifold by De and Ghosh (Desezez *et al*., 2001). Again, if $D = A \otimes B + B \otimes A$, then a PQE_n turns into a generalized quasi-Einstein manifold by Chaki . The object of the present paper is to generalize the notion of PQE_n and is said to be generalized pseudo quasi-Einstein manifold. Let $(M^n, g), n \geq 3$, be a Riemannian or semi-Riemannian manifold. Let $U_S = \{x \in M : S - \alpha g - \beta A \otimes A \neq \gamma D \text{ at } x\}$. Then the manifold (M^n, g) is said to be a generalized pseudo quasi-Einstein manifold if on $U_S \subset M$, we have

$$S - \alpha g - \beta A \otimes A - \gamma D = \delta E, \tag{1.3}$$

where A is an 1-form on U_S and α, β, γ, δ are some smooth functions on U_S and D, E are two trace free symmetric tensors of type (0, 2) such that $D(X, U) = 0$, $E(X, U) = 0$ for any vector field X. Such an n-dimensional manifold will be denoted by $GPQE_n$. It follows that every QE_n as well as PQE_n is a $GPQE_n$ but not conversely as shown by the example in section 5. We note that if $D = B \otimes B$, B being a non-zero 1-form, then a $GPQE_n$ turns into a pseudo generalized quasi-Einstein manifold by Shaikh and Jana. Also, if $D = A \otimes B + B \otimes A$ and $E = A \otimes C + C \otimes A$, C being a non-zero 1-form, then a $GPQE_n$ turns into a hyper generalized quasi-Einstein manifold.

The paper is organized as follows. Section 2 deals with some geometric properties of $GPQE_n$. Section 3 is concerned with conformally flat $GPQE_n$ and obtained various interesting geometric properties of such a manifold. It is shown that a $GPQE_n$ with certain condition is a product manifold. Section 4 is devoted to the study of global properties of $GPQE_n$ and it is shown that in such a manifold under certain conditions there exists no non-zero Killing, projective Killing and conformal Killing vector fields. Also the harmonic vector field in such a manifold reduces to a parallel vector field. The last section deals with an example of $GPQE_n$ which is neither QE_n nor PQE_n.

2. Some geometric properties of $GPQE_n$

From (1.3) it follows that

$$r = n\alpha + \beta, \tag{2.1}$$

where r is the scalar curvature of the manifold,

$$S(X, U) = (\alpha + \beta) A(X), \tag{2.2}$$

$$S(U, U) = \alpha + \beta. \tag{2.3}$$

We now prove the following:

Theorem 2.1- Let (M^n, g), $n > 2$, be a connected orientable Riemannian manifold which is either non-compact or compact with vanishing Euler number. If the Ricci tensor S of type (0, 2) of a Riemannian manifold is of rank > 1 which satisfies the relation

$$S(Y,Z)S(X,W)-S(X,Z)S(Y,W)=p_1[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \tag{2.4}$$
$$+p_2g(LX,W)g(Y,Z)+p_3g(TX,W)g(Y,Z),$$

where p_1, p_2, p_3 are non-zero scalars and L, T are the symmetric endomorphisms, with vanishing trace, of the tangent space at any point of the manifold corresponding to the tensors of type (0, 2) such that LX and TX are orthogonal to a unit vector field U, then the manifold is a generalized pseudo quasi-Einstein manifold.

Proof: To prove the theorem we first state a well-known result (Shaikh *et al.,* 2006) as follows:

Proposition 2.1. For a connected orientable manifold M^n the following assertions are equivalent:
There is a nowhere vanishing vector field V on M^n.
Either M^n is non-compact, or M^n is compact and has Euler number $\chi(M^n)=0$.
From the Proposition 2.1, it follows that there is a nowhere vanishing vector field U on the manifold (M^n, g) under consideration such that $g(X, U) = A(X)$ for any vector field X. We also assume that $g(U, U) = 1$. Then setting $Y = Z = U$ in (2.4), we get

$$S(U,U)S(X,W)-S(X,U)S(W,U)=p_1[g(U,U)g(X,W)-g(X,U)g(W,U)]$$
$$+p_2g(LX,W)g(U,U)+p_3g(TX,W)g(U,U),$$

which can be written as

$$aS(X,W)-A(QX)A(QW)=p_1g(X,W)-p_1A(X)A(W) \tag{2.5}$$
$$+p_2g(LX,W)+p_3g(TX,W),$$

where $a = S(U, U)$ and $A(QX) = g(QX, U) = S(X, U)$. Since U is a unit vector field and the Ricci tensor is nowhere vanishing, we have $a \neq 0$. From (2.5) it follows that

$$S(X,W)=\alpha_1g(X,W)+\alpha_2A(X)A(W) \tag{2.6}$$
$$+\alpha_3F(X)F(W)+\alpha_4D(X,W)+\alpha_5E(X,W),$$

where $\alpha_1=\frac{p_1}{a},\ \alpha_2=-\frac{p_1}{a},\ \alpha_3=\frac{1}{a},\ \alpha_4=\frac{p_2}{a},\ \alpha_5=\frac{p_3}{a}$; $F(X)=A(QX)$, $D(X,W)$ $=g(LX,W)$ and $E(X, W) = g(TX, W)$ for all vector fields X and W. Since LX and TX are orthogonal to U, we have $D(X, U) = 0$ and $E(X, U) = 0$ for all X. Since U is nowhere vanishing, $S \neq 0$, p_1, p_2 and p_3 are non-zero scalars, it follows that $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non-zero scalars.
Again putting $Y = U$ in (2.4) we obtain

$$S(U,Z)S(X,W)-S(X,Z)S(U,W)=p_1[g(U,Z)g(X,W)-g(X,Z)g(U,W)]$$
$$+p_2g(LX,W)g(U,Z)+p_3g(TX,W)g(U,Z),$$

which implies

$$F(Z)S(X,W)-S(X,Z)F(W)=p_1[A(Z)g(X,W)-g(X,Z)A(W)]$$
$$+p_2g(LX,W)A(Z)+p_3g(TX,W)A(Z),$$

which yields by virtue of (2.6) that

$$\alpha_1[F(Z)g(X,W)-F(W)g(X,Z)]+\alpha_2[A(X)F(Z)A(W)-A(X)A(Z)F(W)] \tag{2.7}$$
$$+\alpha_4[F(Z)g(LX,W)-F(W)g(LX,Z)]+\alpha_5[F(Z)g(TX,W)-F(W)g(TX,Z)]$$
$$=p_1[A(Z)g(X,W)-g(X,Z)A(W)]+p_2g(LX,W)A(Z)+p_3g(TX,W)A(Z).$$

Setting $X = W = U$ in (2.7), we obtain

$$F(Z)=a\,A(Z) \text{ for all } Z. \tag{2.8}$$

Using (2.8) in (2.6) we obtain

$$S(X,W) = \alpha g(X,W) + \beta A(X)A(W) + \gamma D(X,W) + \delta E(X,W),$$

where $\alpha = \alpha_1$, $\beta = \alpha_2 + \alpha_3 a^2$, $\gamma = \alpha_4$ and $\delta = \alpha_5$. Thus the manifold under consideration is a $GPQE_n$.

Proposition 2.2- In a Ricci semi-symmetric $GPQE_n$, n > 2, the relation γA (R(X, Y)LZ) + δA(R(X, Y)TZ) = βA(R(X, Y)Z) holds for all X, Y.

Proof. We consider a $GPQE_n$ which is Ricci semi-symmetric. Now we have

$$\begin{aligned}(R(X,Y).S)(Z,W) &= -S(R(X,Y)Z,W) - S(R(X,Y)W,Z)\\ &= -\alpha\left[g(R(X,Y)Z,W) + g(R(X,Y)W,Z)\right]\\ &\quad -\beta\left[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)\right]\\ &\quad -\gamma\left[D(R(X,Y)Z,W) + D(R(X,Y)W,Z)\right]\\ &\quad -\delta\left[E(R(X,Y)Z,W) + E(R(X,Y)W,Z)\right]\\ &= -\beta\left[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)\right]\\ &\quad -\gamma\left[D(R(X,Y)Z,W) + D(R(X,Y)W,Z)\right]\\ &\quad -\delta\left[E(R(X,Y)Z,W) + E(R(X,Y)W,Z)\right].\end{aligned}$$

The above relation implies by virtue of R(X, Y).S = 0, that

$$\begin{aligned}&\beta\left[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)\right]\\ &+\gamma\left[D(R(X,Y)Z,W) + D(R(X,Y)W,Z)\right]\\ &+\delta\left[E(R(X,Y)Z,W) + E(R(X,Y)W,Z)\right] = 0.\end{aligned} \tag{2.9}$$

Setting W = U in (2.9) we get

$$\gamma A(R(X,Y)LZ) + \delta A(R(X,Y)TZ) = \beta A(R(X,Y)Z) \text{ for all } X,Y,Z. \tag{2.10}$$

This proves the result.

Theorem 2.2-A Ricci semi-symmetric $GPQE_n$ satisfying the relation

$$R(X, Y)U = A(Y)X - A(X)Y \tag{2.11}$$

for all X, Y is an Einstein manifold.

Proof. From (2.11) we get

S(Y, U) = (n − 1) g (Y, U).

Also from (1.2) we have

S(Y, U) = (α + β) g (Y, U).

Comparing the last two relations we obtain

α + β = n − 1 (as A is a non-zero 1-form).

By virtue of (2.11), the relation (2.10) can be written as

$$\begin{aligned}&\gamma\left[A(Y)D(X,Z) - A(X)D(Y,Z)\right] + \delta\left[A(Y)E(X,Z) - A(X)E(Y,Z)\right]\\ &= \beta\left[A(Y)g(X,Z) - A(X)g(Y,Z)\right].\end{aligned} \tag{2.12}$$

Setting Y = U in (2.12) and noting that D(X, U) = E(X, U) = 0 for all X, we get

γD(X, Z) + δE(X, Z) = β[g(X, Z) − A(X) A(Z)],

which yields, on contraction, β = 0 (since Tr.D = Tr.E = 0).

Hence α = n − 1 and γD(X, Z) + δE(X, Z) = 0 for all X, Z. Consequently, (1.3) takes the form

S(X, Y) = (n − 1)g(X, Y) for all X, Y

and hence the manifold under consideration is Einstein. This proves the theorem.

3. Conformally flat $GPQE_n$

The Kulkarni-Nomizu product $E \wedge F$ of two (0, 2) tensors E and F is defined by

$(E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4) F(X_2, X_3) + E(X_2, X_3) F(X_1, X_4) - E(X_1, X_3) F(X_2, X_4) - E(X_2, X_4) F(X_1, X_3),$

for all vector fields X_i, $i = 1, 2, 3, 4$. Let $G = \frac{1}{2}(g \wedge g)$.

Definition 3.1. A Riemannian manifold (M^n, g), n > 3, is said to be of pseudo quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) satisfies the condition (Deszez *et al.,* 1999)

$$R = a_1 G + a_2 g \wedge V + a_3 g \wedge D,$$

where a_1, a_2, a_3 are non-zero scalars, D is a symmetric tensor of type (0, 2) and V = A ⊗ A.

In particular, if $a_3 = 0$, the manifold reduces to the notion of a manifold of quasi-constant curvature introduced by Chen and Yano (Deszez et al., 1999); and if $a_2 = a_3 = 0$, then the manifold turns into a manifold of constant curvature.

Generalizing this notion of pseudo quasi-constant curvature we define the notion of a manifold of generalized pseudo quasi-constant curvature.

Definition 3.2- A Riemannian manifold (M^n, g), n > 3, is said to be of generalized pseudo quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) satisfies the condition

$$R = a_1 G + a_2 g \wedge V + a_3 g \wedge D + a_4 g \wedge E, \tag{3.1}$$

where a_1, a_2, a_3, a_4 are non-zero scalars and D, E are symmetric tensors of type (0, 2).

Especially, if $a_4 = 0$, then the notion reduces to the manifold of pseudo quasi-constant curvature. We now prove the following theorem.

Theorem 3.1- A conformally flat $GPQE_n$, n > 3, is a manifold of generalized pseudo quasi-constant curvature.

Proof. If a $GPQE_n$, $n > 3$, is conformally flat, then its curvature tensor R of type (0, 4) takes the following form

$$R = \frac{1}{n-2} g \wedge S - \frac{r}{(n-1)(n-2)} G. \tag{3.2}$$

Using (1.3) and (2.1) in (3.2) we obtain

$$R = \frac{\alpha(n-2) - \beta}{(n-1)(n-2)} G + \frac{\beta}{n-2} g \wedge V + \frac{\gamma}{n-2} g \wedge D + \frac{\delta}{n-2} g \wedge E. \tag{3.3}$$

Now the relation (3.3) can be written as

$$R = b_1 G + b_2 g \wedge V + b_3 g \wedge D + b_4 g \wedge E, \tag{3.4}$$

where $b_1 = \frac{\alpha(n-2) - \beta}{(n-1)(n-2)}$, $b_2 = \frac{\beta}{n-2}$, $b_3 = \frac{\gamma}{n-2}$ and $b_4 = \frac{\delta}{n-2}$ are non-zero scalars. Comparing (3.1) and (3.4), it follows that the manifold is of generalized pseudo quasi-constant curvature.

Corollary 3.1- *A* $GPQE_3$ is a manifold of pseudo quasi-constant curvature.

Corollary 3.2- A manifold (M^n, g), n > 2, of generalized pseudo quasi-constant curvature is a $GPQE_n$.

Proof. If the manifold is of generalized pseudo quasi-constant curvature, then we have (3.1), which yields on contraction over X and W that

$$S(Y, Z) = \bar{\alpha} g(Y, Z) + \bar{\beta} A(Y) A(Z) + \bar{\gamma} D(Y, Z) + \bar{\delta} E(Y, Z)$$

for all Y, Z, where $\bar{\alpha} = (n-1)a_1 + a_2$, $\bar{\beta} = (n-2)a_2$, $\bar{\gamma} = (n-2)a_3$ and $\bar{\delta} = (n-2)a_4$ are non-zero scalars. Hence the result.

Lemma 3.1- In a conformally flat $GPQE_n$, n > 3, the curvature tensor R of type (1, 3) satisfies the following:

$$R(X,Y)Z = \frac{\alpha(n-2)-\beta}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] + \frac{\gamma}{n-2}[D(Y,Z)X - D(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] + \frac{\delta}{n-2}[E(Y,Z)X - E(X,Z)Y + g(Y,Z)TX - g(X,Z)TY], \tag{3.5}$$

$$R(X,U)Z = -\frac{\alpha(n-2)-\beta}{(n-1)(n-2)}g(X,Z)U - \frac{\gamma}{n-2}D(X,Z)U - \frac{\delta}{n-2}E(X,Z)U \tag{3.6}$$

and

$$R(X,U)U = \frac{\alpha+\beta}{(n-1)}X + \frac{\gamma}{n-2}LX + \frac{\delta}{n-2}TX \tag{3.7}$$

for all X, Y, $Z \in U^{\perp}$, the (n − 1)-dimensional distribution orthogonal to the generator U.

Proof. In a conformally flat $GPQE_n$ we have the relation (3.3). Since $U^{\perp}$ is the $(n-1)$-dimensional distribution orthogonal to the generator U we have $g(X, U) = 0$ if and only if $X \in U^{\perp}$. Hence (3.3) yields the relations (3.5)-(3.7) for all X, Y, $Z \in U^{\perp}$. This proves the theorem.

Theorem 3.2-If a conformally flat $GPQE_n$, n > 3, is homogenous with respect to the structure tensors D, E in the direction of X as well as Y, then the sectional curvature of all planes determined by X, $Y \in U^{\perp}$ is $\frac{\alpha(n-2)-\beta+2(n-1)(\gamma c+d\delta)}{(n-1)(n-2)}$, δ being a scalar.

Proof. Let κ_1 be the sectional curvature of the plane determined by X and Y, where X, $Y \in U^{\perp}$. If the manifold is homogenous with respect to the structure tensors D, E in the direction of X, Y, then we have

$D(X, X) = cg(X, X)$, $D(X, Y) = cg(X, Y)$, and $D(Y, Y) = cg(Y, Y)$, c being a scalar.

$E(X, X) = dg(X, X)$, $E(X, Y) = dg(X, Y)$, and $E(Y, Y) = dg(Y, Y)$, d being a scalar.

Thus by virtue of (3.5) we obtain

$$\kappa_1 = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y)-\{g(X,Y)\}^2} = \frac{\alpha(n-2)-\beta+2(n-1)(\gamma c+d\delta)}{(n-1)(n-2)}.$$

This proves the theorem.

We note that $\kappa_1 = 0$ (resp. constant, non-constant) according as $\alpha(n-2)-\beta+2(n-1)(c\gamma+d\delta)=0$ (resp. $\alpha(n-2)-\beta+2(n-1)(c\gamma+d\delta) =$ constant, $\alpha(n-2)-\beta+2(n-1)(c\gamma+\delta d) \neq$ constant). This leads to the following:

Corollary 3.3- If a conformally flat $GPQE_n$, n > 3, is homogenous with respect to the structure tensor D, E in the direction of X as well as Y , the sectional curvature of all planes determined by X and Y is zero (resp. constant) if and only if $\alpha(n-2)-\beta+2(c\gamma+d\delta)(n-1)=0$ (resp. constant).

Theorem 3.3- If a conformally flat $GPQE_n$, n > 3, is homogenous with respect to the structure tensors D, E in the direction of X, the sectional curvature of all planes determined by X and U is $\frac{\alpha(n-2)+\beta(n-2)+(\gamma c+\delta d)(n-1)}{(n-1)(n-2)}$, c, d being scalars, for all $X \in U^{\perp}$.

Proof. Let κ_2 be the sectional curvature of the plane determined by *X* and *U*, where $X \in U^{\perp}$.
If the manifold is homogenous with respect to the structure tensor *D* in the direction of *X*, then we have

$D(X, X) = cg(X, X)$, $E(X, X) = dg(X, X)$ *c*, *d* being scalars.

Thus by virtue of (3.7) we obtain

$$\kappa_2 = \frac{g(R(X,U)U,X)}{g(X,X)g(U,U)-\{g(X,U)\}^2}$$
$$= \frac{\alpha(n-2)+\beta(n-2)+(\gamma c+\delta d)(n-1)}{(n-1)(n-2)}.$$

This proves the theorem.

We note that $\kappa_2 = 0$ (resp. constant, non-constant) according as $(\alpha+\beta)(n-2)+(c\gamma+d\delta)(n-1)=0$ (resp. $(\alpha+\beta)(n-2)+(c\gamma+d\delta)(n-1)=$ constant, $(\alpha+\beta)(n-2)+(c\gamma+d\delta)(n-1)\neq$ constant). This leads to the following:

Corollary 3.4- If a conformally flat $GPQE_n$, n > 3, is homogenous with respect to the structure tensors D, E in the direction of X, the sectional curvature of all planes determined by X and U is zero (resp. constant) if and only if $(\alpha+\beta)(n-2)+(c\gamma+d\delta)(n-1)=0$ (resp. constant).

Definition 3.3- A Riemannian manifold (M^n, g), n > 3, is said to be conformally con-servative if the divergence of the conformal curvature tensor vanishes (Schouten, 1934)

From the definition of conformal curvature tensor *C*, it can be easily seen that

$$div\, C(X,Y)Z = \frac{n-3}{n-2}\left[(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)-\frac{1}{2(n-1)}\left[dr(X)g(Y,Z)-dr(Z)g(Y,X)\right]\right] \quad (3.8)$$

Then we prove the following:

Theorem 3.4- If in a $GPQE_n$, n > 3, the associated scalars are constants, the structure tensors are of Codazzi type and the generator U is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is conformally conservative.

Proof. If the associated scalars *α*, *β*, *γ* and *δ* are constants, then (2.1) yields that the scalar curvature is constant and hence $dr(X) = 0$ for all *X*. Consequently (3.8) takes the form

$$div\, C(X,Y)Z = \frac{n-3}{n-2}\left[(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)\right]. \quad (3.9)$$

From (1.3) it follows that

$$(\nabla_X S)(Y,Z) = d\alpha(X)g(Y,Z)+\beta\left[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)\right]$$
$$+d\beta(X)A(Y)A(Z)+d\gamma(X)D(Y,Z)+\gamma(\nabla_X D)(Y,Z)$$
$$+d\delta(X)E(Y,Z)+\delta(\nabla_X E)(Y,Z). \quad (3.10)$$

Since *α*, *β*, *γ* and *δ* are constants, (3.10) reduces to

$$(\nabla_X S)(Y,Z) = \beta\left[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)\right]+\gamma(\nabla_X D)(Y,Z)+\delta(\nabla_X E)(Y,Z) \quad (3.11)$$

We now assume that the structure tensors *D*, *E* of $GPQE_n$ are of Coddazi type (Neill, 2003). Then for all vector fields *X*, *Y*, *Z*, we have

$$(\nabla_X D)(Y,Z) = (\nabla_Z D)(Y,X); \quad (\nabla_X E)(Y,Z) = (\nabla_Z E)(Y,X). \quad (3.12)$$

In view of (3.11), (3.9) can be written as

$$div\, C(X,Y)Z = \frac{n-3}{n-2}\Big[\beta\big[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) \tag{3.13}$$
$$-(\nabla_Z A)(Y)A(X) - (\nabla_Z A)(X)A(Y)\big] + \gamma\big[(\nabla_X D)(Y,Z) - (\nabla_Z D)(Y,X)\big]$$
$$+\delta\big[(\nabla_X E)(Y,Z) - (\nabla_Z E)(Y,X)\big]\Big].$$

By virtue of (3.12), (3.13) takes the form

$$div\, C(X,Y)Z = \frac{n-3}{n-2}\Big[\beta\big[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) \tag{3.14}$$
$$-(\nabla_Z A)(Y)A(X) - (\nabla_Z A)(X)A(Y)\big]\Big].$$

Next, if the generator U of the manifold under consideration is a recurrent vector field (Tanno, 1988), then we have $\nabla_X U = \pi(X)U$, where π is called the 1-form of recurrence such that π is different from A. Consequently we get

$$g(\nabla_X U, Y) = g(\pi(X)U, Y) \text{ and hence } (\nabla_X A)(Y) = \pi(X)A(Y). \tag{3.15}$$

In view of (3.15), (3.14) reduces to

$$\frac{n-2}{n-3} div\, C(X,Y)Z = \beta\big[\pi(X)A(Y)A(Z) + \pi(X)A(Z)A(Y) \tag{3.16}$$
$$-\pi(Z)A(Y)A(X) - \pi(Z)A(X)A(Y)\big].$$

Also since $g(U, U) = 1$, it follows that $(\nabla_X A)(U) = g(\nabla_X U, U) = 0$ and hence (3.15) yields $\pi(X) = 0$ for all X. Therefore from (3.16), we have $div\, C(X, Y)Z = 0$. This proves the theorem.

Theorem 3.5- If in a $GPQE_n$, n > 3, the associated scalars are non-constants but their sum vanishes, the structure tensor is of Codazzi type and the generator U satisfy the conditions (3.20) and (3.21), then the manifold is conformally conservative.

Proof. If the associated scalars of $GPQE_n$ are not constants and $\alpha + \beta + \gamma + \delta = 0$, then (3.10) yields

$$(\nabla_X S)(Y,Z) = d\alpha(X)\big[g(Y,Z) - E(Y,Z)\big] - (\alpha+\beta+\gamma)(\nabla_X E)(Y,Z) \tag{3.17}$$
$$+d\beta(X)\big[A(Y)A(Z) - E(Y,Z)\big]$$
$$+\beta\big[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)\big]$$
$$+d\gamma(X)\big[D(Y,Z) - E(Y,Z)\big] + \gamma(\nabla_X D)(Y,Z).$$

From (2.1) we have

$$dr(X) = nd\alpha(X) + d\beta(X). \tag{3.18}$$

Using (3.17) and (3.18) in (3.8) we obtain by virtue of (3.12) that

$$\frac{n-2}{n-3} div\, C(X,Y)Z = \frac{n-2}{2(n-1)}\big[d\alpha(X)g(Y,Z) - d\alpha(Z)g(X,Y)\big] \tag{3.19}$$
$$-\big[d\alpha(X)E(Y,Z) - d\alpha(Z)E(X,Y)\big] - \frac{1}{2(n-1)}\big[d\beta(X)g(Y,Z) - d\beta(Z)g(X,Y)\big]$$
$$+d\beta(X)\big[A(Y)A(Z) - E(Y,Z)\big] - d\beta(Z)\big[A(Y)A(X) - E(Y,X)\big]$$
$$+\big[d\gamma(X)D(Y,Z) - d\gamma(Z)D(Y,X)\big] - \big[d\gamma(X)E(Y,Z) - d\gamma(Z)E(Y,X)\big]$$
$$+\beta\big[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) - (\nabla_Z A)(Y)A(X) - (\nabla_Z A)(X)A(Y)\big].$$

We now assume the following conditions

$$U = \frac{1}{2\alpha} grad\,\alpha = \frac{1}{2\beta} grad\,\beta = \frac{1}{2\gamma} grad\,\gamma, \tag{3.20}$$

$$\beta\nabla_X U = \frac{\alpha(n-2)-\beta}{n-1}X + 2\gamma LX - 2(\alpha+\beta+\gamma)TX \tag{3.21}$$

for all X. Then (3.20) implies that

$$g(X,U) = g\left(X, \frac{1}{2\alpha} grad\ \alpha\right) = \frac{1}{2\alpha} g(X, grad\ \alpha) = \frac{1}{2\alpha} d\alpha(X),$$ i.e.,

$$d\alpha(X) = 2\alpha g(X, U) = 2\alpha A(X). \tag{3.22}$$

Similarly we have

$$d\beta(X) = 2\beta g(X, U) = 2\beta A(X) \text{ and } d\gamma(X) = 2\beta g(X, U) = 2\gamma A(X). \tag{3.23}$$

Again from (3.21) it follows that

$$\beta(\nabla_X A)(Y) = \frac{\alpha(n-2)-\beta}{n-1} g(X,Y) + 2\gamma D(X,Y) - 2(\alpha+\beta+\gamma)E(X,Y). \tag{3.24}$$

In view of (3.22)-(3.23), the relation (3.19) reduces to $div\ C(X, Y)Z = 0$. This proves the theorem.
Now the geometric significance of the condition (3.21) is determined by the following:

Theorem 3.6- A $GPQE_n$, n > 3, satisfying the condition (3.21) is a product manifold.

Proof. For any X, $Y \in U^{\perp}$, $(\nabla_X g)(Y,U) = 0$ implies that

$$\beta g(\nabla_X Y, U) = -\beta g(\nabla_X U, Y),$$

which yields by virtue of (3.21) that

$$\beta g(\nabla_X Y, U) = -\frac{\alpha(n-2)-\beta}{n-1} g(X,Y) - 2\gamma D(X,Y)$$
$$+2(\alpha+\beta+\gamma)E(X,Y) = \beta g(\nabla_Y X, U).$$

Hence

$$g(\nabla_X Y, U) = g(\nabla_Y X, U), \quad \text{i.e.,} \quad g(\nabla_X Y - \nabla_Y X, U) = 0$$

for all X, $Y \in U^{\perp}$. Consequently $g([X, Y], U) = 0$. Thus $[X, Y]$ is orthogonal to U and hence $[X,Y] \in U^{\perp}$. Therefore the distribution $U^{\perp}$ is involutive. Hence from Frobenius theorem [2] it follows that $U^{\perp}$, the $(n-1)$-dimensional distribution orthogonal to U, is integrable. Consequently $GPQE_n$ is a product manifold. This completes the proof.

4. Some global properties of $GPQE_n$

This section is concerned with a compact, orientable $GPQE_n$, $n > 2$, without boundary with vanishing Euler number, and also with α, β, γ, δ as associated scalars, U as the generator and D, E as the structure tensors. Then we prove the following:

Theorem 4.1- If in a compact, orientable $GPQE_n$, n > 2, without boundary and with vanishing Euler number, the associated scalars and the structure tensor are such that $\beta > 0$, $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$ for any X, then there exists no non-zero Killing vector field in this manifold.

Proof. It is known that [34] for any vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \left[S(X,X) - |\nabla X|^2 - (div\ X)^2\right] dv \le 0, \tag{4.1}$$

where dv denotes the volume element of M. If X is a Killing vector field, then div $X = 0$. Hence (4.1) takes the following form

$$\int_M \left[S(X,X) - |\nabla X|^2\right] dv = 0. \tag{4.2}$$

Let θ be the angle between the generator U and any vector X of $GPQE_n$. Then $\cos\theta = \frac{g(X,U)}{\sqrt{g(X,X)}} \le 1$. Therefore $g(X,U) \le \sqrt{g(X,X)}$, and consequently from (1.3) it follows that

$$S(X,X) \le (\alpha+\beta)|X|^2 + \gamma D(X,X) + \delta E(X,X) \quad \text{for } \beta > 0. \tag{4.3}$$

Let us consider $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$. Hence by virtue of (4.3) we have

$$\int_{M=P(QE)_n} \left[(\alpha + \beta)|X|^2 + \gamma D(X, X) + \delta E(X, X) - |\nabla X|^2 \right] dv$$

$$\geq \int_M \left[S(X, X) - |\nabla X|^2 \right] dv,$$

which yields by virtue of (4.2) that

$$\int_M \left[(\alpha + \beta)|X|^2 + \gamma D(X, X) + \delta E(X, X) - |\nabla X|^2 \right] dv \geq 0 .$$

If $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$, then the last relation reduces to

$$\int_M \left[(\alpha + \beta)|X|^2 + \gamma D(X, X) + \delta E(X, X) - |\nabla X|^2 \right] dv = 0 .$$

Hence $X = 0$. This proves the theorem.

Definition 4.1- A vector field X in a Riemannian manifold (M^n, g), n > 2, is said to be projective Killing vector field if it satisfies

$$(\pounds_X \nabla)(Y, Z) = \omega(Y)Z + \omega(Z)Y$$

for any vector fields Y and Z, ω being a certain 1*-form and £ is the operator of Lie differentiation*.

Theorem 4.2- If in a compact, orientable $GPQE_n$, n > 2, without boundary and with vanishing Euler number, the associated scalars and the structure tensors are such that $\beta > 0$, $\alpha + \beta \leq 0$ and $\gamma D(X, X) + \delta E(X, X) \leq 0$ for any X, then a projective Killing vector field has vanishing covariant derivative; and if $\beta > 0$, $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$ for any X, then there exists no non-zero projective Killing vector field in this manifold.

Proof. We know that for any vector field *X* in a Riemannian manifold *M*, the following relation holds

$$\int_M \left[S(X, X) - \frac{1}{4}|d\xi|^2 - \frac{n-1}{2(n+1)}(div\ X)^2 \right] dv = 0, \tag{4.4}$$

where ξ is an 1-form corresponding to the vector field *X*. We now assume the conditions $\beta > 0$, $\alpha + \beta \leq 0$ and $\gamma D(X, X) + \delta E(X, X) \leq 0$ for any *X*. Therefore (4.3) yields $S(X, X) \leq 0$ and hence from (4.4) we obtain $d\xi = 0$ and $div\ X = 0$. This implies that *X* is harmonic as well as a Killing vector field. Consequently its covariant derivative vanishes. This proves the theorem.

Definition 4.2. A vector field X in a Riemannian manifold (M^n, g), n > 2, is said to be conformal Killing vector field if it satisfies

$$\pounds_X g = 2\rho g$$

for any vector field X, where ρ is given by $\rho = -\frac{1}{n}(div\ X)$ and £ is the operator of Lie differentiation.

Theorem 4.3. If in a compact, orientable $GPQE_n$, n > 2, without boundary and with vanishing Euler number, the associated scalars and the structure tensors are such that $\beta > 0$, $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$ for any X, then there exists no non-zero conformal Killing vector field in this manifold.

Proof. It is known that for any vector field *X* in a Riemannian manifold *M*, the following relation holds

$$\int_M \left[S(X, X) - |\nabla X|^2 - \frac{n-2}{n}(div\ X)^2 \right] dv = 0, \tag{4.5}$$

where *dv* denotes the volume element of *M*. Now we assume the conditions $\beta > 0$, $\alpha + \beta < 0$ and $\gamma D(X, X) + \delta E(X, X) < 0$ for any *X*. Then proceeding similarly as before we obtain

$$\nabla X = 0,\ div\ X = 0.$$

This proves the theorem.

Theorem 4.4. If in a compact, orientable $GPQE_n$, n > 2, without boundary and with vanishing Euler number, the associated scalars and the structure tensors are such that β > 0, $\alpha |X|^2 + \gamma D(X,X) + \delta E(X,X) = 0$ for any X, then any harmonic vector field is orthogonal to the generator U and also it is a parallel vector field.

Proof. It is known that for any vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \left[S(X,X) - |\nabla X|^2 \right] dv = 0, \tag{4.6}$$

where dv denotes the volume element of M. Now we assume the conditions $\beta > 0$, $\alpha |X|^2 + \gamma D(X,X) + \delta E(X,X) = 0$. Then by virtue of (1.3) we obtain $S(X,X) = \beta |g(X,U)|^2$ and hence (4.6) reduces to

$$\int_M \left[\beta |g(X,U)|^2 - |\nabla X|^2 \right] dv = 0,$$

which implies that

$$g(X, U) = 0, \text{ and } \nabla X = 0.$$

Hence the theorem follows.

5. A proper Example of $GPQE_n$

This section deals with a proper example of $GPQE_n$.

Example 5.1- A $(2m+1)$-dimensional smooth manifold M is said to have a contact structure and is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^m \neq 0$$

everywhere on M, where the exponent denotes the m-th exterior power. We call η a contact form on M. A Riemannian metric g is said to be an associated metric if there exists a (1, 1) tensor field ϕ, and a vector field ξ such that

$$d\eta(X,Y) = g(X,\phi Y), \quad \eta(\xi) = 1,$$

$$\phi^2 X = -X + \eta(X)\xi$$

for all vector fields X, Y. A contact structure with an associated metric is called a contact metric structure and the contact manifold equipped with a contact metric structure is called a contact metric manifold. Given a contact metric manifold, we define a (1, 1) tensor field h by $h = \frac{1}{2}£_\xi \phi$, where £ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$, $Tr.h = Tr.\phi h = 0$.

In 1998, S. Tanno introduced the notion of k-nullity distribution of a Riemannian manifold as a distribution

$$p \to N_p(k) = \left[Z \in T_p M : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} \right]$$

for any X, $Y \in T_p M$.

A contact metric manifold with ξ belonging to $N(k)$ satisfies the relation

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y]$$

for all X, Y. In particular, if k = 1 then the manifold is Sasakian. Generalizing this notion of k-nullity distribution (Blair, *et al.,* 1995) introduced the notion of (k, μ)-nullity distribution of a contact metric manifold as a distribution

$$p \to N_p(k,\mu) = \left[Z \in T_p M : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} \right.$$

$$+\mu\{g(Y,Z)hX - g(X,Z)hY\}]$$

for any X, $Y \in T_pM$, k, μ are real constants. A contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If we assume k, μ as smooth functions then a (k, μ)-contact metric manifold is said to be generalized (k, μ)-contact metric manifold (Spivak, 1970). However, such a manifold exists only for dimension three (Shaikh *et. al*., 2006 and Yano, 1970).

A ($2m$ + 1)-dimensional contact metric manifold M (η, ξ, φ, g) whose characteristic vector field ξ is a harmonic vector field is called an H-contact metric manifold. D. Perrone (Spivak, 1970) proved that $M(\eta, \xi, \varphi, g)$ is an H-contact metric manifold if and only if ξ is an eigenvector of the Ricci operator, which generalizes the results of González-Dávila and Vanhecke (Perrone, 201) for m = 1. It is important to mention that the class of H-contact metric manifolds includes several interesting classes of contact metric manifolds such as Sasakian and η-Einstein manifolds, K-contact manifolds, strongly φ-symmetric spaces, (k, μ)-contact metric manifolds, and generalized (k, μ)-contact metric manifolds. Perrone (Spain, 19995) also gave a geometric interpretation of generalized (k, μ) in terms of harmonic maps.

Again in (Shaikh, *et al.,* 2009) Koufogiorgos *et al.,* 2008 introduced the notion of (k, μ, v)-contact metric manifold, defined as follows:

A ($2m$ + 1)-dimensional contact metric manifold $M(\eta, \xi, \varphi, g)$ is said to be a (k, μ, v)-contact metric manifold if its curvature tensor satisfies

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + v[\eta(Y)\varphi hX - \eta(X)\varphi hY]$$

for all X, $Y \in T_pM$, where k, μ, v are smooth functions on M. Then Koufogiorgos *et. al.,* 2008 and Shaiks *et al*., 2009 proved that if a 3-dimensional contact metric manifold $M(\eta, \xi, \varphi, g)$ is a (k, μ, v)-contact metric manifold, then M is an H-contact metric manifold and conversely; if M is a 3-dimensional H-contact metric manifold, then M is a (k, μ, v)-contact metric manifold on an everywhere open and dense subset of M. They also proved that such a manifold exists only for dimension 3. Then Koufogiorgs *et. al.,* 2008 and Shaiks *et al*., 2009 obtained the Ricci tensor of a (k, μ, v)-contact metric manifold as follows:

$$S(Y, Z) = \alpha g(Y, Z) + \beta\eta(Y)\,\eta(Z) + \gamma D(Y, Z) + \delta E(Y, Z),$$

where $\alpha = \frac{r}{2} - k,\ \beta = -\frac{r}{2} + 3k,\ \gamma = \mu,\ \delta = v,\ D(Y,Z) = g(hY,Z)$ and $E(Y, Z) = g(\varphi hY, Z)$ are symmetric tensor of type (0, 2) such that $Tr.D = Tr.E = 0$. Also since in a contact metric manifold $h\xi = \varphi\xi = 0$, it follows that $D(X, \xi) = 0 = E(X, \xi)$. Hence if we take $U = \xi$, then a (k, μ, v)-contact metric manifold is a $GPQE_3$ which is neither PQE_3 nor QE_3 . Thus we can state the following:

Theorem 5.1- A 3-dimensional (k, μ, ν)-contact metric manifold is a generalized pseudo quasi-Einstein manifold, which is neither quasi-Einstein nor pseudo-quasi-Einstein.

References

1. Blair, D. E., Koufogiorgos, T. and Papantoniou, B. J. (1995). Contact metric manifolds satisfying a nullity condition. *Israel Journal of Math.* 91: 189-214.
2. Brickell, F. and Clark, R. S. (1978). Differentiable manifold. *Van. Nostrand. Reinhold Comp.*, London.
3. Chaki, M. C. (2001). On generalized quasi Einstein manifolds. *Publ. Math. Debrecen*. 58: 683-691.
4. Chaki, M. C. and Maity, R. K. (2000). On quasi-Einstein manifolds. *Publ. Math. Debrecen*.57: 297-306.
5. Chen, B. Y. and Yano, K. (1972). Hypersurfaces of conformally flat spaces. *Tensor N. S.* 26: 318-322.
6. De, U. C. and De, B. K. (2008). On quasi Einstein manifolds, Commun. *Korean Math. Soc.* 23: 413-420.

7. De, U. C. and Ghosh, G. C. (2004). On quasi Einstein manifolds. *Period. Math. Hungar.* 48(1-2): 223-231.
8. De, U. C. and Ghosh, G. C. (2004). On generalized quasi Einstein manifolds. *Kyungpook Math. Journal.* 44(4): 607-615.
9. De, U. C. and Ghosh, G. C. (2005). Some global properties of generalized quasi-Einstein manifolds. *Ganita.* 56(1): 65-70.
10. De, U. C., Shaikh, A. A. and Sengupta, J. (2007). Tensor Calculus, Second *Edn*. *Narosa Publ. Pvt. Ltd*.
11. Deszcz, R., Dillen, F., Verstraelen, L. and Vrancken, L. (1999). Quasi-Einstein totally real submanifolds of $S^6(1)$. *Tohoku Math. Journal.* 51: 461-478.
12. Deszcz, R., Glogowska, M., Hotlo´s, M. and Sentürk, Z. (1998). On certain quasi-Einstein semi-symmetric hypersurfaces. *Annales Univ. Sci. Budapest.* 41: 153-166.
13. Deszcz, R., Hotloś, M. and Sentürk, Z. (2001). Quasi-Einstein hypersurfaces in semi-Riemannian space forms. *Colloq. Math.*, 81: 81-97.
14. Deszcz, R., Hotloś, M. and Sentürk, Z. (2001). On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces. *Soochow Journal Math*., 27(4): 375-389.
15. Deszcz, R., Verheyen, P. and Verstraelen, L. (1996). On some generalized Einstein metric conditions. *Publ. Inst. Math*., 60:74: 108-120.
16. Ferus, D. (1981). A remark on Codazzi tensors on constant curvature space, Lecture Notes in Math.. 838, *Global Differential Geometry and Global Analysis, Springer-Verlag* , New York.
17. Glogowska, M. (2008). On quasi-Einstein Cartan type hypersurfaces. *Journal Geom. and Phys.*, 58: 599-614.
18. González-Dávila, J. C. and Vanhecke, L. (2001). Minimal and harmonic characteristic vector fields on three dimensional contact metric three-manifolds. *Journal of Geom*. 72(1-2): 65-76.
19. Hicks, N. J. (1969). Notes on Differential Geometry. *Affiliated East West Press Pvt. Ltd*.
20. Koufogiorgos, T., Markellos, M. and Papantoniou, V. J. (2008). The harmonicity of the reeb vector field on contact metric 3-manifolds. *Pacific Journal of Math*. 234(2): 325-344.
21. Koufogiorgos, T. and Tsichlias, C. (2003). Generalized (k, μ)-contact metric manifolds with ||grad k|| =constant. *Journal of Geom*. 78: 83-91.
22. O' Neill, B. (1983). Semi-Riemannian Geometry with application to relativity. *Academic Press*, New York.
23. Perrone, D. (2003), Harmonic characteristic vector fields on contact metric three-manifolds, *Bull. Austral. Math. Soc*., 67(2): 305-315.
24. Perrone, D. (2004), Contact metric manifold whose characteristic vector fields is a harmonic vector field, *Diff. Geom. Appl*. 20(3): 367-378.
25. Schouten, J. A. (1934). Ricci Calculus (2nd Edn.). *Springer-Verlag*, Berlin.
26. Shaikh, A. A. (2009). On pseudo quasi-Einstein manifolds. *Period. Math. Hungarica*. 59(2): 119-146.
27. Shaikh, A. A. and Baishya, K. K. (2006). On a contact metric manifold, *Diff. Geom-Dynamical System*.8: 253-261.
28. Shaikh, A. A. and Baishya, K. K. and Eyasmin, S. (2006). On ϕ-recurrent generalized (k, μ)-contact metric manifolds. *Lobachevski Journal of Math*. 27: 3-13.
29. Shaikh, A. A. and Jana, S. K. (2008). On pseudo generalized quasi-Einstein manifolds. *Tamkang Journal of Math*. 39: 9-24.
30. Shaikh, A. A., Özgur, C. and Patra, A. (2011). On hyper-generalized quasi-Einstein manifolds. *Int. J. of Math. Sci. and Engg. Appl*. 5(3): 189-206.
31. Spain, B. (1995).Tensor calculas (3rd Edn.). *Diff. Radha Publishing House*. Kolkata.
32. Spivak, M. (1970), A comprehensive Introduction to Differential Geometry, *Publish and Perish*,
33. Tanno, S. (1988). Ricci curvatures of Riemannian manifolds. *Tohoku Math. Journal*.40: 441-448.
34. Watanabe, Y. (1968). Integral inequalities in compact orientable manifolds. Riemannian or Kahlerian, *Kodai Math. Sem. Rep.*, 20: 261-271.
35. Yano, K. (1970). Integral formulas in Riemannian Geometry. *Marcel Dekker*, New York.

Received on 05.11.2010 and accepted on 13.06.2011